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A NOTE ON TORIC DELIGNE-MUMFORD STACKS

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Abstract. We give a new description of the data needed to specify a morphism from a scheme to a toric Deligne-Mumford stack. The description is given in terms of a collection of line bundles and sections which satisfy certain conditions. As applications, we characterize any toric Deligne-Mumford stack as a product of roots of line bundles over the rigidified stack, describe the torus action, describe morphisms between toric Deligne-Mumford stacks with complete coarse moduli spaces in terms of homogeneous polynomials, and compare two different definitions of toric stacks.

1. Introduction. A map from a scheme Y to the projective space $\boldsymbol{P}^d$ is determined by a line bundle L on Y together with $d+1$ sections which do not vanish simultaneously. More generally, when X is a smooth toric variety a map $Y \to X$ is determined by a collection of line bundles and sections on Y which satisfy certain compatibility and nondegeneracy conditions [6]. An analogous result holds in the case of a simplicial toric variety X: there is a natural orbifold structure $\mathcal{X}$ on X such that a map $Y \to \mathcal{X}$ is determined by a collection of line bundles and sections on Y precisely as in the case of X smooth [7]. More recently, toric Deligne-Mumford stacks have been defined [4]. They are smooth separated Deligne-Mumford stacks whose coarse moduli spaces are simplicial toric varieties and such that the automorphism group of the generic point is not necessarily trivial.

The goal of the present paper is to generalize the results in [6] and [7] in order to describe morphisms, $Y \to \mathcal{X}$ where $\mathcal{X}$ is a toric Deligne-Mumford stack in the sense of [4] (Theorem 2.6). As an application of this result we deduce some geometric properties of toric Deligne-Mumford stacks. We show that, given $\mathcal{X}$, the rigidification with respect to the generic automorphism group $\mathcal{X} \to \mathcal{X}_{\mathrm{rig}}$ is isomorphic to the fibered product of roots of certain line bundles over $\mathcal{X}_{\mathrm{rig}}$ (Proposition 3.1). This result is used to obtain a classification of toric Deligne-Mumford stacks in terms of the combinatorial data $\boldsymbol{\Delta}$ (Theorem 3.3). In Section 4 we describe the torus action. In Section 5 we show how homogeneous polynomials can be used to describe all maps $\mathcal{Y} \to \mathcal{X}$ between the toric Deligne-Mumford stacks $\mathcal{Y}$ and $\mathcal{X}$ whose coarse moduli spaces are complete varieties (Theorem 5.1). Finally, we compare two different definitions of toric stacks: the one of [4] and that of [11].

The construction presented in [4] is inspired by the quotient construction for toric varieties. On the other hand, a toric variety X is a separated, normal variety containing a torus as dense open subvariety such that the multiplication of the torus extends to an action on X. This point of view inspired the work [8], where the definition of "smooth toric Deligne-Mumford

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stack" is given. Toric Deligne-Mumford stacks are smooth stacks and they are "smooth" in the sense of [8]. Throughout the paper and in the appendix we will make some remarks on the relations between the two constructions.

NOTATION. We denote by $\boldsymbol{\Delta}$ the following set of data:

(i) a free abelian group N of rank d,

(ii) a simplicial fan Δ in $N_{\boldsymbol{Q}}$ in the sense of [9], where $N_{\boldsymbol{Q}} := N \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$,

(iii) an element $a_\rho \in \rho \cap N$ for any $\rho \in \Delta(1)$, where $\Delta(1)$ is the set of 1-dimensional cones of Δ,

(iv) a sequence of R positive non zero integers $r_1, \ldots, r_R$,

(v) and an integer $b_{i\rho} \in \boldsymbol{Z}$, for any $i \in \{1, \ldots, R\}$ and $\rho \in \Delta(1)$.

We denote by M the dual lattice of N, $M := \mathrm{Hom}_{\boldsymbol{Z}}(N, \boldsymbol{Z})$, and by $\Delta_{\max}$ the set of maximal cones in Δ with respect to the inclusion. We will use $\otimes_\rho$ to denote $\otimes_{\rho \in \Delta(1)}$, $\otimes_i$ to denote $\otimes_{i=1}^R$, and similarly for $\sum_\rho$ and $\sum_i$.

We work over the field of complex numbers $\boldsymbol{C}$. The site given by the category of $\boldsymbol{C}$-schemes with the étale topology is denoted by (Sch). We write DM-stack instead of Deligne-Mumford stack. For any scheme Y, we denote by $\underline{\boldsymbol{C}}$ the trivial line bundle over it.

