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ON THE IMAGE OF GALOIS l-ADIC REPRESENTATIONS FOR ABELIAN VARIETIES OF TYPE III

GRZEGORZ BANASZAK, WOJCIECH GAJDA AND PIOTR KRASOŃ

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Abstract. In this paper we investigate the image of the l-adic representation attached to the Tate module of an abelian variety defined over a number field. We consider simple abelian varieties of type III in the Albert classification. We compute the image of the l-adic and mod l Galois representations and we prove the Mumford-Tate and Lang conjectures for a wide class of simple abelian varieties of type III.

1. Introduction. Our main objective in this paper is the computation of the images of the Galois representations:

$$\rho_l \,:\, G_F \,\rightarrow \mathrm{GL}(T_l(A))\,,$$

$$\overline{\rho_l} \,:\, G_F \,\rightarrow \mathrm{GL}(A[l])\,,$$

attached to certain abelian varieties of type III according to the Albert classification list (cf. [20, p. 201, Theorem 2]). We also prove the Mumford-Tate and Lang conjectures for these varieties. To be more precise, the main results of this paper concern the following class of abelian varieties:

DEFINITION 1.1. Abelian variety A/F defined over a number field F is of class $\mathcal{B}$, if the following conditions hold:

(i) A is a simple abelian variety of dimension g.

(ii) $\mathcal{R} = \mathrm{End}_{\bar{F}}(A) = \mathrm{End}_F(A)$ and the endomorphism algebra $D = \mathcal{R} \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ is of type III in the Albert list of division algebras with involution.

(iii) The field F is such that, for every l, the Zariski closure G_l^{alg} of the image $\rho_l(G_F)$ in $\mathrm{GL}_{V_l(A)}/\boldsymbol{Q}_l$ is a connected algebraic group.

(iv) $g = 2eh$, where h is an odd integer and $e = [E : \boldsymbol{Q}]$ is the degree of the center E of D.

The organization of the paper and its main results are as follows. In Sections 2 and 3, we give an explicit description of the endomorphism algebra and its involution for an abelian variety of type III as well as the relation to various bilinear forms coming from Weil pairing. This detailed treatment of endomorphism algebras and bilinear forms differs significantly

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from that of [6] and [2]. Due to our approach the proof of Theorem 3.29, in Section 3, is achieved in an explicit way. Theorem 3.29 is an important tool which gives us symmetric nondegenerate forms out of symplectic forms coming from the Weil pairing. These symmetric forms are defined over the rings of integers in the completions of the center of D at primes over l for $l \gg 0$. In Section 4 we compute Lie algebras that lead to the determination of $(G_l^{\mathrm{alg}})'$ (Theorem 4.19). In Section 5 we apply Theorem 4.19 in the proof of the Mumford-Tate conjecture for the abelian varieties of class $\mathcal{B}$:

THEOREM 5.11. *If A is an abelian variety of class $\mathcal{B}$, then*

$$G_l^{\mathrm{alg}} = \mathrm{MT}(A) \otimes \boldsymbol{Q}_l\,,$$

for every prime number l, where $\mathrm{MT}(A)$ denotes the Mumford-Tate group of A, i.e., the Mumford-Tate conjecture is true for A.

This generalizes the result of Tankeev [32] who proved the Mumford-Tate conjecture for abelian varieties of type III, with similar dimension restrictions, such that $\mathrm{End}(A) \otimes \boldsymbol{Q}$ has center equal to $\boldsymbol{Q}$. In particular, Theorem 5.11 implies the result of Tankeev [32] for abelian varieties over number fields such that G_l^{alg} is connected for every l. We have been very recently informed by A. Vasiu about his results [35] where he proves some cases of the Mumford-Tate conjecture for abelian varieties of types I through IV.

On the way of the proof of Mumford-Tate conjecture, we also compute explicitly the Hodge group and prove that it is equal to the Lefschetz group. However this is not enough to get the Hodge conjecture for abelian varieties of type III of class $\mathcal{B}$ (cf. [21]). Note that the proof of Mumford-Tate conjecture and equality of Hodge and Lefschetz groups for abelian varieties of type I and II of class $\mathcal{A}$ in [2] gave us the Hodge and Tate conjectures for these abelian varieties. In Section 6 (Theorem 6.29) we estimate the images $\rho_l(G'_F)$ and $\overline{\rho_l}(G'_F)$ where $G' := \overline{[G,\, G]}$ denote the closure of the commutator subgroup for any profinite group G. This estimation gives the following theorem.

THEOREM 6.31. *If A is an abelian variety of class $\mathcal{B}$, then for $l \gg 0$*

$$\rho_l(G''_F) = \prod_{\lambda | l} \mathrm{SO}_{(T_\lambda,\, \psi_\lambda)}(\mathcal{O}_\lambda)'\,,$$

$$\overline{\rho_l}(G''_F) = \prod_{\lambda | l} \mathrm{SO}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}(k_\lambda)'\,.$$

Let κ be the $\boldsymbol{Z}$-bilinear, non-degenerate, alternating pairing $\kappa\,:\,\Lambda \times \Lambda \to \boldsymbol{Z}$ given by the polarization of A, where Λ is the Riemann lattice such that $A(\boldsymbol{C}) = \boldsymbol{C}^g/\Lambda$. Let $C_{\mathcal{R}}(\mathrm{Sp}_{(\Lambda,\,\kappa)})$ be the centralizer of $\mathcal{R}$ in $\mathrm{Sp}_{(\Lambda,\,\kappa)}$. In the proof of Proposition 6.23 we show that:

$$C_{\mathcal{R}}(\mathrm{Sp}_{(\Lambda,\,\kappa)}) \otimes_{\boldsymbol{Z}} \boldsymbol{Z}_l \;\cong\; \prod_{\lambda | l} \mathrm{SO}_{(T_\lambda,\, \psi_\lambda)} \quad \text{for } l \gg 0\,.$$

In Section 7 we prove the following generalization of the open image theorem of Serre [27], [29].

THEOREM 7.2. *Let A be an abelian variety of class $\mathcal{B}$ and let $r(l)$ be the number of primes over l in $\mathcal{O}_E$. Then*:

(i) $\rho_l(G_F)$ *is open in* $C_{\mathcal{R}}(\mathrm{GSp}_{(\Lambda,\kappa)})(\boldsymbol{Z}_l)$ *for every prime number* l,

(ii) $\rho_l(G'_F)$ *has index dividing* $2^{r(l)}$ *in* $C_{\mathcal{R}}(\mathrm{Sp}_{(\Lambda,\kappa)})(\boldsymbol{Z}_l)$ *for* $l \gg 0$,

(iii) $\rho_l(G''_F) = C_{\mathcal{R}}(\mathrm{Sp}_{(\Lambda,\kappa)})(\boldsymbol{Z}_l)'$ *for* $l \gg 0$.

For other results concerning the images of Galois representations coming from abelian varieties, see also [33], [34].

2. Abelian varieties of type III and their endomorphism algebras. Let A/F be a simple abelian variety of dimension g such that $D = \mathrm{End}_{\bar{F}}(A) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} = \mathrm{End}_F(A) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ and the polarization of A is defined over F. We assume that A/F is an abelian variety over F of type III according to the Albert's classification list. Hence D is a definite quaternion algebra over E with center E, a totally real extension of $\boldsymbol{Q}$ of degree e such that, for every imbedding $E \subset \boldsymbol{R}$,

$$D \otimes_E \boldsymbol{R} = \boldsymbol{H}\,.$$

Observe that in this case $[D : E] = 4$ so $g = 2eh$ where $e = [E : \boldsymbol{Q}]$ and h is an integer. We take $l \gg 0$ such that A has good reduction at all primes over l (cf. [30]) and the algebra D splits over all primes over l and l does not divide the degree of the polarization. Let $\mathcal{R}_D$ be a maximal order in D. Since $\mathcal{R} = \mathrm{End}_{\bar{F}}(A)$ is an order in D, we observe that $\mathcal{R} \otimes_{\boldsymbol{Z}} \boldsymbol{Z}_l = \mathcal{R}_D \otimes_{\boldsymbol{Z}} \boldsymbol{Z}_l$ for l that does not divide the index $[\mathcal{R}_D : \mathcal{R}]$. Since $\mathcal{R}$ is a finitely generated free $\boldsymbol{Z}$-module, we check that $\mathcal{R} \cap E = \mathcal{O}_E^0$ is an order in $\mathcal{O}_E$.

To get explicit information about the algebra D we start with a more general framework. Let D be a division algebra with two involutions $*_1$ and $*_2$ and the center E. For each $x \in D$ we will denote x^{*_i} to be the image of the involution $*_i$ acting on x. By Skolem-Noether Theorem [24, p. 103], there is an element $a \in D$ such that for each $x \in D$ we have:

$$x^{*_2} = a x^{*_1} a^{-1}\,. \tag{2.1}$$

Because $*_i \circ *_i = \mathrm{id}_D$, applying $*_2$ to (2.1), we get

$$a^{*_1} = \varepsilon a \tag{2.2}$$

for $\varepsilon \in E$ and applying $*_1$, we check that $\varepsilon^2 = 1$. Hence $\varepsilon = 1$ or $\varepsilon = -1$ (cf. [20, p. 195]). Observe that the center of D is invariant under any involution of D. Hence, by (2.1), $c^{*_1} = c^{*_2}$ for every $c \in E$. Let $E_0 = \{c \in E\,;\ c^{*_1} = c^{*_2} = c\}$. Then E/E_0 is an extension of degree at most 2.

For a simple abelian variety of type III, $E = E_0$ and E is totally real (cf. [20, p. 194]). Also in this case $\varepsilon = 1$ in (2.2) (cf. [20, pp. 193–196]). Hence $a \in E$ and $*_2 = *_1$. Therefore the division algebra D coming from a simple abelian variety of type III has a unique positive involution $*$, i.e., the Rosati involution. Moreover the map $D \to D$ given by $\alpha \to \alpha^*$ is an isomorphism of E-algebras so by [24, p. 96, Corollary 7.14], the algebra D gives an element of order 1 or 2 in $\mathrm{Br}(E)$. Since D is a noncommutative division algebra, it gives an element of order 2 in $\mathrm{B}r(E)$.

By [24, Theorem 32.20], every central simple E-algebra is cyclic. This shows that D is isomorphic, as an E-algebra, to the division algebra

$$D(c,d) := \{a_0 + a_1\alpha + a_2\beta + a_3\alpha\beta\,;\ \alpha^2 = c,\ \beta^2 = d,\ \alpha\beta = -\beta\alpha\}\,. \tag{2.3}$$

This isomorphism induces the unique positive involution on $D(c,d)$ which will also be denoted by $*$. Therefore $*$ must be the natural positive involution

$$(a_0 + a_1\alpha + a_2\beta + a_3\alpha\beta)^* = a_0 - a_1\alpha - a_2\beta - a_3\alpha\beta$$

on $D(c,d)$. From now on we identify D with $D(c,d)$. Since $D \otimes_E \boldsymbol{R} = \boldsymbol{H}$ for every imbedding $E \to \boldsymbol{R}$, we observe that c and d are totally negative numbers. Put $L = E(\alpha)$. Let $\eta = a_0 + a_1\alpha$ and $\gamma = a_2 + a_3\alpha$. Hence

$$\eta + \gamma\beta = a_0 + a_1\alpha + a_2\beta + a_3\alpha\beta\,.$$

For an element $\delta = e + f\alpha \in L$ with $e, f \in E$, put $\bar{\delta} = e - f\alpha$. The field L splits the algebra $D(c,d)$. Namely we have an isomorphism of L algebras:

$$D(c,d) \otimes_E L \to M_{2,2}(L) \tag{2.4}$$

$$(\eta + \gamma\beta) \otimes 1 \mapsto \begin{bmatrix} \eta & \gamma \\ d\bar{\gamma} & \bar{\eta} \end{bmatrix}.$$

From this isomorphism it is clear that

$$(\eta + \gamma\beta)^* = \mathrm{Tr}^0(\eta + \gamma\beta) - (\eta + \gamma\beta)$$

because by definition

$$\mathrm{Tr}^0(\eta + \gamma\beta) = \mathrm{Tr}\begin{bmatrix} \eta & \gamma \\ d\bar{\gamma} & \bar{\eta} \end{bmatrix} = 2a_0\,,$$

where Tr^0 denotes the reduced trace (see [24, pp. 112–116]) from $D(c,d)$ to E. The involution on $M_{2,2}(L)$ induced by $*$ is of the following form:

$$B^* = J\,{}^tBJ^{-1} \tag{2.5}$$

where $B \in M_{2,2}(L)$ and

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

REMARK 2.6. It is clear that if we take in the above computations, instead of $L = E(\alpha)$, the field $E(\beta)$ or $E(\alpha\beta)$, then they also split the algebra D by a formula similar to (2.4), and the involution $*$ will induce on $M_{2,2}(E(\beta))$ and $M_{2,2}(E(\alpha\beta))$ the involution given by formula (2.5).

Note that any maximal commutative subfield of $D(c,d)$ has form $E(a_1\alpha+a_2\beta+a_3\alpha\beta)$ for some $a_1, a_2, a_3 \in E$ not all equal to zero. If $\mathrm{Nr}^0 : D(c,d) \to E$ denotes the reduced norm, then, for every $\eta+\gamma\beta \in D(c,d)$, we have

$$\begin{aligned}\mathrm{Nr}^0(\eta+\gamma\beta) &= \det\begin{bmatrix} \eta & \gamma \\ d\bar{\gamma} & \bar{\eta}\end{bmatrix} = (\eta+\gamma\beta)^*(\eta+\gamma\beta) \\ &= a_0^2 - a_1^2c - a_2^2d + a_3^2cd = a_0^2 - (a_1\alpha+a_2\beta+a_3\alpha\beta)^2 .\end{aligned} \tag{2.7}$$

For some $a_1, a_2, a_3 \in E$ not all equal to zero, put $\alpha' := a_1\alpha + a_2\beta + a_3\alpha\beta$. If $\beta' := b_1\alpha + b_2\beta + b_3\alpha\beta$ is an element of $D(c,d)$, put $c_1 := a_3b_2 - a_2b_3$, $\quad c_2 := a_1b_3 - a_3b_1$ and $c_3 := a_1b_2 - a_2b_1$. Then

$$\alpha'\beta' = a_1b_1c + a_2b_2d - a_3b_3cd + c_1d\alpha + c_2c\beta + c_3\alpha\beta \tag{2.8}$$

and

$$\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ dc_1 & cc_2 & c_3\end{bmatrix} = -d\,c_1^2 - c\,c_2^2 + c_3^2 \geq 0 . \tag{2.9}$$

Since $c < 0$ and $d < 0$, the determinant in (2.9) is zero if and only if the elements α' and β' are linearly dependent over E. Hence it is possible to find β' in such a way that $a_1b_1c + a_2b_2d - a_3b_3cd = 0$ and the determinant in (2.9) is nonzero. With this choice of β', we see that $c' := \alpha'^2 < 0$, $d' := \beta'^2 < 0$ and $\alpha'\beta' = -\beta'\alpha'$. We observe that, for any $a_0', a_1', a_2', a_3' \in E$

$$(a_0' + a_1'\alpha' + a_2'\beta' + a_3'\alpha'\beta')^* = a_0' - a_1'\alpha' - a_2'\beta' - a_3'\alpha'\beta' . \tag{2.10}$$

Hence $D(c,d) = D(c',d')$, and we can use the field $L = E(\alpha')$ and the isomorphism (2.4) for this field to split our algebra $D(c',d')$. Recall that $D(c,d)\otimes_E \boldsymbol{R} \cong \boldsymbol{H}$ for any imbedding $E \to \boldsymbol{R}$, so $D(c',d')\otimes_E \boldsymbol{R} \cong \boldsymbol{H}$ for any imbedding $E \to \boldsymbol{R}$. Hence all numbers c, d, c', d' are negative in any imbedding $E \to \boldsymbol{R}$.

For a given prime number l, throughout the paper, λ will denote an ideal in $\mathcal{O}_E$ such that $\lambda | l$ and w will denote an ideal of $\mathcal{O}_L$ such that $w|\lambda$.

Let S be a finite set of primes of $\boldsymbol{Z}$ such that it contains 2 and all prime numbers divisible by primes in the decomposition of c and d. Moreover we assume that S contains the prime numbers divisible by prime ideals of $\mathcal{O}_E$ that are ramified primes for the algebra D (cf. [24, Theorem 32.1]). We can also assume that S is big enough so that $\mathcal{R}_S := \mathcal{R}\otimes_{\mathcal{O}_E^0}\mathcal{O}_{E,S}$ is a maximal $\mathcal{O}_{E,S}$ order of D with $\mathcal{R}_S \cap E = \mathcal{O}_{E,S}$ and $\mathcal{R}_S = \mathcal{O}_{E,S} + \mathcal{O}_{E,S}\alpha + \mathcal{O}_{E,S}\beta + \mathcal{O}_{E,S}\alpha\beta$.

LEMMA 2.11. *Let $l \notin S$ and $\lambda\,|\,l$. There is a finite set S' of prime numbers such that $S \subset S'$ and $l \notin S'$, and there are elements $\alpha' := a_1\alpha + a_2\beta + a_3\alpha\beta \in \mathcal{R}_S$ and $\beta' := b_1\alpha + b_2\beta + b_3\alpha\beta \in \mathcal{R}_{S'} := \mathcal{R}_S \otimes_{\mathcal{O}_{E,S}} \mathcal{O}_{E,S'}$ such that*

(i) *$c' := \alpha'^2$ and $d' := \beta'^2$ are relatively prime to λ and $\alpha'\beta' = -\beta'\alpha'$,*

(ii) *$D(c,d) = D(c',d')$ and $\mathcal{R}_{S'} = \mathcal{O}_{E,S'} + \mathcal{O}_{E,S'}\alpha' + \mathcal{O}_{E,S'}\beta' + \mathcal{O}_{E,S'}\alpha'\beta'$,*

(iii) *the maximal commutative subfield* $L = E(\alpha')$ *of* $D(c,d)$ *gives the isomorphism* (2.4) *which induces the imbedding of* $\mathcal{O}_{E,S'}$*-algebras*

$$\mathcal{R}_{S'} \to M_{2,2}(\mathcal{O}_{E,S'})\,, \tag{2.12}$$

(iv) *for* $\mathcal{R}_\lambda := \mathcal{R}_S \otimes_{\mathcal{O}_{E,S}} \mathcal{O}_\lambda$ *the imbedding* (2.12) *yields, after tensoring with* $\mathcal{O}_\lambda$*, the isomorphism of* $\mathcal{O}_\lambda$*-algebras*

$$\mathcal{R}_\lambda \simeq M_{2,2}(\mathcal{O}_\lambda)\,. \tag{2.13}$$

PROOF. By [24, Theorems 22.4, 22.15 and 24.13], there is a maximal ideal $M \subset \mathcal{R}$ such that $\mathrm{Nr}^0(M) = \lambda$. Let $\mathcal{P} \subset M$ be the unique prime ideal of $\mathcal{R}$ corresponding to M (cf. [24, Theorem 22.15]). By our choice of l and [24, Theorem 32.1], we get $\lambda\mathcal{R} = \mathcal{P}$. It follows by [24, Theorem 22.10 and Corollary 24.12] that there is an element $t \in \lambda \setminus \lambda^2$ such that $\mathrm{Nr}^0(m) = t$ for some $m = a_0 + a_1\alpha + a_2\beta + a_3\alpha\beta \in M$. Formula (2.7) gives

$$t = a_0^2 - (a_1\alpha + a_2\beta + a_3\alpha\beta)^2 = a_0^2 - ca_1^2 - da_2^2 + cda_3^2\,.$$

Since $t \in \lambda \setminus \lambda^2$, it is clear that $a_i \notin \lambda$ for some $0 \le i \le 3$. By multiplying the last formula by $-c$, $-d$ and cd, we get additional three formulas

$$-ct = (ca_1)^2 - (a_0\alpha + ca_3\beta + a_2\alpha\beta)^2 = (ca_1)^2 - ca_0^2 - d(ca_3)^2 + cda_2^2\,,$$

$$-dt = (da_2)^2 - (da_3\alpha + a_0\beta + a_1\alpha\beta)^2 = (da_2)^2 - c(da_3)^2 - da_0^2 + cda_1^2\,,$$

$$cdt = (cda_3)^2 - (da_2\alpha + ca_1\beta + a_0\alpha\beta)^2 = (cda_3)^2 - c(da_2)^2 - d(ca_1)^2 + cda_0^2\,.$$

Based on these four formulas, we put

$$\begin{aligned}
\alpha_0 &:= a_1\alpha + a_2\beta + a_3\alpha\beta\,, & L_0 &:= E(\alpha_0)\,,\\
\alpha_1 &:= a_0\alpha + ca_3\beta + a_2\alpha\beta\,, & L_1 &:= E(\alpha_1)\,,\\
\alpha_2 &:= da_3\alpha + a_0\beta + a_1\alpha\beta\,, & L_2 &:= E(\alpha_2)\,,\\
\alpha_3 &:= da_2\alpha + ca_1\beta + a_0\alpha\beta\,, & L_3 &:= E(\alpha_3)\,.
\end{aligned}$$

If $a_0 \notin \lambda$, then the equality $t = a_0^2 - (\alpha_0)^2$ shows that $(\alpha_0)^2$ is a square in $\mathcal{O}_\lambda^\times$. So λ splits in L_0.

If $a_1 \notin \lambda$, then $ca_1 \notin \lambda$ and the equality $-ct = (ca_1)^2 - (\alpha_1)^2$ shows that $(\alpha_1)^2$ is a square in $\mathcal{O}_\lambda^\times$. So λ splits in L_1.

If $a_2 \notin \lambda$, then $da_2 \notin \lambda$ and the equality $-dt = (da_2)^2 - (\alpha_2)^2$ shows that $(\alpha_2)^2$ is a square in $\mathcal{O}_\lambda^\times$. So λ splits in L_2.

If $a_3 \notin \lambda$, then $cda_3 \notin \lambda$ and the equality $cdt = (cda_3)^2 - (\alpha_3)^2$ shows that $(\alpha_3)^2$ is a square in $\mathcal{O}_\lambda^\times$. So λ splits in L_3.

Thus we can choose $\alpha' = a_1\alpha + a_2\beta + a_3\alpha\beta$ to be an appropriate α_i, and L equal to corresponding L_i for some elements $a_1, a_2, a_3 \in \mathcal{O}_{E,S}$. Observe that $c' := \alpha'^2 = ca_1^2 + da_2^2 - cda_3^2 \notin \lambda$ by above constructions. We will construct $\beta' := b_1\alpha + b_2\beta + b_3\alpha\beta \in D$ such that:

$$ca_1b_1 + da_2b_2 - cda_3b_3 = 0\,, \tag{2.14}$$

$$d' := \beta'^2 = cb_1^2 + db_2^2 - cdb_3^2 \notin \lambda \,. \tag{2.15}$$

Because $c' \notin \lambda$, without loss of generality, we can assume that $a_1 \notin \lambda$. The case $a_2 \notin \lambda$ is done in the same way and ditto the case $a_3 \notin \lambda$ under observation that $(\alpha\beta)^2 = -cd$. Because $c < 0$, $d < 0$, $-cd < 0$ in every real imbedding $E \to \boldsymbol{R}$, the equation (2.14) shows that α' and β' are linearly independent over E and $\alpha'\,\beta' = -\beta'\,\alpha'$.

Consider the following cases.

(1) If $a_2, a_3 \in \lambda$, we can take any $b_2 \in \lambda$ and $b_3 \notin \lambda$, and compute b_1 from (2.14) to find out that $b_1 \in \lambda$ and (2.15) holds.

(2) If $a_2 \notin \lambda$ and $a_3 \in \lambda$, we can take any $b_2 \notin \lambda$, and $b_3 \in \lambda$, and compute b_1 from (2.14) to find out that $b_1 \in \lambda$ and (2.15) also holds. Similarly we treat the case $a_2 \in \lambda$ and $a_3 \notin \lambda$.

(3) If $a_2 \notin \lambda$ and $a_3 \notin \lambda$ and if c is not a square mod λ , then taking any $b_2, b_3 \notin \lambda$ such that $b_1 = d(ca_3b_3 - a_2b_2)/(a_1c) \in \lambda$ we find out that (2.15) holds. Note that, in the case c is a square mod λ we can simply take $\alpha' = \alpha$, $\beta' = \beta$ and $L = E(\alpha)$ from the very beginning to prove the lemma.

Define $S' := S \cup \{p \; ; \; p$ divisible by primes of $\mathcal{O}_E$ dividing a_1, c' and $d'\}$. Note that with this choice of S' we get $\beta' \in \mathcal{R}_{S'}$. Observe that using formula (2.8) and (2.14),

$$-c'd' = (\alpha'\beta')^2 = c_1^2d^2c + c_2^2c^2d - c_3^2cd = -cd(-c_1^2d - c_2^2c + c_3^2)\,.$$

By formula (2.9) and definition of S', we get equality of free $\mathcal{O}_{E,S'}$-modules: $\mathcal{O}_{E,S'}\alpha + \mathcal{O}_{E,S'}\beta + \mathcal{O}_{E,S'}\alpha\beta = \mathcal{O}_{E,S'}\alpha' + \mathcal{O}_{E,S'}\beta' + \mathcal{O}_{E,S'}\alpha'\beta'$. This gives $\mathcal{R}_{S'} := \mathcal{R}_S \otimes_{\mathcal{O}_{E,S}} \mathcal{O}_{E,S'} = \mathcal{O}_{E,S'} + \mathcal{O}_{E,S'}\alpha' + \mathcal{O}_{E,S'}\beta' + \mathcal{O}_{E,S'}\alpha'\beta'$.