2. $\boldsymbol{\Delta}$-collections. Δ-collections were introduced in [6] as the data needed to specify a morphism from a scheme to a smooth toric variety (see Example 2.3 below). The generalization to the case of a simplicial fan Δ has been studied in [7]: when $R = 0$ the content of Theorem 2.6 below coincides with that of Theorem 16 in [7]. In this Section we investigate the case of morphisms to a toric DM-stack (in the sense of [4]). We define $\boldsymbol{\Delta}$-collections and morphisms between them as collections of line bundles and sections over a given scheme which satisfy certain compatibility and nondegeneracy conditions. The main result is Theorem 2.6 where we prove that the category of $\boldsymbol{\Delta}$-collections $\mathcal{C}_{\boldsymbol{\Delta}}$ is a separated smooth DM-stack and, if the 1-dimensional cones of Δ span $N_{\boldsymbol{Q}}$, is isomorphic to the toric DM-stack $\mathcal{X}_\Sigma$, where Σ is a stacky fan determined by $\boldsymbol{\Delta}$.

DEFINITION 2.1. Let Y be a scheme. A $\boldsymbol{\Delta}$-*collection* on Y is the data of

(i) a line bundle L_ρ on Y and a section $u_\rho \in H^0(Y, L_\rho)$ for any $\rho \in \Delta(1)$,

(ii) isomorphisms $c_m : \otimes_\rho L_\rho^{\otimes \langle m, a_\rho \rangle} \to \underline{\boldsymbol{C}}$ for any $m \in M$,

(iii) a line bundle M_i on Y and an isomorphism $d_i : \otimes_\rho L_\rho^{\otimes b_{i\rho}} \otimes M_i^{\otimes r_i} \to \underline{\boldsymbol{C}}$ for any $i \in \{1, \ldots, R\}$,

such that the following conditions are satisfied:

(1) $c_m \otimes c_{m'} = c_{m+m'}$ for any $m, m' \in M$,

(2) for any $y \in Y$, there exists a cone $\sigma \in \Delta_{\max}$ such that $u_\rho(y) \neq 0$ for all $\rho \notin \sigma$.

A $\boldsymbol{\Delta}$-collection on Y is written $(L_\rho, u_\rho, c_m, M_i, d_i)/Y$.

DEFINITION 2.2. Let $(L'_\rho, u'_\rho, c'_m, M'_i, d'_i)/Y'$ and $(L_\rho, u_\rho, c_m, M_i, d_i)/Y$ be two $\boldsymbol{\Delta}$-collections. A *morphism* from $(L'_\rho, u'_\rho, c'_m, M'_i, d'_i)/Y'$ to $(L_\rho, u_\rho, c_m, M_i, d_i)/Y$ is given by a morphism $f : Y' \to Y$ of schemes, morphisms $\gamma_\rho : L'_\rho \to L_\rho$ and $\delta_i : M'_i \to M_i$ of

line bundles for any $\rho \in \Delta(1)$ and any $i \in \{1, \ldots, R\}$, such that the following conditions are satisfied:

(i) the γ_ρ's induce isomorphisms $L'_\rho \to f^* L_\rho$ and the same holds for the δ_i's;

(ii) $\gamma_\rho \circ u'_\rho = u_\rho \circ f$ for all $\rho \in \Delta(1)$;

(iii) for any $m \in M$ the following diagram commutes

$$\begin{array}{ccc} \otimes_\rho L_\rho'^{\otimes\langle m, a_\rho\rangle} & \xrightarrow{c'_m} & \underline{\boldsymbol{C}} \\ {\scriptstyle \otimes_\rho \gamma_\rho^{\otimes\langle m, a_\rho\rangle}}\downarrow & & \downarrow{\scriptstyle f\times \mathrm{id}_{\boldsymbol{C}}} \\ \otimes_\rho L_\rho^{\otimes\langle m, a_\rho\rangle} & \xrightarrow{c_m} & \underline{\boldsymbol{C}} \end{array}$$

(iv) for any $i \in \{1, \ldots, R\}$ the following diagram commutes

$$\begin{array}{ccc} \otimes_\rho L_\rho'^{\otimes b_{i\rho}} \otimes M_i'^{\otimes r_i} & \xrightarrow{d'_i} & \underline{\boldsymbol{C}} \\ {\scriptstyle \otimes_\rho \gamma_\rho^{\otimes b_{i\rho}} \otimes \delta_i^{\otimes r_i}}\downarrow & & \downarrow{\scriptstyle f\times \mathrm{id}_{\boldsymbol{C}}} \\ \otimes_\rho L_\rho^{\otimes b_{i\rho}} \otimes M_i^{\otimes r_i} & \xrightarrow{d_i} & \underline{\boldsymbol{C}}. \end{array}$$

A morphism from $(L'_\rho, u'_\rho, c'_m, M'_i, d'_i)/Y'$ to $(L_\rho, u_\rho, c_m, M_i, d_i)/Y$ is denoted by $(f, \gamma_\rho, \delta_i)$.