Observe that the elements $1\otimes 1$, $\alpha'\otimes 1$, $\beta'\otimes 1$ and $\alpha'\beta'\otimes 1$ are mapped correspondingly, via the imbeding (2.12), to elements

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \alpha' & 0 \\ 0 & -\alpha' \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ d' & 0 \end{bmatrix}, \begin{bmatrix} 0 & \alpha' \\ -d'\alpha' & 0 \end{bmatrix}.$$

Since λ splits completely in $L = E(\alpha')$ and λ does not divide c' we get $\alpha' \in \mathcal{O}_\lambda{}^\times$. Since λ does not divide d' either, we observe that the matrices $e_{ij} \in M_{2,2}(\mathcal{O}_\lambda)$ are in the image of the map (2.13), where e_{ij} has the (i, j)-entry equal to 1 and all other entries are 0. Hence (2.13) is an isomorphism of $\mathcal{O}_\lambda$-algebras. □

3. Bilinear forms associated with abelian varieties of type III. Put $\mathcal{R}_l = \mathcal{R} \otimes \boldsymbol{Z}_l$ and $D_l = D \otimes \boldsymbol{Q}_l$. The polarization of A gives a $\boldsymbol{Z}$-bilinear non-degenerate alternating pairing

$$\kappa \; : \; \Lambda \times \Lambda \to \boldsymbol{Z}\,, \tag{3.1}$$

where Λ is the Riemann lattice such that $A(\boldsymbol{C}) = \boldsymbol{C}^g/\Lambda$. This pairing, upon tensoring with $\boldsymbol{Z}_l$ ([19, diagram on page 133]), becomes $\boldsymbol{Z}_l$-bilinear non-degenerate alternating pairing

$$\kappa_l \; : \; T_l(A) \times T_l(A) \to \boldsymbol{Z}_l\,, \tag{3.2}$$

derived easily from the Weil pairing. If l does not divide the degree of the polarisation of A, then for any $\alpha \in \mathcal{R}_l$ we get $\alpha^* \in \mathcal{R}_l$ (see [19, Chapters 13 and 17]) where α^* is the image of α via the Rosati involution. Hence for any $v, w \in T_l(A)$, we have $\kappa_l(\alpha v, w) = \kappa_l(v, \alpha^* w)$ (see loc. cit.). Let $V_l(A) = T_l(A) \otimes_{\mathbf{Z}_l} \boldsymbol{Q}_l$, and let $\kappa_l^0 : V_l(A) \times V_l(A) \rightarrow \boldsymbol{Q}_l$ be the bilinear form $\kappa_l \otimes_{\mathbf{Z}_l} \boldsymbol{Q}_l$. For any l that is unramified in E, by [2, Lemma 3.1], there is a unique $\mathcal{O}_{E_l}$-bilinear form

$$\phi_l : T_l(A) \times T_l(A) \rightarrow \mathcal{O}_{E_l} \tag{3.3}$$

such that $\mathrm{Tr}_{E_l/\boldsymbol{Q}_l}(\phi_l(v_1, v_2)) = \kappa_l(v_1, v_2)$ for all $v_1, v_2 \in T_l(A)$. Put

$$\phi_l^0 = \phi_l \otimes_{\mathbf{Z}_l} \boldsymbol{Q}_l : V_l(A) \times V_l(A) \rightarrow E_l\,. \tag{3.4}$$

By uniqueness of the form ϕ_l, for each $\alpha \in \mathcal{R}_l$ and for all $v_1, v_2 \in T_l(A)$, we have

$$\phi_l(\alpha v_1,\ v_2) = \phi_l(v_1,\ \alpha^* v_2)\,, \tag{3.5}$$

hence $\phi_l^0(\alpha v_1,\ v_2) = \phi_l^0(v_1,\ \alpha^* v_2)$ for each $\alpha \in D_l$ and for all $v_1, v_2 \in V_l(A)$.

Let S be the set of primes which contains all the primes described in the hypotheses of Lemma 2.11. We can enlarge S so that it also contains all primes that ramify in E and all primes that divide the polarisation degree of A. Now, for such an S and for any $l \notin S$, we apply Lemma 2.11 to construct the appropriate field L.

Define $T_w(A) = T_l(A) \otimes_{\mathcal{O}_E^0} \mathcal{O}_w$, $V_w(A) = V_l(A) \otimes_E L_w$ and

$$\phi_w = \phi_l \otimes_{\mathcal{O}_E^0} \mathcal{O}_w : T_w(A) \times T_w(A) \rightarrow \mathcal{O}_w\,. \tag{3.6}$$

Hence $\phi_w^0 := \phi_w \otimes_{\mathcal{O}_w} L_w$ is the L_w-bilinear form:

$$\phi_w^0 : V_w(A) \times V_w(A) \rightarrow L_w\,. \tag{3.7}$$

The form ϕ_w is non-degenerate if ϕ_l is non-degenerate.

Let e_λ be the idempotent corresponding to the decomposition $\mathcal{O}_{E_l} \cong \prod_{\lambda | l} \mathcal{O}_\lambda$. Put $T_\lambda(A) = e_\lambda T_l(A) \cong T_l(A) \otimes_{\mathcal{O}_{E_l}} \mathcal{O}_\lambda$ and $V_\lambda(A) = T_\lambda(A) \otimes_{\mathcal{O}_\lambda} E_\lambda$. Define $\mathcal{O}_\lambda$-bilinear form ϕ_λ by $\phi_\lambda = \phi_l \otimes_{\mathcal{O}_E^0} \mathcal{O}_\lambda$.

For $l \notin S$ we have $\mathcal{O}_\lambda = \mathcal{O}_w$. Hence $\phi_\lambda = \phi_w$.

DEFINITION 3.8. Define a new bilinear form ψ_λ as follows.

$$\psi_\lambda : T_\lambda(A) \times T_\lambda(A) \rightarrow \mathcal{O}_\lambda\,, \tag{3.9}$$

$$\psi_\lambda(v_1, v_2) = \phi_\lambda(J v_1, v_2)$$

for all $v_1, v_2 \in T_\lambda(A)$.

This gives us the corresponding k_λ-bilinear form

$$\overline{\psi}_\lambda = \psi_\lambda \otimes_{\mathcal{O}_\lambda} k_\lambda : A[\lambda] \times A[\lambda] \rightarrow k_\lambda \tag{3.10}$$

and the E_λ-bilinear form

$$\psi_\lambda^0 = \psi_\lambda \otimes_{\mathcal{O}_\lambda} E_\lambda : V_\lambda(A) \times V_\lambda(A) \rightarrow E_\lambda\,. \tag{3.11}$$

By (2.4) and Lemma 2.11, we get the following isomorphisms

$$D_\lambda := D \otimes_E E_\lambda \cong M_{2,2}(E_\lambda)\,, \tag{3.12}$$

which obviously induces isomorphisms

$$\mathcal{R}_\lambda \cong \mathcal{R} \otimes_{\mathcal{O}_E^0} \mathcal{O}_\lambda \cong M_{2,2}(\mathcal{O}_\lambda)\,. \tag{3.13}$$

REMARK 3.14. We should note that an isomorphism between both sides of (3.13) can be obtained by [24, Corollary 11.6 and Theorem 17.3] for $l \gg 0$. However these results give an isomorphism which comes from a conjugation by an element of $D \otimes_E L_l \cong M_{2,2}(L_l)$ (see [24, loc. cit.]). To keep track of the action of the involution $*$, we prefer to use the isomorphism (3.13).

PROPOSITION 3.15. *The involution $*$ induced on $\mathcal{R}_\lambda \cong M_{2,2}(\mathcal{O}_\lambda)$ (resp. on $D_\lambda \cong M_{2,2}(E_\lambda)$) from D has the form $B^* = J\,{}^tBJ^{-1}$ for any $B \in \mathcal{R}_\lambda$ (resp. for any $B \in D_\lambda$).*

PROOF. By (2.4) and (2.5), for any $B \in M_{2,2}(L)$, we get $B^* = J\,{}^tBJ^{-1}$. Hence the claim follows by (3.12) and (3.13) □

Observe that, by (2.5) for each $B \in \mathcal{R}_\lambda$ and for all $v_1, v_2 \in T_\lambda(A)$, we have

$$\phi_\lambda(Bv_1,\ v_2) = \phi_\lambda(v_1, B^*v_2) = \phi_\lambda(v_1, J\,{}^tBJ^{-1}v_2)\,.$$

Therefore, for each $B \in M_{2,2}(E_\lambda)$ and for all $v_1, v_2 \in V_\lambda(A)$, we have

$$\phi_\lambda(Bv_1,\ v_2) = \phi_\lambda(v_1, B^*v_2) = \phi_\lambda(v_1,\ J\,{}^tBJ^{-1}v_2)\,.$$

PROPOSITION 3.16. *For any $v_1, v_2 \in T_\lambda(A)$ and $B \in \mathcal{R}_\lambda$, we have*

$$\psi_\lambda(Bv_1, v_2) = \psi_\lambda(v_1,\ {}^tBv_2)\,.$$

Hence for any $v_1, v_2 \in A[\lambda]$ and any $B \in \mathcal{R}_\lambda \otimes_{\mathcal{O}_\lambda} k_\lambda \cong M_{2,2}(k_\lambda)$ (resp. for any $v_1, v_2 \in V_\lambda(A)$ and any $B \in M_{2,2}(E_\lambda)$), we have

$$\overline{\psi}_\lambda(Bv_1, v_2) = \psi_\lambda(v_1,\ {}^tBv_2)$$

$$(resp.\ \psi_\lambda^0(Bv_1, v_2) = \psi_\lambda(v_1,\ {}^tBv_2))\,.$$

Moreover ψ_λ (resp. ψ_λ^0) is symmetric (resp. antisymmetric) if and only if ϕ_λ (resp. ${\phi_\lambda}^0$) is antisymmetric (resp. symmetric).

PROOF. We get the first equality as

$$\begin{aligned}\psi_\lambda(Bv_1, v_2) &= \phi_\lambda(JBv_1, v_2) = \phi_\lambda(v_1,\ J\,{}^tB\,{}^tJJ^{-1}v_2) = \phi_\lambda(v_1,\ -J\,{}^tBv_2)\\ &= \phi_\lambda(v_1,\ J\,{}^tJJ^{-1}\,{}^tBv_2) = \phi_\lambda(Jv_1,\ {}^tBv_2) = \psi_\lambda(v_1,\ {}^tBv_2)\,.\end{aligned}$$

The remaining claim follows by Definition 3.8 and by the observation that ${}^tJ = J^{-1} = -J$ and $J\,{}^tJJ^{-1} = -J$. □

REMARK 3.17. All bilinear forms ψ_λ, $\overline{\psi}_\lambda$ and ψ_λ^0 are symmetric and non-degenerate. This follows by results of this section, [2, Lemmas 3.1 and 3.2] and by the non-degeneracy of the pairing (3.1) which is independent of l.

We proceed to investigate some natural Galois actions. From now on, we assume that $\mathcal{R} = \mathrm{End}_{\bar{F}}(A) = \mathrm{End}_F(A)$. Consider the representations

$$\begin{aligned} \rho_l &: G_F \to \mathrm{GL}(T_l(A)), \\ \rho_l^0 &: G_F \to \mathrm{GL}(V_l(A)), \\ \overline{\rho}_l &: G_F \to \mathrm{GL}(A[l]). \end{aligned}$$

Let $\mathcal{G}_l^{\mathrm{alg}}$ be the Zariski closure of $\rho_l(G_F)$ in $\mathrm{GL}_{T_l(A)}$, and let G_l^{alg} be the Zariski closure of $\rho_l^0(G_F)$ in $\mathrm{GL}_{V_l(A)}$. Let $G(l)^{\mathrm{alg}}$ be the special fiber of $\mathcal{G}_l^{\mathrm{alg}}/\mathbf{Z}_l$. Note that G_l^{alg} is the general fiber of $\mathcal{G}_l^{\mathrm{alg}}/\mathbf{Z}_l$. This gives natural representations

$$\begin{aligned} \rho_\lambda &: G_F \to \mathrm{GL}(T_\lambda(A)), \\ \rho_\lambda^0 &: G_F \to \mathrm{GL}(V_\lambda(A)), \\ \overline{\rho}_\lambda &: G_F \to \mathrm{GL}(A[\lambda]). \end{aligned}$$

We define $\mathcal{G}_\lambda^{\mathrm{alg}}$ to be the Zariski closure of $\rho_\lambda(G_F)$ in $\mathrm{GL}_{T_\lambda(A)}$, and G_λ^{alg} the Zariski closure of $\rho_\lambda^0(G_F)$ in $\mathrm{GL}_{V_\lambda(A)}$. Let $G(\lambda)^{\mathrm{alg}}$ be the special fiber of $\mathcal{G}_\lambda^{\mathrm{alg}}/\mathcal{O}_\lambda$. Then, G_λ^{alg} is the general fiber of $\mathcal{G}_\lambda^{\mathrm{alg}}/\mathcal{O}_\lambda$.

LEMMA 3.18. *Let $\chi_\lambda : G_F \to \mathbf{Z}_l \subset \mathcal{O}_\lambda$ be the composition of the cyclotomic character with the natural imbedding $\mathbf{Z}_l \subset \mathcal{O}_\lambda$. Let $l \gg 0$ be such that $\lambda | l$ be a prime of E which splits in L.*

(i) *For any $\sigma \in G_F$ and all $v_1, v_2 \in T_\lambda(A)$, we have*

$$\psi_\lambda(\sigma v_1, \sigma v_2) = \chi_\lambda(\sigma)\psi_\lambda(v_1, v_2).$$

(ii) *For any $B \in \mathcal{R} \otimes_{\mathcal{O}_E^0} \mathcal{O}_\lambda$ and all $v_1, v_2 \in T_\lambda(A)$, we have*

$$\psi_\lambda(Bv_1, v_2) = \psi_\lambda(v_1, {}^tBv_2).$$

PROOF. (i) follows by [6, Lemma 2.3] or [2, Lemma 4.7] which concern the pairing ϕ_λ and by (3.9) and definition 3.8, because the G_F-action commutes with the action of $\mathcal{R}$ on $T_l(A)$. Indeed, since $\psi_\lambda(v, w) = \phi_\lambda(Jv, w)$ and J commutes with the G_F-action by assumption, we get immediately statement (i) for ψ_λ. Part (ii) follows by Proposition 3.16. □

By [10, Theorem 3] and [2, Lemma 4.17] G_F acts on both $V_\lambda(A)$ and $A[\lambda]$ semi-simply and G_λ^{alg} and $G(\lambda)^{\mathrm{alg}}$ are reductive algebraic groups. Hence $\mathcal{G}_\lambda^{\mathrm{alg}}$ is a reductive group scheme over $\mathcal{O}_\lambda$ for $l \gg 0$ by [18, Prop. 1.3] (cf. [36, Theorem 1]).