Let us consider some examples.

EXAMPLE 2.3. Let Δ be a fan and N be a lattice which determine a smooth toric variety X. Set the a_ρs the minimal lattice points of the rays, $R = 0$. In this case $\boldsymbol{\Delta}$ is determined by Δ and N, so we talk about Δ-collections. On X there is a canonical Δ-collection defined as follows. For any $\rho \in \Delta(1)$, let D_ρ be the corresponding effective Weil divisor stable under the torus action. Since X is smooth, it corresponds to an effective Cartier divisor, hence it gives a line bundle L_ρ with a section $u_\rho \in H^0(X, L_\rho)$. For any $m \in M$, the character χ^m is a rational function on X such that $\mathrm{div}(\chi^m) = \sum_\rho \langle m, a_\rho\rangle D_\rho$, hence we get an isomorphism

$$c_{\chi^m} : \otimes_\rho L_\rho^{\otimes\langle m, a_\rho\rangle} \to \underline{\boldsymbol{C}}.$$

Then, $(L_\rho, u_\rho, c_{\chi^m})$ is a Δ-collection on X (Lemma 1.1 in [6]). This Δ-collection is called *universal* because of the following result. Let

$$C_\Delta : (\mathrm{Sch}) \to (\mathrm{Sets})$$

be the contravariant functor that associates to any scheme Y the set of equivalence classes of Δ-collections on Y. Then X is the fine moduli space for C_Δ, and $(L_\rho, u_\rho, c_{\chi^m})$ is the universal family [6].

EXAMPLE 2.4. Let $N := \boldsymbol{Z}$, $\Delta := \{\{0\}, \rho := \boldsymbol{Q}_{\geq 0}\}$, $a_\rho := a$, and $R := 0$. A $\boldsymbol{\Delta}$-collection is given by a line bundle L on Y, a section $u \in H^0(Y, L)$, and an isomorphism $c : L^{\otimes a} \to \underline{\boldsymbol{C}}$. Then the category of $\boldsymbol{\Delta}$-collections is equivalent to the stack $[\boldsymbol{A}^1_{\boldsymbol{C}}/\mu_a]$, where μ_a is the group of a-th roots of the unity acting by multiplication on $\boldsymbol{A}^1_{\boldsymbol{C}}$.

The next example is the analog of Example 3.5 in [4].

EXAMPLE 2.5. Let $N := \boldsymbol{Z}$, $\Delta := \{\{0\}, \rho := \boldsymbol{Q}_{\leq 0}, \tau := \boldsymbol{Q}_{\geq 0}\}$, $a_\rho := -3$, $a_\tau := 2$, $r = 2$, $b_\rho := 0$ and $b_\tau := 1$.
A $\boldsymbol{\Delta}$-collection over Y is given by

(i) line bundles L_ρ and L_τ over Y with sections $u_\rho \in H^0(Y, L_\rho)$ and $u_\tau \in H^0(Y, L_\tau)$ which do not vanish simultaneously,

(ii) an isomorphism $c : (L_\rho^\vee)^{\otimes 3} \otimes L_\tau^{\otimes 2} \to \underline{\boldsymbol{C}}$,

(iii) a line bundle M over Y and an isomorphism $d : L_\tau \otimes M^{\otimes 2} \to \underline{\boldsymbol{C}}$.

Let $\boldsymbol{P}(3, 2)$ be the weighted projective line $[\boldsymbol{C}^2 - \{0\}/\boldsymbol{C}^*]$, where $\boldsymbol{C}^*$ acts with weights 2, 3. Let $\mathcal{O}(1)$ be the line bundle on $\boldsymbol{P}(3, 2)$ associated to the character $\mathrm{id}_{\boldsymbol{C}^*}$ as in [5]. We define a functor from the category of $\boldsymbol{\Delta}$-collections to $\sqrt{\mathcal{O}(1)/\boldsymbol{P}(3, 2)}$ (in Section 3 we will review the definition of roots of line bundles).