Let

$$t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $f = (1+u)/2$, $\mathcal{X} = f\,T_\lambda(A)$ and $\mathcal{Y} = (1-f)\,T_\lambda(A)$. Put $X = \mathcal{X} \otimes_{\mathcal{O}_\lambda} E_\lambda$, $Y = \mathcal{Y} \otimes_{\mathcal{O}_\lambda} E_\lambda$, $\overline{\mathcal{X}} = \mathcal{X} \otimes_{\mathcal{O}_\lambda} k_\lambda$ and $\overline{\mathcal{Y}} = \mathcal{Y} \otimes_{\mathcal{O}_\lambda} k_\lambda$. Because $tft = 1 - f$, the matrix t gives an $\mathcal{O}_\lambda[G_F]$-isomorphism between $\mathcal{X}$ and $\mathcal{Y}$, hence it gives an $E_\lambda[G_F]$-isomorphism between X and Y and a $k_\lambda[G_F]$-isomorphism between $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$. Using the computations of endomorphism algebras by [10, Satz 4] and [37, Corollary 5.4.5], we get:

$$\operatorname{End}_{\mathcal{O}_\lambda[G_F]}(\mathcal{X}) = \mathcal{O}_\lambda\,, \tag{3.19}$$

$$\operatorname{End}_{E_\lambda[G_F]}(X) = E_\lambda\,, \tag{3.20}$$

$$\operatorname{End}_{k_\lambda[G_F]}(\overline{\mathcal{X}}) = k_\lambda\,. \tag{3.21}$$

So the representations of G_F on the spaces X and Y (resp. $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$) are absolutely irreducible over E_λ (resp. over k_λ). Hence, the bilinear form ψ_λ^0 (resp. $\overline{\psi}_\lambda$) when restricted to either of the spaces X, Y (resp. spaces $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$) is non-degenerate or isotropic.

LEMMA 3.22. *The modules $\mathcal{X}$ and $\mathcal{Y}$ are orthogonal with respect to ψ_λ. Consequently, the modules X and Y (resp. $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$) are orthogonal with respect to ψ_λ^0 (resp. $\overline{\psi}_\lambda$).*

PROOF. Note that $uf = f$ and $u(1-f) = -(1-f)$. Hence for every $v \in \mathcal{X}$ and for every $w \in \mathcal{Y}$, we get $uv = v$ and $uw = -w$. Hence

$$\psi_\lambda(v, w) = \psi_\lambda(uv, w) = \psi_\lambda(v, u^t w) = \psi_\lambda(v, uw) = \psi_\lambda(v, -w) = -\psi_\lambda(v, w)\,.$$

Hence $\psi_\lambda(v, w) = 0$ for every $v \in \mathcal{X}$ and for every $w \in \mathcal{Y}$. □

THEOREM 3.23. *Let A be of type III and $l \gg 0$. Then there is a free $\mathcal{O}_\lambda$-module $\mathcal{W}_\lambda(A)$ of rank $2h$ with the following properties.*

(i) *$T_\lambda(A) \cong \mathcal{W}_\lambda(A) \oplus \mathcal{W}_\lambda(A)$ as $\mathcal{O}_\lambda[G_F]$-modules.*

(ii) *There exists a symmetric, non-degenerate pairing $\psi_\lambda : \mathcal{W}_\lambda(A) \times \mathcal{W}_\lambda(A) \to \mathcal{O}_\lambda$.*

(ii′) *For $W_\lambda(A) = \mathcal{W}_\lambda(A) \otimes_{\mathcal{O}_\lambda} E_\lambda$, the induced symmetric pairing*

$$\psi_\lambda^0 : W_\lambda(A) \times W_\lambda(A) \to E_\lambda$$

is non-degenerate. The G_F module $W_\lambda(A)$ is absolutely irreducible.

(ii″) *For $\overline{\mathcal{W}}_\lambda(A) = \mathcal{W}_\lambda(A) \otimes_{\mathcal{O}_\lambda} k_\lambda$, the induced symmetric pairing*

$$\overline{\psi}_\lambda : \overline{\mathcal{W}}_\lambda(A) \times \overline{\mathcal{W}}_\lambda(A) \to k_\lambda$$

is non-degenerate. The G_F module $\overline{\mathcal{W}}_\lambda(A)$ is absolutely irreducible.

Pairings (ii), (ii′) *and* (ii″) *are compatible with the G_F-action in the same way as the pairing in Lemma* 3.18 (i).

PROOF. (i) follows by taking $\mathcal{W}_\lambda(A) = \mathcal{X}$. We get (ii) by restricting ψ_λ to $\mathcal{X}$. To finish the proof, observe that the form (3.2) is non-degenerate, so $\overline{\psi}_l = \psi_l \otimes \boldsymbol{F}_l$ is non-degenerate for any abelian variety with polarization degree prime to l. By [2, Lemma 3.2], the form $\overline{\psi}_\lambda$ is non-degenerate for all λ, hence the forms ψ_λ^0 and $\overline{\psi}_\lambda$ are simultaneously non-degenerate. Hence (ii′) and (ii″) follow by (ii), (3.20) and (3.21) and also by Remark 3.17 and Lemma 3.22. □

4. Representations associated with Abelian varieties of type III. Let A/F be an abelian variety of type III. The field of definition F is such that G_l^{alg} is a connected algebraic group. Let us put $T_\lambda = \mathcal{W}_\lambda(A)$, $V_\lambda = T_\lambda \otimes_{\mathcal{O}_\lambda} E_\lambda$ and $A_\lambda = V_\lambda / T_\lambda$. With this notation, by Theorem 3.23 we have $V_l(A) = \bigoplus_{\lambda | l} \big(V_\lambda \oplus V_\lambda\big)$. We put

$$V_l = \bigoplus_{\lambda | l} V_\lambda \,. \tag{4.1}$$

Let V_{Φ_λ} be the space V_λ considered over $\boldsymbol{Q}_l$. Then there is the following equality of $\boldsymbol{Q}_l$-vector spaces:

$$V_l = \bigoplus_{\lambda | l} V_{\Phi_\lambda} \,. \tag{4.2}$$

The l-adic representation

$$\rho_l^0 \, : \, G_F \to \mathrm{GL}(V_l(A)) \tag{4.3}$$

induces the following representations (note that we use the notation ρ_l^0 for both representations (4.3) and (4.4)) (cf. [2, Remark 5.13]):

$$\rho_l^0 : \, G_F \to \mathrm{GL}(V_l) \,, \tag{4.4}$$

$$\rho_\lambda^0 : \, G_F \to \mathrm{GL}(V_\lambda) \,. \tag{4.5}$$

Consider the representation ρ_{Φ_λ} defined in [2, p. 54]:

$$\rho_{\Phi_\lambda} \, : \, G_F \to \mathrm{GL}(V_{\Phi_\lambda}) \,. \tag{4.6}$$

By Theorem 3.23 (cf. [2, Remark 5.13]), the group scheme G_l^{alg} (resp. G_λ^{alg}) is naturally isomorphic to the Zariski closure in GL_{V_l} (resp. GL_{V_λ}) of the image of the representation ρ_l of (4.4) (resp. ρ_λ of (4.5)). Let $G_{\Phi_\lambda}^{\mathrm{alg}}$ denote the Zariski closure in $\mathrm{GL}_{V_{\Phi_\lambda}}$ of the image of the representation ρ_{Φ_λ} of (4.6). Let $\mathfrak{g}_l = \mathrm{Lie}(G_l^{\mathrm{alg}})$, $\mathfrak{g}_\lambda = \mathrm{Lie}(G_\lambda^{\mathrm{alg}})$ and $\mathfrak{g}_{\Phi_\lambda} = \mathrm{Lie}(G_{\Phi_\lambda}^{\mathrm{alg}})$. By definition $G_l^{\mathrm{alg}} \subset \prod_{\lambda | l} G_{\Phi_\lambda}^{\mathrm{alg}}$ so $\mathfrak{g}_l \subset \bigoplus_{\lambda | l} \mathfrak{g}_{\Phi_\lambda}$. This implies:

$$(G_l^{\mathrm{alg}})' \subset \prod_{\lambda | l} (G_{\Phi_\lambda}^{\mathrm{alg}})', \tag{4.7}$$

$$\mathfrak{g}_l^{ss} \subset \bigoplus_{\lambda | l} \mathfrak{g}_{\Phi_\lambda}^{ss} \,. \tag{4.8}$$

In the remainder of this section, we compute the Lie algebras corresponding to representations we consider. Some results that we proved in [2] for abelian varieties of type I and II work as well for abelian varieties of type III. Since the detailed proofs of these results were given in [2], we will merely reformulate corresponding results for abelian varieties of type III. For example the proof of Lemma 4.9 (resp. Lemma 4.10) below is essentially the same as the proof of [2, Lemma 5.20] (resp. [2, Lemma 5.22]).

LEMMA 4.9. *The natural map of Lie algebras*

$$\mathfrak{g}_l^{ss} \to \mathfrak{g}_{\Phi_\lambda}^{ss}$$

is surjective.

LEMMA 4.10. *Let A/F be an abelian variety over F of type III such that* $\mathrm{End}_F(A) = \mathrm{End}_{\overline{F}}(A)$. *Then*

$$\mathrm{End}_{\mathfrak{g}_\lambda}(V_\lambda) \cong \mathrm{End}_{E_\lambda[G_F]}(V_\lambda) \cong E_\lambda ,$$

$$\mathrm{End}_{\mathfrak{g}_{\Phi_\lambda}}(V_{\Phi_\lambda}) \cong \mathrm{End}_{\boldsymbol{Q}_l[G_F]}(V_{\Phi_\lambda}) \cong E_\lambda .$$

We define the subgroups of GL_{V_λ} by

$$\mathrm{GO}_{(V_\lambda,\psi_\lambda)} = \{A \in \mathrm{GL}_{V_\lambda};\ \psi_\lambda(Av_1, Av_2) = c_\lambda(A)\psi_\lambda(v_1, v_2) \text{ for all } v_1, v_2 \in V_\lambda\} ,$$

$$\mathrm{O}_{(V_\lambda,\psi_\lambda)} = \{A \in \mathrm{GL}_{V_\lambda};\ \psi_\lambda(Av_1, Av_2) = \psi_\lambda(v_1, v_2) \text{ for all } v_1, v_2 \in V_\lambda\} .$$

Denote by $\mathrm{SO}_{(V_\lambda,\psi_\lambda)}$ the connected component of the identity in $\mathrm{O}_{(V_\lambda,\psi_\lambda)}$. By Lemma 3.18, we see that $\rho_\lambda(G_F) \subset \mathrm{GO}_{(V_\lambda,\psi_\lambda)}$ and therefore $G_\lambda^{\mathrm{alg}} \subset \mathrm{GO}_{(V_\lambda,\psi_\lambda)}$. This of course implies that $(G_\lambda^{\mathrm{alg}})' \subset \mathrm{O}_{(V_\lambda,\psi_\lambda)}$. Extending the base field F, if necessary, one can assume that G_λ^{alg} and hence $(G_\lambda^{\mathrm{alg}})'$ are connected (cf. [5, Proposition 3.6]). This gives the inclusions

$$(G_\lambda^{\mathrm{alg}})' \subset \mathrm{SO}_{(V_\lambda,\psi_\lambda)} \tag{4.11}$$

and

$$\mathfrak{g}_\lambda^{ss} \subset \mathrm{so}_{(V_\lambda,\psi_\lambda)} . \tag{4.12}$$

From now on, in this section we assume that A is an abelian variety of class $\mathcal{B}$.

LEMMA 4.13. *The equality $\mathfrak{g}_\lambda^{ss} = \mathrm{so}_{(V_\lambda,\psi_\lambda)}$ holds.*

PROOF. The proof is similar to the proofs of [1, Lemma 3.2] and [2, Lemma 5.33]. Since type III is more exotic than types I and II, we will give here a complete proof. Observe that the minuscule conjecture for the λ-adic representations $\rho_F : G_F \to \mathrm{GL}(V_\lambda)$ holds. Namely by [P, Corollary 5.11], we know that $\mathfrak{g}_l^{ss} \otimes \bar{\boldsymbol{Q}}_l$ may only have simple factors of types A, B, C or D with minuscule weights. By Lemma 4.9, the natural map of Lie algebras

$$\mathfrak{g}_l^{ss} \to \mathfrak{g}_{\Phi_\lambda}^{ss} \tag{4.14}$$

is surjective. Hence by the semisimplicity of $\mathfrak{g}_l^{ss}$ the simple factors of $\mathfrak{g}_{\Phi_\lambda}^{ss} \otimes \overline{\boldsymbol{Q}}_l$ are also of types A, B, C or D with minuscule weights. By [2, Proposition 2.12] and [2, Lemmas 2.21, 2.22, 2.23], there is an isomorphism of $\boldsymbol{Q}_l$-Lie algebras

$$\mathfrak{g}_{\Phi_\lambda}^{ss} \cong R_{E_\lambda/\boldsymbol{Q}_l}\mathfrak{g}_\lambda^{ss} . \tag{4.15}$$

The isomorphisms $\mathfrak{g}_{\Phi_\lambda}^{ss} \otimes_{\boldsymbol{Q}_l} \overline{\boldsymbol{Q}}_l \cong \mathfrak{g}_\lambda^{ss} \otimes_{E_\lambda} E_\lambda \otimes_{\boldsymbol{Q}_l} \overline{\boldsymbol{Q}}_l \cong \bigoplus_{E_\lambda \hookrightarrow \overline{\boldsymbol{Q}}_l} \mathfrak{g}_\lambda^{ss} \otimes_{E_\lambda} \overline{\boldsymbol{Q}}_l$ imply that simple factors of $\mathfrak{g}_\lambda^{ss} \otimes_{E_\lambda} \overline{\boldsymbol{Q}}_l$ are of types A, B, C or D with minuscule weights. Put $\bar{V}_\lambda = V_\lambda \otimes \overline{\boldsymbol{Q}}_l$. We have the decomposition

$$\bar{V}_\lambda = E(\omega_1) \otimes_{\overline{\boldsymbol{Q}}_l} \cdots \otimes_{\overline{\boldsymbol{Q}}_l} E(\omega_r) ,$$

where $E(\omega_i)$, for all $1 \leq i \leq r$, are the irreducible Lie algebra modules of the highest weight ω_i. The modules $E(\omega_i)$ correspond to simple Lie algebras $\mathfrak{g}_i$, which are summands of the image

$$\mathrm{Im}(\mathfrak{g}_\lambda^{ss} \otimes \overline{\boldsymbol{Q}}_l \to \mathrm{so}_{2h}(\overline{V}_\lambda)) = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r .$$

By [4, Chap. VIII Proposition 12], $E(\omega_i)$ are symplectic or orthogonal. By [23, Corollary 5.11], all simple factors of $\mathfrak{g}_\lambda^{ss} \otimes \overline{\boldsymbol{Q}}_l$ are of classical types A, B, C and D, and the weights $\omega_1, \ldots, \omega_r$ are minuscule. All minuscule weights and dimensions of representations are listed in [4, Chap. VIII, Tables 1 and 2] and in [14, p. 72]. Since h is odd, the investigation of the tables of minuscule weights and the dimensions of associated representations shows that the tensor product can contain only one factor which is orthogonal and is either of type D_n, weight w_1 and dimension $2n$ or of type A_{4k+3}, weight w_{2k+2} and dimension $\binom{4k+4}{2k+2}$. Hence V_λ is an irreducible $\mathfrak{g}_\lambda^{ss}$-module and we get

$$\mathfrak{g}_\lambda^{ss} = \mathrm{so}_{(V_\lambda, \psi_\lambda)} . \qquad \square$$

The following lemma has the proof analogous to that of [2, Lemma 5.35].

LEMMA 4.16. *There are natural isomorphisms of $\boldsymbol{Q}_l$-algebras.*

$$\mathrm{End}_{\mathfrak{g}_{\Phi_\lambda}^{ss}}(V_{\Phi_\lambda}) \cong \mathrm{End}_{\mathfrak{g}_\lambda^{ss}}(V_\lambda) \cong E_\lambda$$

PROPOSITION 4.17. *There is an equality of Lie algebras*:

$$\mathfrak{g}_l^{ss} = \bigoplus_{\lambda | l} \mathfrak{g}_{\Phi_\lambda}^{ss} . \tag{4.18}$$

PROOF. By use of (4.8) and Lemma 4.16, the proof is the same as that of [2, Proposition 5.39]. $\square$

THEOREM 4.19. *There is an equality of group schemes over $\boldsymbol{Q}_l$*:

$$(G_l^{\mathrm{alg}})' = \prod_{\lambda | l} R_{E_\lambda / \boldsymbol{Q}_l} \mathrm{SO}_{(V_\lambda, \psi_\lambda)} . \tag{4.20}$$

PROOF. By [2, Proposition 2.12], we get:

$$G_{\Phi_\lambda}^{\mathrm{alg}} \cong R_{E_\lambda / \boldsymbol{Q}_l} G_\lambda^{\mathrm{alg}} \subset R_{E_\lambda / \boldsymbol{Q}_l} \mathrm{GO}_{(V_\lambda, \psi_\lambda)} . \tag{4.21}$$

Hence it follows from [2, Lemma 2.23], that

$$(G_{\Phi_\lambda}^{\mathrm{alg}})' \subset R_{E_\lambda / \boldsymbol{Q}_l} \mathrm{SO}_{(V_\lambda, \psi_\lambda)} . \tag{4.22}$$

By (4.7) and (4.22), we have a closed immersion of two connected group schemes over $\boldsymbol{Q}_l$:

$$(G_l^{\mathrm{alg}})' \subset \prod_{\lambda | l} R_{E_\lambda / \boldsymbol{Q}_l} \mathrm{SO}_{(V_\lambda, \psi_\lambda)} .$$

But this imbedding induces the Lie algebra isomorphism of Proposition 4.17, hence the theorem follows by Proposition 4.17 and [13, Theorem on p. 87 and Proposition on p. 110]. $\square$

5. Mumford-Tate conjecture for abelian varieties of type III. Choose an imbedding of F into the field of complex numbers $\boldsymbol{C}$. Define $V := H^1(A(\boldsymbol{C}), \boldsymbol{Q})$ to be the singular cohomology group with rational coefficients. Consider the Hodge decomposition

$$V \otimes_{\boldsymbol{Q}} \boldsymbol{C} = H^{1,0} \oplus H^{0,1},$$

where $H^{p,q} = H^p(A;\ \Omega^q_{A/\boldsymbol{C}})$ and $\overline{H^{p,q}} = H^{q,p}$. Observe that $H^{p,q}$ are invariant subspaces with respect to $D = \mathrm{End}_{\overline{F}}(A) \otimes \boldsymbol{Q}$ action on $V \otimes_{\boldsymbol{Q}} \boldsymbol{C}$. Hence, in particular, $H^{p,q}$ are E-vector spaces. Tensoring (3.1) with $\boldsymbol{Q}$, we get the $\boldsymbol{Q}$-bilinear nondegenerate alternating form $\kappa^0 := \kappa \otimes_{\boldsymbol{Z}} \boldsymbol{Q} : V \times V \to \boldsymbol{Q}$. Abusing notation sligthly, we will denote by κ the form κ^0 i.e., we have the form:

$$\kappa : V \times V \to \boldsymbol{Q}.$$

Define the cocharacter

$$\mu_\infty : \boldsymbol{G}_m(\boldsymbol{C}) \to \mathrm{GL}(V \otimes_{\boldsymbol{Q}} \boldsymbol{C}) = \mathrm{GL}_{2g}(\boldsymbol{C})$$

such that, for any $z \in \boldsymbol{C}^\times$, the automorphism $\mu_\infty(z)$ is the multiplication by z on $H^{1,0}$ and the identity on $H^{0,1}$.

DEFINITION 5.1. The Mumford-Tate group of the abelian variety A/F is the smallest algebraic subgroup $\mathrm{MT}(A) \subset \mathrm{GL}_{2g}$, defined over $\boldsymbol{Q}$, such that $\mathrm{MT}(A)(\boldsymbol{C})$ contains the image of μ_∞. The Hodge group $H(A)$ is by definition the connected component of the identity in $\mathrm{MT}(A) \cap \mathrm{SL}_V$.

$\mathrm{MT}(A)$ is a reductive group (see [8], [11]). Since by definition

$$\mu_\infty(\boldsymbol{C}^\times) \subset \mathrm{GSp}_{(V,\kappa)}(\boldsymbol{C}),$$

it follows that the group $\mathrm{MT}(A)$ is a subgroup of the group of symplectic similitudes $\mathrm{GSp}_{(V,\kappa)}$ and that

$$H(A) \subset \mathrm{Sp}_{(V,\kappa)}. \tag{5.2}$$

DEFINITION 5.3. The algebraic group $L(A) = C^\circ_D(\mathrm{Sp}_{(V,\kappa)})$, which is the connected component of the identity of the centralizer of D in $\mathrm{Sp}_{(V,\kappa)}$ (cf. Remark 5.4), is called the *Lefschetz group* of an abelian variety A. Note that the group $L(A)$ does not depend on the form κ (cf. [26]).

Before investigating Mumford-Tate group further, let us make two general remarks concerning centralizers of group schemes which we will often use.

REMARK 5.4. Let $B_1 \subset B_2$ be two commutative rings with identity. Let Λ be a free, finitely generated B_1-module such that it is also an R-module for a B_1-algebra R. Let G be a B_1-group subscheme of GL_Λ. Then $C_R(G)$ will denote the centralizer of R in G. The symbol $C^\circ_R(G)$ will denote the connected component of identity in $C_R(G)$. Let $\beta : \Lambda \times \Lambda \to B_1$ be a bilinear form and let $G_{(\Lambda,\beta)} \subset \mathrm{GL}_\Lambda$ be the subscheme of GL_Λ of the isometries with respect to the form β. Then we check that $C_R(G_{(\Lambda,\beta)}) \otimes_{B_1} B_2 \cong C_{R\otimes_{B_1} B_2}(G_{(\Lambda\otimes_{B_1} B_2,\, \beta\otimes_{B_1} B_2)})$.

REMARK 5.5. Let L/K be a finite separable field extension. Let V be a finite dimensional vector space and let $\phi : V \times V \to L$ be a nondegenerate bilinear form. Assume that $G_{(V,\phi)}$ is a connected algebraic group. Then there is a natural isomorphism $R_{L/K}G_{(V,\phi)} \cong C_L^{\circ}(G_{(V,\,\mathrm{Tr}_{L/K}\phi)})$. Let $B \subset K$ be a subring of K, integrally closed in K, and let $C \subset L$ be the integral closure of B in L. Assume that C is a free B-module which has a basis over B, such that the dual basis with respect to $\mathrm{Tr}_{L/K}$ is also in C. Let T be a finitely generated free C-module. Let $\phi : T \times T \to C$ be a nondegenerate bilinear form. Assume that $G_{(T,\phi)}$ is a connected algebraic group scheme over C. Then there is a natural isomorphism $R_{C/B}G_{(T,\phi)} \cong C_C^{\circ}(G_{(T,\,\mathrm{Tr}_{C/B}\phi)})$ of group schemes over B.

By [8, Sublemma 4.7], there is a unique E-bilinear, nondegenerate, alternating pairing

$$\phi : V \times V \to E$$

such that $\mathrm{Tr}_{E/\boldsymbol{Q}}(\phi) = \kappa$. Since the actions of $H(A)$ and $L(A)$ on V commute with the D-structure, and since $R_{E/\boldsymbol{Q}}(\mathrm{Sp}_{(V,\,\phi)}) = C_E(\mathrm{Sp}_{(V,\,\kappa)})$, by Remark 5.5, we get

$$H(A) \subset L(A) = C_D^{\circ}(R_{E/\boldsymbol{Q}}(\mathrm{Sp}_{(V,\,\phi)})) \subset C_D(R_{E/\boldsymbol{Q}}(\mathrm{Sp}_{(V,\,\phi)}))\,. \tag{5.6}$$

If $L/\boldsymbol{Q}$ is a field extension of $\boldsymbol{Q}$, we put

$$\mathrm{MT}(A)_L := \mathrm{MT}(A) \otimes_{\boldsymbol{Q}} L\,,\quad H(A)_L := H(A) \otimes_{\boldsymbol{Q}} L\,,\quad L(A)_L := L(A) \otimes_{\boldsymbol{Q}} L\,.$$

CONJECTURE 5.7 (Mumford-Tate, cf. [28, C.3.1]). *If A/F is an abelian variety over a number field F, then for any prime number l*

$$(G_l^{\mathrm{alg}})^{\circ} = \mathrm{MT}(A)_{\boldsymbol{Q}_l}, \tag{5.8}$$

where $(G_l^{\mathrm{alg}})^{\circ}$ denotes the connected component of the identity.

THEOREM 5.8 (Deligne [8, I, Proposition 6.2]). *If A/F is an abelian variety over a number field F and l is a prime number, then*

$$(G_l^{\mathrm{alg}})^{\circ} \subset \mathrm{MT}(A)_{\boldsymbol{Q}_l}\,. \tag{5.10}$$

THEOREM 5.11. *The Mumford-Tate conjecture is true for abelian varieties of class $\mathcal{B}$.*

PROOF. It is enough to verify (5.8) for a single prime l by [18, Theorem 4.3]. Hence we can use the equality (4.20) by taking l big enough. The proof goes similarly to that of [2, Theorem 7.12]. The important step is the transition (see 5.15 below) from symplectic forms to the symmetric forms to which we can apply the results of previous sections of this paper. It is known that $H(A)$ is semisimple (cf. [11, Proposition B.63]) and the center of $\mathrm{MT}(A)$ is $\boldsymbol{G}_m$ (cf. [11, Corollary B.59]). In addition $\mathrm{MT}(A) = \boldsymbol{G}_m H(A)$, and hence

$$(\mathrm{MT}(A)_{\boldsymbol{Q}_l})' = (H(A)_{\boldsymbol{Q}_l})' = H(A)_{\boldsymbol{Q}_l}\,. \tag{5.12}$$

By (4.20), (5.9) and (5.12), we have

$$\prod_{\lambda | l} R_{E_\lambda/\boldsymbol{Q}_l}(\mathrm{SO}_{(V_\lambda,\psi_\lambda^0)}) \subset H(A)_{\boldsymbol{Q}_l}\,. \tag{5.13}$$