For any $\boldsymbol{\Delta}$-collection $(L_\rho, L_\tau, u_\rho, u_\tau, M, d)/Y$, we define a morphism $Y \to \sqrt{\mathcal{O}(1)/\boldsymbol{P}(3, 2)}$ as follows: the morphism $Y \to \boldsymbol{P}(3, 2)$ is determined by the line bundle $L := L_\rho^\vee \otimes L_\tau$ and the sections $u_\rho \in H^0(Y, L^{\otimes 2})$ and $u_\tau \in H^0(Y, L^{\otimes 3})$, where we have used c to identify $L^{\otimes 2}$ with L_ρ and $L^{\otimes 3}$ with L_τ; the square root of $\mathcal{O}(1)$ is $(M \otimes (L_\rho^\vee \otimes L_\tau), d)$. The correspondence between arrows is defined in an analogous way. Then the resulting functor is an equivalence of categories.

The main result of the present paper (Theorem 2.6) states that, in general, there is a correspondence between combinatorial data $\boldsymbol{\Delta}$ and toric DM-stacks. Let $\mathcal{C}_{\boldsymbol{\Delta}}$ be the category whose objects are $\boldsymbol{\Delta}$-collections and morphisms are morphisms between $\boldsymbol{\Delta}$-collections. The functor

$$p : \mathcal{C}_{\boldsymbol{\Delta}} \to (\mathrm{Sch})$$

which sends the $\boldsymbol{\Delta}$-collection $(L_\rho, u_\rho, c_m, M_i, d_i)/Y$ to Y and the morphism $(f, \gamma_\rho, \delta_i)$ to f makes $\mathcal{C}_{\boldsymbol{\Delta}}$ a category fibered in groupoids (a CFG) over the site (Sch).

THEOREM 2.6. *The category fibered in groupoids $p : \mathcal{C}_{\boldsymbol{\Delta}} \to (\mathrm{Sch})$ is a separated smooth Deligne-Mumford stack whose coarse moduli space is the toric variety associated to the fan Δ and the lattice N.*

Moreover, if the 1-dimensional cones of Δ span $N_{\boldsymbol{Q}}$, set

$$\Sigma := (N \oplus_{i=1}^R \boldsymbol{Z}/r_i, \Delta, \{(a_\rho, \overline{b_{1\rho}}, \ldots, \overline{b_{R\rho}})\}_\rho),$$

where $\overline{b_{i\rho}}$ denotes the class of $b_{i\rho}$ in $\boldsymbol{Z}/r_i$. Then $\mathcal{C}_{\boldsymbol{\Delta}}$ is isomorphic to the toric Deligne-Mumford stack $\mathcal{X}_\Sigma$ associated to the stacky fan Σ as defined in [4].

We notice that one can define $\boldsymbol{\Delta}$-collections over a stack formally in the same way as for schemes. Moreover, on $\mathcal{C}_{\boldsymbol{\Delta}}$ there is a tautological $\boldsymbol{\Delta}$-collection:

$$(\mathfrak{L}_\rho, \mathfrak{u}_\rho, \mathfrak{c}_m, \mathfrak{M}_i, \mathfrak{d}_i)/\mathcal{C}_{\boldsymbol{\Delta}}.$$

As a result the theory of descent gives the following

COROLLARY 2.7. *Let $\mathcal{Y}$ be a DM-stack. Then the category of morphisms $\mathcal{Y} \to \mathcal{C}_{\boldsymbol{\Delta}}$ is equivalent to the category of $\boldsymbol{\Delta}$-collections over $\mathcal{Y}$.*

Before to proceed with the proof of the theorem, let me remark that the stacks $\mathcal{C}_{\boldsymbol{\Delta}}$ defined in this paper are exactly the "smooth toric DM stacks" introduced in [8] (see the appendix, Theorem 7.2).

We collect below some general results we need in order to prove the above Theorem.

NOTATION 2.8. Following the notations used in [3], for any quasi-affine group scheme G, we denote by

$$p_G : BG \to (\mathrm{Sch})$$

the structure morphism of the stack BG, and by

$$\pi_G : \mathrm{Spec}\boldsymbol{C} \to BG$$

the covering. We denote by

$$b_G : \mathrm{pre}BG \to BG$$

the stackification morphism from the pre-stack preBG to BG. For any morphism $\varphi : G \to H$ of group schemes, we denote by

$$\mathrm{pre}B\varphi : \mathrm{pre}BG \to \mathrm{pre}BH$$

the induced morphism of pre-stacks.

We quote a remark from [3].