By (5.6) and Remark 5.5, we have

$$
\begin{aligned}
H(A)_{\boldsymbol{Q}_l} &\subset L(A)_{\boldsymbol{Q}_l} \subset C_D(R_{E/\boldsymbol{Q}}(\mathrm{Sp}_{(V,\phi)})) \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l \\
&\cong \prod_{\lambda|l} C_{D_\lambda}(R_{E_\lambda/\boldsymbol{Q}_l}(\mathrm{Sp}_{(V_\lambda(A),\,\phi^0_\lambda)}))\,,
\end{aligned}
\tag{5.14}
$$

where $\kappa_l = \kappa \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l$, and κ_l is essentially the Weil pairing (cf. [19, diagram on p. 133]). By definitions of the forms ϕ_λ and ψ_λ, we have

$$
C_{D_\lambda}(\mathrm{Sp}_{(V_\lambda(A),\,\mathrm{Tr}_{E_\lambda/\boldsymbol{Q}_l}\phi^0_\lambda)}) \cong C_{D_\lambda}(\mathrm{SO}_{(V_\lambda(A),\,\mathrm{Tr}_{E_\lambda/\boldsymbol{Q}_l}\psi^0_\lambda)})\,.
\tag{5.15}
$$

So by (5.13), (5.14) and (5.15), we have

$$
\prod_{\lambda|l} R_{E_\lambda/\boldsymbol{Q}_l}(\mathrm{SO}_{(V_\lambda,\psi^0_\lambda)}) \subset H(A)_{\boldsymbol{Q}_l} \subset L(A)_{\boldsymbol{Q}_l} \subset \prod_{\lambda|l} C_{D_\lambda}(R_{E_\lambda/\boldsymbol{Q}_l}(\mathrm{SO}_{(V_\lambda(A),\,\psi^0_\lambda)}))\,.
\tag{5.16}
$$

Observe that $V_\lambda(A) \cong V_\lambda \oplus V_\lambda$ by Theorem 3.23. Moreover $D_\lambda = M_{2,2}(E_\lambda)$ by assumption on λ. Hence evaluating left and right ends of the inclusions (5.16) on the $\overline{\boldsymbol{Q}}_l$-points, we get equalities of the both ends with

$$
\prod_{\lambda|l} \prod_{E_\lambda \hookrightarrow \overline{\boldsymbol{Q}}_l} (\mathrm{SO}_{(V_\lambda,\psi_\lambda)})(\overline{\boldsymbol{Q}}_l)
$$

which is an irreducible algebraic variety over $\overline{\boldsymbol{Q}}_l$. Then we use [12, Propositions II, 2.6 and II, 4.10] in order to conclude that the groups $H(A)_{\overline{\boldsymbol{Q}}_l}$ and $L(A)_{\overline{\boldsymbol{Q}}_l}$ as well as the groups over $\boldsymbol{Q}_l$

$$
\prod_{\lambda|l} C_{D_\lambda}(R_{E_\lambda/\boldsymbol{Q}_l}(\mathrm{SO}_{(V_\lambda(A),\,\psi^0_\lambda)})) = \prod_{\lambda|l} C_{D_\lambda}(\mathrm{SO}_{(V_\lambda(A),\,\mathrm{Tr}_{E_\lambda/\boldsymbol{Q}_l}\psi^0_\lambda)})
$$

are connected. Then (5.16) gives the following equalities by use of Remark 5.5:

$$
\prod_{\lambda|l} R_{E_\lambda/\boldsymbol{Q}_l}(\mathrm{SO}_{(V_\lambda,\psi^0_\lambda)}) = H(A)_{\boldsymbol{Q}_l} = L(A)_{\boldsymbol{Q}_l} = \prod_{\lambda|l} C_{D_\lambda}(\mathrm{SO}_{(V_\lambda(A),\,\mathrm{Tr}_{E_\lambda/\boldsymbol{Q}_l}\psi^0_\lambda)})\,.
\tag{5.17}
$$

The equalities (4.20), (5.17) and [3, p. 702, Corollary 1] give

$$
\mathrm{MT}(A)_{\boldsymbol{Q}_l} = \boldsymbol{G}_m H(A)_{\boldsymbol{Q}_l} = \boldsymbol{G}_m (G_l^{\mathrm{alg}})' \subset G_l^{\mathrm{alg}}\,.
\tag{5.18}
$$

The theorem follows by (5.10) and (5.18). □

COROLLARY 5.19. *If A is an abelian variety of class $\mathcal{B}$, then*

$$
H(A) = L(A) = C^\circ_D(R_{E/\boldsymbol{Q}}(\mathrm{Sp}_{(V,\phi)})) = C_D(R_{E/\boldsymbol{Q}}(\mathrm{Sp}_{(V,\phi)}))\,.
\tag{5.20}
$$

PROOF. By (5.6) and (5.17) we get equality of Lie algebras

$$
\mathcal{L}ie\, H(A) = \mathcal{L}ie\, L(A) = \mathcal{L}ie\, C^\circ_D(R_{E/\boldsymbol{Q}}(\mathrm{Sp}_{(V,\phi)})) = \mathcal{L}ie\, C_D(R_{E/\boldsymbol{Q}}(\mathrm{Sp}_{(V,\phi)}))
$$

of connected group schemes. Hence (5.20) follows by (5.6) and [13, Theorem p. 87]. □

CONJECTURE 5.21 (Lang). *Let A be an abelian variety over a number field F. Then for $l \gg 0$ the group $\rho_l(G_F)$ contains the group of all homotheties in* $\mathrm{GL}_{T_l(A)}(\boldsymbol{Z}_l)$.

THEOREM 5.22 (Wintenberger [36, p. 5, Corollary 1]). *Let A be an abelian variety over a number field F. The Lang conjecture holds for A if the Mumford-Tate conjecture holds for A or if* $\dim A < 5$.

THEOREM 5.23. *The Lang's conjecture is true for abelian varieties of class $\mathcal{B}$.*

PROOF. It follows by Theorems 5.11 and 5.22. □

Consider again the bilinear form $\phi : V \times V \to E$.
We have:

$$H^1(A(\boldsymbol{C}); \boldsymbol{C}) \cong V \otimes_{\boldsymbol{Q}} \boldsymbol{C} \cong \bigoplus_{\sigma : E \hookrightarrow \boldsymbol{C}} V \otimes_{E,\sigma} \boldsymbol{C}\,.$$

Put $V_\sigma(A) = V \otimes_{E,\sigma} \boldsymbol{C}$ and let ϕ_σ be the form

$$\phi \otimes_{E,\sigma} \boldsymbol{C} : V_\sigma(A) \otimes_{\boldsymbol{C}} V_\sigma(A) \to \boldsymbol{C}\,.$$

Since A is of type III, there are isomorphisms $D \otimes_E \boldsymbol{C} \cong D \otimes_E \boldsymbol{R} \otimes_{\boldsymbol{R}} \boldsymbol{C} \cong \boldsymbol{H} \otimes_{\boldsymbol{R}} \boldsymbol{C} \cong M_{2,2}(\boldsymbol{C})$. Define the bilinear form

$$\psi_\sigma : V_\sigma(A) \times V_\sigma(A) \to \boldsymbol{C} \quad \text{by} \quad \psi_\sigma(v_1, v_2) := \phi_\sigma(J v_1, v_2)\,. \tag{5.24}$$

LEMMA 5.25. *If A is simple abelian variety of type III, then for each $\sigma : E \hookrightarrow \boldsymbol{C}$ there is a $\boldsymbol{C}$-vector space $W_\sigma(A)$ of dimension $g/e = 4\dim A/[D : \boldsymbol{Q}]$ such that*

(i) $V_\sigma(A) \cong W_\sigma(A) \oplus W_\sigma(A)$,

(ii) *the restriction of ψ_σ to $W_\sigma(A)$ gives a nondegenerate, symmetric pairing*

$$\psi_\sigma : W_\sigma(A) \times W_\sigma(A) \to \boldsymbol{C}\,.$$

PROOF. The idea of the proof is the same as that of Theorem 3.23. Namely, using some arguments that we used in the proof of Proposition 3.18, we can prove as follows.

$$\psi_\sigma(B v_1, v_2) = \psi_\sigma(v_1, B^t v_2) \quad \text{for every } B \in M_{2,2}(\boldsymbol{C})\,.$$

Let $t, u, f, e \in M_{2,2}(\boldsymbol{C})$ be the matrices defined in Section 3. Define $W_\sigma(A) := f V_\sigma(A)$. We get the proof by repeating the argument of Lemma 3.22. □

COROLLARY 5.26. *If A is an abelian variety of class $\mathcal{B}$, then*

$$H(A)_{\boldsymbol{C}} = L(A)_{\boldsymbol{C}} = \prod_{\sigma : E \hookrightarrow \boldsymbol{C}} \mathrm{SO}_{(W_\sigma(A),\, \psi_\sigma)}\,. \tag{5.27}$$

PROOF. With use of Lemma 5.25 and the argument similar to the proof of formula (5.17), we obtain

$$C_D^\circ(R_{E/\boldsymbol{Q}}(\mathrm{Sp}_{(V,\,\phi)})) \otimes_{\boldsymbol{Q}} \boldsymbol{C} \cong \prod_{\sigma E \hookrightarrow \boldsymbol{C}} \mathrm{SO}_{(W_\sigma(A),\, \psi_\sigma)}\,.$$

Hence (5.27) follows by (5.20). □

6. Images of the Galois representations ρ_l and $\overline{\rho}_l$. In this section we explicitly compute the images of the l-adic representations induced by the action of the absolute Galois group on the Tate module of abelian varieties of type III .

By Theorem 3.23 (i) the representation ρ_λ induces naturally the representation (denoted in the same way)

$$\rho_\lambda \, : \, G_F \to \mathrm{GL}(T_\lambda)\,.$$

Moreover, by Theorem 3.23 (ii), we have

$$\rho_l(G_F) \subset \prod_{\lambda | l} \mathrm{GO}_{(T_\lambda, \psi_\lambda)}(\mathcal{O}_\lambda) = \prod_{\lambda | l} R_{\mathcal{O}_\lambda / \boldsymbol{Z}_l}(\mathrm{GO}_{(T_\lambda, \psi_\lambda)})(\boldsymbol{Z}_l)\,. \tag{6.1}$$

By (6.1) there is a closed immersion

$$\mathcal{G}_l^{\mathrm{alg}} \subset \prod_{\lambda | l} R_{\mathcal{O}_\lambda / \boldsymbol{Z}_l}(\mathrm{GO}_{(T_\lambda, \psi_\lambda)})\,, \tag{6.2}$$

which implies

$$\rho_l(G_F) \subset \mathcal{G}_l^{\mathrm{alg}}(\boldsymbol{Z}_l) \subset \prod_{\lambda | l} R_{\mathcal{O}_\lambda / \boldsymbol{Z}_l}(\mathrm{GO}_{(T_\lambda, \psi_\lambda)})(\boldsymbol{Z}_l)\,. \tag{6.3}$$

Since l is unramified in E, there is a natural isomorphism $R_{\mathcal{O}_\lambda/\boldsymbol{Z}_l}(.) \otimes_{\boldsymbol{Z}_l} \boldsymbol{F}_l \cong R_{k_\lambda/\boldsymbol{F}_l}(.)$. To see this isomorphism in an elementary way, we can use [2, Remark 2.8] and a modification of [2, Lemma 2.1] to the case of $R_{\mathcal{O}_\lambda/\boldsymbol{Z}_l}$. Changing base in (6.2), we get a natural closed immersion of group schemes

$$G(l)^{\mathrm{alg}} \subset \prod_{\lambda | l} R_{k_\lambda / \boldsymbol{F}_l}(\mathrm{GO}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)})\,, \tag{6.4}$$

where $A_\lambda[\lambda] = \overline{\mathcal{W}}_\lambda(A)$ and $A[\lambda] \cong A_\lambda[\lambda] \oplus A_\lambda[\lambda]$ (cf. Theorem 3.23 (i), (ii$''$)). Hence, by reducing mod l in (6.3), we get

$$\overline{\rho}_l(G_F) \subset G(l)^{\mathrm{alg}}(\boldsymbol{F}_l) \subset \prod_{\lambda | l} R_{k_\lambda / \boldsymbol{F}_l}(\mathrm{GO}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)})(\boldsymbol{F}_l)\,. \tag{6.5}$$

Because extracting derived subgroup commutes with base change (see [2, Remark 6.8]), and because $(\mathcal{G}_l^{\mathrm{alg}})'$ (resp. $(G(l)^{\mathrm{alg}})'$ are connected, by (6.2) (resp. by (6.4)) we get

$$(\mathcal{G}_l^{\mathrm{alg}})' \subset \prod_{\lambda | l} R_{\mathcal{O}_\lambda / \boldsymbol{Z}_l}(\mathrm{SO}_{(T_\lambda, \psi_\lambda)})\,, \tag{6.6}$$

$$(G(l)^{\mathrm{alg}})' \subset \prod_{\lambda | l} R_{k_\lambda / \boldsymbol{F}_l}(\mathrm{SO}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)})\,. \tag{6.7}$$

PROPOSITION 6.8. *Let A/F be an abelian variety of class $\mathcal{B}$. Then for all $l \gg 0$, we have the equalitiy of ranks of group schemes over $\boldsymbol{F}_l$*

$$\mathrm{rank}(G(l)^{\mathrm{alg}})' = \mathrm{rank} \, \prod_{\lambda | l} R_{k_\lambda / \boldsymbol{F}_l}(\mathrm{SO}_{(A_\lambda[\lambda],\overline{\psi}_\lambda)})\,. \tag{6.9}$$

PROOF. Using (6.7) and Theorem 4.19 we apply [2, Lemma 6.1] to finish the proof in the same way as that of [2, Theorem 6.6]. □

THEOREM 6.10. *Let A/F be an abelian variety of class $\mathcal{B}$. Then for all $l \gg 0$, we have the equality of group schemes*

$$(G(l)^{\mathrm{alg}})' = \prod_{\lambda | l} R_{k_\lambda/\boldsymbol{F}_l}(\mathrm{SO}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}) \,. \tag{6.11}$$

PROOF. Projecting onto the λ-component in (6.7), we obtain the representation

$$\underline{\rho}_{\Phi_\lambda} \; : \; (G(l)^{\mathrm{alg}})' \to \; R_{k_\lambda/\boldsymbol{F}_l}(\mathrm{SO}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}) \; . \tag{6.12}$$

This gives the representation

$$(G(l)^{\mathrm{alg}})' \otimes_{\boldsymbol{F}_l} k_\lambda \to \; \mathrm{SO}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)} \,. \tag{6.13}$$

By (3.21) we have the natural isomorphism

$$(\mathrm{End}_{k_\lambda[G_F]} \; A_\lambda[\lambda]) \otimes_{k_\lambda} L \cong L \tag{6.14}$$

for any field extension L/k_λ. Hence, by (6.13), (6.14) and the Schur's Lemma, it follows that

$$\underline{\rho}_{\Phi_\lambda}(Z((G(l)^{\mathrm{alg}})' \otimes_{\boldsymbol{F}_l} k_\lambda)) \subset k_\lambda^\times \mathrm{Id}_{A_\lambda[\lambda]} \,.$$

Hence by (6.13)

$$\underline{\rho}_{\Phi_\lambda}(Z((G(l)^{\mathrm{alg}})' \otimes_{\boldsymbol{F}_l} k_\lambda)) \subset \; \mu_2 \,,$$

which implies that

$$\underline{\rho}_{\Phi_\lambda}(Z((G(l)^{\mathrm{alg}})')) \subset R_{k_\lambda/\boldsymbol{F}_l}(\mu_2) \,.$$

Hence

$$Z((G(l)^{\mathrm{alg}})') \subset \prod_{\lambda | l} R_{k_\lambda/\boldsymbol{F}_l}(\mu_2) \subset Z\bigg(\prod_{\lambda | l} R_{k_\lambda/\boldsymbol{F}_l}(\mathrm{SO}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}) \bigg) \,.$$

Since both groups $(G(l)^{\mathrm{alg}})'$ and $\prod_{\lambda | l} R_{k_\lambda/\boldsymbol{F}_l}(\mathrm{SO}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)})$ are reductive, the proof is finished in the same way as that of [1, Lemma 3.4]. □

REMARK 6.15. Let $\widetilde{G}$ denote the universal cover for a semisimple group scheme G. The existence of the universal cover for a semisimple group scheme over a field was proven by Chevalley [7] (cf. [31]). In general, the existance of the universal cover for a semisimple group scheme over a base scheme S follows from [9, Exposé XXV]. The universal cover is compatible with the base change.

Let $\mathrm{Spin}_{(T_\lambda,\, \psi_\lambda)}$ (resp. $\mathrm{Spin}_{(V_\lambda,\, \psi_\lambda)}$, $\mathrm{Spin}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}$) denote the universal cover of the group scheme $\mathrm{SO}_{(T_\lambda,\, \psi_\lambda)}$ (resp. $\mathrm{SO}_{(V_\lambda,\, \psi_\lambda)}$, $\mathrm{SO}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}$). Consider the following, short exact sequences of group schemes:

$$1 \longrightarrow \; \mu_2 \; \longrightarrow \; \mathrm{Spin}_{(T_\lambda,\, \psi_\lambda)} \; \xrightarrow{\pi_\lambda} \; \mathrm{SO}_{(T_\lambda,\, \psi_\lambda)} \; \longrightarrow \; 1 \,, \tag{6.16}$$

$$1 \longrightarrow \mu_2 \longrightarrow \mathrm{Spin}_{(V_\lambda,\,\psi_\lambda)} \xrightarrow{\pi_\lambda} \mathrm{SO}_{(V_\lambda,\,\psi_\lambda)} \longrightarrow 1\,, \tag{6.17}$$

$$1 \longrightarrow \mu_2 \longrightarrow \mathrm{Spin}_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)} \xrightarrow{\overline{\pi}_\lambda} \mathrm{SO}_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)} \longrightarrow 1\,. \tag{6.18}$$

The sequences (6.17) and (6.18) are obtained by base change from the sequence (6.16). Evaluating the exact sequence (6.16) on $\mathcal{O}_\lambda$-points (resp. (6.18) on k_λ-points), we get

$$\mathrm{SO}_{(T_\lambda,\,\psi_\lambda)}(\mathcal{O}_\lambda)\,/\,\pi_\lambda(\mathrm{Spin}_{(T_\lambda,\,\psi_\lambda)}(\mathcal{O}_\lambda)) \cong \boldsymbol{Z}/2 \tag{6.19}$$

$$(\text{resp. } \mathrm{SO}_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}(k_\lambda)\,/\,\overline{\pi}_\lambda(\mathrm{Spin}_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}(k_\lambda)) \cong \boldsymbol{Z}/2)\,. \tag{6.20}$$

Evaluating the exact sequence (6.17) on E_λ-points we get

$$\mathrm{SO}_{(V_\lambda,\,\psi_\lambda)}(E_\lambda)\,/\,\pi_\lambda(\mathrm{Spin}_{(V_\lambda,\,\psi_\lambda)}(E_\lambda)) \cong \boldsymbol{Z}/2 \oplus \boldsymbol{Z}/2\,. \tag{6.21}$$

Indeed, the theorem of Steinberg (cf. [16, Theorem 2.1]) gives $H^1(k_\lambda,\ \mathrm{Spin}_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}) = 0$ and the theorem of Kneser (cf. [16, Theorem 2.2]) gives $H^1(E_\lambda,\ \mathrm{Spin}_{(V_\lambda,\,\psi_\lambda)}) = 0$. In addition, by a theorem of Tits (cf. [22, Theorem 4.1]), the natural map

$$H^1(\mathcal{O}_\lambda,\ \mathrm{Spin}_{(T_\lambda,\,\psi_\lambda)}) \to H^1(E_\lambda,\ \mathrm{Spin}_{(V_\lambda,\,\psi_\lambda)})$$

is an imbedding. Hence $H^1(\mathcal{O}_\lambda,\ \mathrm{Spin}_{(T_\lambda,\,\psi_\lambda)}) = 0$.

LEMMA 6.22 (Integral Gram-Schmidt). *Let (R, m_R) be a local integral domain with* char $R \neq 2$. *Let $k := R/m_R$ be the residue field. Let T be a free finitely generated R-module and let $\overline{T} := T \otimes_R k$. Consider a symmetric bilinear form*

$$\beta\ :\ T \times T \to R$$

such that the form

$$\overline{\beta} := \beta \otimes_R k\ :\ \overline{T} \times \overline{T} \to k$$

is nondegenerate. Assume that $1 + m_R = (1 + m_R)^2$. Then the map

$$\mathrm{SO}_{(T,\,\beta)}(R) \to \mathrm{SO}_{(\overline{T},\,\overline{\beta})}(k)$$

is surjective.

PROOF. The proof is an analogue of Gram-Schmidt algorithm. □

Let $G' := \overline{[G,\ G]}$ denote the closure of the commutator subgroup for any profinite group G.

PROPOSITION 6.23. *Let A/F be an abelian variety of class $\mathcal{B}$. Then for $l \gg 0$, the equalities*

$$\prod_{\lambda | l} \pi_\lambda(\mathrm{Spin}_{(T_\lambda,\,\psi_\lambda)}(\mathcal{O}_\lambda)) = \prod_{\lambda | l} \mathrm{SO}_{(T_\lambda,\,\psi_\lambda)}(\mathcal{O}_\lambda)'\,, \tag{6.24}$$

$$\prod_{\lambda | l}\ \overline{\pi}_\lambda(\mathrm{Spin}_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}(k_\lambda)) = \prod_{\lambda | l} \mathrm{SO}_{(A_\lambda[\lambda],\,\overline{\psi}_\lambda)}(k_\lambda)' \tag{6.25}$$

hold.

PROOF. Observe that

$$\prod_{\lambda | l} \mathrm{SO}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}(k_\lambda)' \subset \prod_{\lambda | l} \overline{\pi}_\lambda(\mathrm{Spin}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}(k_\lambda))$$

by (6.20). On the other hand $\mathrm{Spin}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}(k_\lambda)$ is a perfect group for all $\lambda \mid l$. So

$$\prod_{\lambda | l} \overline{\pi}_\lambda(\mathrm{Spin}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}(k_\lambda)) \subset \prod_{\lambda | l} \mathrm{SO}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}(k_\lambda)'$$

by (6.18). Hence it proves (6.25). Consider the group scheme $\mathcal{G} := C_{\mathcal{R}}(\mathrm{Sp}_{(A,\,\kappa)})$ over Spec $\mathbf{Z}$. Take a natural number N big enough so that, for $l \geq N$, the condition $l \gg 0$ holds. Let $\mathcal{G}_N := \mathcal{G} \otimes_{\mathbf{Z}} \mathbf{Z}[1/N]$. The scheme $\mathcal{G}_N$ is semisimple. By [9, Exposé XXII, Proposition 4.3.4], the scheme $\widetilde{\mathcal{G}_N}$ is semisimple. Remark 5.4 and the universality of the fiber product give

$$C_{\mathcal{R}}(\mathrm{Sp}_{(A,\,\kappa)}) \otimes_{\mathbf{Z}} \mathbf{Z}_l = C_{\mathcal{R} \otimes_{\mathbf{Z}} \mathbf{Z}_l}(\mathrm{Sp}_{(T_l(A),\,\kappa_l)})\,. \tag{6.26}$$

By definition of the forms ψ_λ, ϕ_λ, we have

$$C_{\mathcal{R}_\lambda}(\mathrm{Sp}_{(T_\lambda(A),\, \mathrm{Tr}_{\mathcal{O}_\lambda/\mathbf{Z}_l}\phi_\lambda)}) \cong C_{\mathcal{R}_\lambda}(\mathrm{SO}_{(T_\lambda(A),\, \mathrm{Tr}_{\mathcal{O}_\lambda/\mathbf{Z}_l}\psi_\lambda)})\,.$$

For $l \gg 0$, we have $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_l \subset \mathcal{R} \otimes_{\mathbf{Z}} \mathbf{Z}_l$, $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_l = \prod_{\lambda | l} \mathcal{O}_\lambda$ and $\mathcal{R} \otimes_{\mathbf{Z}} \mathbf{Z}_l = \prod_{\lambda | l} \mathcal{R}_\lambda$. Moreover, by (2.13) we have natural isomorphism of $R_\lambda \cong M_{2,2}(\mathcal{O}_\lambda)$ of $\mathcal{O}_\lambda$-algebras, and by Theorem 3.23 (i) we have a natural isomorphism $T_\lambda(A) \cong T_\lambda \oplus T_\lambda$ of $\mathcal{O}_\lambda[G_F]$-modules. Hence by Remark 5.5, we get

$$\begin{aligned}
C_{\mathcal{R} \otimes_{\mathbf{Z}} \mathbf{Z}_l}(\mathrm{Sp}_{(T_l(A),\,\kappa_l)}) &\cong C_{\mathcal{R} \otimes_{\mathbf{Z}} \mathbf{Z}_l}(C_{\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_l}(\mathrm{Sp}_{(T_l(A),\,\kappa_l)}))\\
&\cong C_{\mathcal{R} \otimes_{\mathbf{Z}} \mathbf{Z}_l}\Big(\prod_{\lambda | l} C_{\mathcal{O}_\lambda}(\mathrm{Sp}_{(T_\lambda(A),\, \mathrm{Tr}_{\mathcal{O}_\lambda/\mathbf{Z}_l}\phi_\lambda)})\Big)\\
&\cong \prod_{\lambda | l} C_{\mathcal{R}_\lambda}(\mathrm{Sp}_{(T_\lambda(A),\, \mathrm{Tr}_{\mathcal{O}_\lambda/\mathbf{Z}_l}\phi_\lambda)})\\
&\cong \prod_{\lambda | l} C_{\mathcal{R}_\lambda}(\mathrm{SO}_{(T_\lambda(A),\, \mathrm{Tr}_{\mathcal{O}_\lambda/\mathbf{Z}_l}\psi_\lambda)})\\
&\cong \prod_{\lambda | l} C_{\mathcal{O}_\lambda}(\mathrm{SO}_{(T_\lambda,\, \mathrm{Tr}_{\mathcal{O}_\lambda/\mathbf{Z}_l}\psi_\lambda)})\\
&\cong \prod_{\lambda | l} R_{\mathcal{O}_\lambda/\mathbf{Z}_l}(\mathrm{SO}_{(T_\lambda,\,\psi_\lambda)})\,.
\end{aligned} \tag{6.27}$$

Isomorphisms (6.26) and (6.27) give $\mathcal{G}_N \otimes_{\mathbf{Z}[1/N]} \mathbf{Z}_l \cong \prod_{\lambda | l} R_{\mathcal{O}_\lambda/\mathbf{Z}_l}\mathrm{SO}_{(T_\lambda,\,\psi_\lambda)}$. Because the universal cover is unique and commutes with base change, we get

$$\widetilde{\mathcal{G}_N} \otimes_{\mathbf{Z}[1/N]} \mathbf{Z}_l \cong \prod_{\lambda | l} R_{\mathcal{O}_\lambda/\mathbf{Z}_l}\mathrm{Spin}_{(T_\lambda,\,\psi_\lambda)}\,.$$

Consider the commutative diagram

$$\begin{array}{ccc} \prod_{\lambda|l} \mathrm{Spin}_{(T_\lambda, \psi_\lambda)}(\mathcal{O}_\lambda) & \xrightarrow{r_l} & \prod_{\lambda|l} \mathrm{Spin}_{(A_\lambda[\lambda], \overline{\psi}_\lambda)}(k_\lambda) \\ {\scriptstyle \pi_l}\big\downarrow & & {\scriptstyle \overline{\pi_l}}\big\downarrow \\ \prod_{\lambda|l} \mathrm{SO}_{(T_\lambda, \psi_\lambda)}(\mathcal{O}_\lambda) & \xrightarrow{r_l} & \prod_{\lambda|l} \mathrm{SO}_{(A_\lambda[\lambda], \overline{\psi}_\lambda)}(k_\lambda)\,, \end{array} \tag{6.28}$$

where $\pi_l := \prod_{\lambda|l} \pi_\lambda$, $\overline{\pi_l} := \prod_{\lambda|l} \overline{\pi_\lambda}$ and $r_l := \prod_{\lambda|l} r_\lambda$ for the natural reduction maps r_λ. Note that

$$r_l\left(\pi_l^{-1}\left(\prod_{\lambda|l} \mathrm{SO}_{(T_\lambda,\, \psi_\lambda)}(\mathcal{O}_\lambda)'\right)\right) = \prod_{\lambda|l} \mathrm{Spin}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}(k_\lambda)\,.$$

Indeed, using Lemma 6.22 and (6.25), it follows from the diagram (6.28), because the group $\prod_{\lambda|l} \mathrm{Spin}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}(k_\lambda)$ is perfect for $l \gg 0$ (cf. [31, Chapter 7, Corollary 2 (b)]) and, by the theory of Chevalley's groups [7] (cf. [31]), the kernel of the map $\overline{\pi_l}$ is contained in the center of $\prod_{\lambda|l} \mathrm{Spin}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}(k_\lambda)$. Hence, by [17, Proposition 2.6], we get

$$\prod_{\lambda|l} \mathrm{Spin}_{(T_\lambda,\, \psi_\lambda)}(\mathcal{O}_\lambda) = \pi_l^{-1}\left(\prod_{\lambda|l} \mathrm{SO}_{(T_\lambda,\, \psi_\lambda)}(\mathcal{O}_\lambda)'\right). \qquad \square$$

THEOREM 6.29. *Let A/F be an abelian variety of class $\mathcal{B}$. Then for $l \gg 0$, there are inclusions*

$$\prod_{\lambda|l} \mathrm{SO}_{(T_\lambda,\, \psi_\lambda)}(\mathcal{O}_\lambda)' \subset \rho_l(G_F') \subset \prod_{\lambda|l} \mathrm{SO}_{(T_\lambda,\, \psi_\lambda)}(\mathcal{O}_\lambda)\,, \tag{6.30}$$

$$\prod_{\lambda|l} \mathrm{SO}_{(A_\lambda[\lambda], \overline{\psi}_\lambda)}(k_\lambda)' \subset \overline{\rho_l}(G_F') \subset \prod_{\lambda|l} \mathrm{SO}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}(k_\lambda)\,, \tag{6.31}$$

where $\overline{\rho_l}$ is the representation ρ_l mod l.

PROOF. By (6.5) and (6.11), we have

$$\overline{\rho_l}(G_F') = (\overline{\rho_l}(G_F))' \subset \prod_{\lambda|l} \mathrm{SO}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}(k_\lambda)\,.$$

By a theorem of Serre (cf. [36, Theorem 4]), [36, Lemma 5] and Remark 6.15, we get

$$\prod_{\lambda|l} \pi_\lambda(\mathrm{Spin}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}(k_\lambda)) \subset \overline{\rho_l}(G_F)\,.$$

Since $\mathrm{Spin}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}(k_\lambda)$ is a perfect group [31, Chapter 7, Corollary 2 (b)], we have

$$\prod_{\lambda|l} \pi_\lambda(\mathrm{Spin}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}(k_\lambda)) \subset \overline{\rho_l}(G_F')\,.$$

This proves (6.31). From (6.6) we know that the group $\rho_l(G_F') = (\rho_l(G_F))'$ is a closed subgroup of $\prod_{\lambda|l} \mathrm{SO}_{(T_\lambda,\psi_\lambda)}(\mathcal{O}_\lambda)$. Consider the diagram (6.28). Since the finite group

$\prod_{\lambda|l} \mathrm{Spin}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}(k_\lambda)$ is perfect, it follows by (6.25) and (6.31) that

$$\overline{\pi_l}\Bigg(\prod_{\lambda|l} (\mathrm{Spin}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}(k_\lambda))\Bigg) \subset r_l((\rho_l(G'_F))').$$

On the other hand, it follows from (6.16) and (6.19) that

$$(\rho_l(G'_F))' \subset \pi_l\Bigg(\prod_{\lambda|l} (\mathrm{Spin}_{(T_\lambda,\, \psi_\lambda)}(\mathcal{O}_\lambda))\Bigg).$$

So we get the equality

$$r_l(\pi_l^{-1}((\rho_l(G'_F))')) = \prod_{\lambda|l} \mathrm{Spin}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}(k_\lambda)$$

since $\prod_{\lambda|l} \mathrm{Spin}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}(k_\lambda)$ is perfect and the kernel of the map $\overline{\pi_l}$ is contained in the center of the group $\prod_{\lambda|l} \mathrm{Spin}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}(k_\lambda)$. So

$$\pi_l^{-1}((\rho_l(G'_F))') = \pi_l^{-1}(\rho_l(G'_F)) = \prod_{\lambda|l} \mathrm{Spin}_{(T_\lambda,\, \psi_\lambda)}(\mathcal{O}_\lambda)$$

by [17, Proposition 2.6]. It proves (6.30) in view of (6.24). □

REMARK 6.30. Since $\prod_{\lambda|l} \mathrm{SO}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}(k_\lambda)'$ is a perfect group, the proof of 6.23 shows that $\prod_{\lambda|l} \mathrm{SO}_{(T_\lambda,\, \psi_\lambda)}(\mathcal{O}_\lambda)'$ is a perfect group with respect to the operation of taking commutator and then closure in a profinite group.

THEOREM 6.31. *If A is an abelian variety of class $\mathcal{B}$, then the equalities*

$$\rho_l(G''_F) = \prod_{\lambda|l} \mathrm{SO}_{(T_\lambda,\, \psi_\lambda)}(\mathcal{O}_\lambda)', \tag{6.32}$$

$$\overline{\rho_l}(G''_F) = \prod_{\lambda|l} \mathrm{SO}_{(A_\lambda[\lambda],\, \overline{\psi}_\lambda)}(k_\lambda)' \tag{6.33}$$

hold for $l \gg 0$.

PROOF. It follows by (6.19), (6.20), Theorem 6.29 and Remark 6.30. □

7. Open image property of ρ_l. Consider the group scheme $C_{\mathcal{R}}(\mathrm{Sp}_{(\Lambda,\, \kappa)})$ over Spec $\boldsymbol{Z}$. Since $C_{\mathcal{R}}(\mathrm{Sp}_{(\Lambda,\, \kappa)}) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} = C_D(\mathrm{Sp}_{(V,\, \kappa^0)})$ (see Remark 5.4), there is an open imbedding in the l-adic topology

$$C_{\mathcal{R}}(\mathrm{Sp}_{(\Lambda,\, \kappa)})(\boldsymbol{Z}_l) \subset C_D(\mathrm{Sp}_{(V,\, \kappa^0)})(\boldsymbol{Q}_l). \tag{7.1}$$

THEOREM 7.2. *Let A be an abelian variety of class $\mathcal{B}$ and let $r(l)$ be the number of primes over l in $\mathcal{O}_E$. Then*

(i) *$\rho_l(G_F)$ is open in $C_{\mathcal{R}}(\mathrm{GSp}_{(\Lambda,\, \kappa)})(\boldsymbol{Z}_l)$ for every prime number l,*

(ii) *$\rho_l(G'_F)$ has the index dividing $2^{r(l)}$ in $C_{\mathcal{R}}(\mathrm{Sp}_{(\Lambda,\, \kappa)})(\boldsymbol{Z}_l)$ for $l \gg 0$,*

(iii) *$\rho_l(G''_F) = C_{\mathcal{R}}(\mathrm{Sp}_{(\Lambda,\, \kappa)})(\boldsymbol{Z}_l)'$ for $l \gg 0$.*

PROOF. The group $\mathrm{GSp}_{(\Lambda,\kappa)}(\boldsymbol{Z}_l)$ is generated by $\mathrm{Sp}_{(\Lambda,\kappa)}(\boldsymbol{Z}_l)$ and a subgroup which, in the Frobenius basis of Λ, has the form

$$\left\{ \begin{pmatrix} aI_g & 0 \\ 0 & I_g \end{pmatrix};\ a \in \boldsymbol{Z}_l{}^{\times} \right\}.$$

The group $\boldsymbol{Z}_l^{\times}\mathrm{Sp}_{(\Lambda,\kappa)}(\boldsymbol{Z}_l)$ has index 2 (resp. index 4) in $\mathrm{GSp}_{(\Lambda,\kappa)}(\boldsymbol{Z}_l)$, for $l > 2$ (resp. for $l = 2$). By [3, Corollary 1], there is an open subgroup $U \subset \boldsymbol{Z}_l^{\times}$ such that $U \subset \rho_l(G_F)$. Hence $U\, C_{\mathcal{R}}(\mathrm{Sp}_{(\Lambda,\kappa)})(\boldsymbol{Z}_l) = C_{\mathcal{R}}(U\,\mathrm{Sp}_{(\Lambda,\kappa)}(\boldsymbol{Z}_l))$ is an open subgroup of $C_{\mathcal{R}}(\mathrm{GSp}_{(\Lambda,\kappa)})(\boldsymbol{Z}_l) = C_{\mathcal{R}}(\mathrm{GSp}_{(\Lambda,\kappa)}(\boldsymbol{Z}_l))$. By [3, Theorem 1], the group $\rho_l(G_F)$ is open in $G_l^{\mathrm{alg}}(\boldsymbol{Q}_l)$. By Theorem 5.11 and Corollary 5.19

$$\begin{aligned} U\, C_{\mathcal{R}}(\mathrm{Sp}_{(\Lambda,\kappa)})(\boldsymbol{Z}_l) &\subset \boldsymbol{Q}_l^{\times}\, C_D(\mathrm{Sp}_{(V,\kappa)})(\boldsymbol{Q}_l) \\ &= \boldsymbol{G}_m(\boldsymbol{Q}_l)H(A)(\boldsymbol{Q}_l) \\ &\subset \mathrm{MT}(A)(\boldsymbol{Q}_l) = G_l^{\mathrm{alg}}(\boldsymbol{Q}_l)\,. \end{aligned} \tag{7.3}$$

Hence, $U\, C_{\mathcal{R}}(\mathrm{Sp}_{(\Lambda,\kappa)})(\boldsymbol{Z}_l) \cap \rho_l(G_F)$ is open in $U\, C_{\mathcal{R}}(\mathrm{Sp}_{(\Lambda,\kappa)})(\boldsymbol{Z}_l)$ and we get that $\rho_l(G_F)$ is open in $C_{\mathcal{R}}(\mathrm{GSp}_{(\Lambda,\kappa)})(\boldsymbol{Z}_l)$. Moreover, by (6.26) and (6.27), we have a natural isomorphism

$$C_{\mathcal{R}}(\mathrm{Sp}_{(\Lambda,\kappa)})(\boldsymbol{Z}_l) \cong \prod_{\lambda|l} \mathrm{SO}_{(T_\lambda,\,\psi_\lambda)}(\mathcal{O}_\lambda) \tag{7.4}$$

for $l \gg 0$. Hence from (6.19), (6.24) and (6.30), it follows that the subgroup $\rho_l(G'_F)$ is of index dividing $2^{r(l)}$ in the group $C_{\mathcal{R}}(\mathrm{Sp}_{(\Lambda,\kappa)})(\boldsymbol{Z}_l)$. □

THEOREM 7.5. *If A is an abelian variety of class $\mathcal{B}$, then for every prime number l, the group $\rho_l(G_F)$ is open in the group $\mathcal{G}_l^{\mathrm{alg}}(\boldsymbol{Z}_l)$ in the l-adic topology.*

PROOF. By Theorem 7.2, the group $\rho_l(G_F)$ is open in $C_{\mathcal{R}\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_l}(\mathrm{GSp}_{(T_l(A),\,\kappa_l)})(\boldsymbol{Z}_l)$ in the l-adic topology, so $\rho_l(G_F)$ has a finite index in the group $C_{\mathcal{R}\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_l}(\mathrm{GSp}_{(T_l(A),\,\kappa_l)})(\boldsymbol{Z}_l)$. By the definition of $\mathcal{G}_l^{\mathrm{alg}}$, we have:

$$\rho_l(G_F) \subset \mathcal{G}_l^{\mathrm{alg}}(\boldsymbol{Z}_l) \subset C_{\mathcal{R}\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_l}(\mathrm{GSp}_{(T_l(A),\,\kappa_l)})(\boldsymbol{Z}_l)\,.$$

Hence, $\rho_l(G_F)$ is open in $\mathcal{G}_l^{\mathrm{alg}}(\boldsymbol{Z}_l)$. Note that $C_{\mathcal{R}\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_l}(\mathrm{GSp}_{(T_l(A),\kappa_l)})(\boldsymbol{Z}_l)$ is a profinite group. □

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DEPARTMENT OF MATHEMATICS
ADAM MICKIEWICZ UNIVERSITY
61614 POZNAŃ
POLAND

E-mail address: banaszak@amu.edu.pl
E-mail address: gajda@amu.edu.pl

DEPARTMENT OF MATHEMATICS
SZCZECIN UNIVERSITY
70–415 SZCZECIN
POLAND

E-mail address: krason@wmf.univ.szczecin.pl