REMARK 2.9. Let $\varphi : G \to H$ be a morphism of quasi-affine group schemes. Let $P \to Y$ be a principal G-bundle. Consider the right G-action on $P \times H$:

$$(x, h) \cdot g = (x \cdot g, \varphi(g^{-1}) \cdot h)\,. \tag{1}$$

The theory of descent guarantees that there exists a quotient scheme for the previous action, $(P \times H)/G$, together with a morphism $(P \times H)/G \to Y$ inducing a principal H-bundle structure. The scheme $(P \times H)/G$ will be denoted either as $P \times_\varphi H$ or by $P \times_G H$. Notice that $P \times_\varphi H$ is not unique and due to this ambiguity the correspondence $P \mapsto P \times_\varphi H$ defines a functor $B\varphi : BG \to BH$ up to unique canonical 2-isomorphism. In the following this ambiguity will be understood.

An analogous notation will be used to denote the quotients $P \times_G X$ in the case where X is a quasi-affine scheme with a right G-action.

We now recall two well known results in order to be self-contained as much as possible. For complex algebraic varieties the reference is [16]. For a more general context we refer to [10, Ch. III, Proposition 3.2.1.]

LEMMA 2.10. *Let*

$$1 \to G \xrightarrow{\varphi} H \xrightarrow{\psi} K \to 1$$

be a short exact sequence of quasi-affine group schemes. Then

$$(B\varphi, p_G) : BG \to BH \;{}_{B\psi}\times_{\pi_K} \mathrm{Spec}\boldsymbol{C} \tag{2}$$

is an isomorphism of stacks.

LEMMA 2.11. *Let G be a quasi-affine group scheme and let X be a quasi-affine scheme with a right G-action. For any principal G-bundle $\pi : P \to Y$, there is a bijection*

$$\{\text{sections of } P \times_G X \to Y\} \leftrightarrow \{G\text{-equivariant morphisms } t : P \to X\} \tag{3}$$

which is functorial with respect to base change.

PROOF (of Theorem 2.6). We proceed as follows: we first prove that if $\{\rho \in \Delta(1)\}$ spans $N_{\boldsymbol{Q}}$ then $\mathcal{C}_{\boldsymbol{\Delta}}$ is isomorphic to $\mathcal{X}_\Sigma$ as CFG, where $\mathcal{X}_\Sigma$ is the stack defined in the statement of the Theorem, now the result follows from the fact that $\mathcal{X}_\Sigma$ is a separated smooth DM-stack [4]; we proceed afterwards to prove the statement in general.

Identify the lattice N with $\boldsymbol{Z}^d$ and enumerate the 1-dimensional cones of Δ as $\rho_1, \dots, \rho_n$. Then the $a_\rho \in N$ correspond to $(a_{1k}, \dots, a_{dk}) \in \boldsymbol{Z}^d$, $k \in \{1, \dots, n\}$. Let us define the matrices

$$B := \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{d1} & \cdots & a_{dn} \\ b_{11} & \cdots & b_{1n} \\ \vdots & \cdots & \vdots \\ b_{R1} & \cdots & b_{Rn} \end{pmatrix}; \quad Q := \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ r_1 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & r_R \end{pmatrix},$$

where $B \in \mathrm{Mat}((d+R) \times n, \boldsymbol{Z})$, $Q \in \mathrm{Mat}((d+R) \times R, \boldsymbol{Z})$. We consider the exact sequence

$$0 \to (\boldsymbol{Z}^{d+R})^* \xrightarrow{[BQ]^*} (\boldsymbol{Z}^{n+R})^* \to \mathrm{coker}([BQ]^*) \to 0\,,$$

where $[BQ] \in \mathrm{Mat}((d+R) \times (n+R), \boldsymbol{Z})$, and we apply the functor $\mathrm{Hom}_{\boldsymbol{Z}}(\_, \boldsymbol{C}^*)$, we get an exact sequence of affine group schemes:

$$1 \to G \xrightarrow{\varphi} (\boldsymbol{C}^*)^n \times (\boldsymbol{C}^*)^R \xrightarrow{\psi} (\boldsymbol{C}^*)^d \times (\boldsymbol{C}^*)^R \to 1 \tag{4}$$

where

$$\begin{aligned} &\psi(\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_R) \\ &\quad = (\lambda_1^{a_{11}} \cdots \lambda_n^{a_{1n}}, \dots, \lambda_1^{a_{d1}} \cdots \lambda_n^{a_{dn}}, \mu_1^{r_1} \cdot \lambda_1^{b_{11}} \cdots \lambda_n^{b_{1n}}, \dots, \mu_R^{r_R} \cdot \lambda_1^{b_{R1}} \cdots \lambda_n^{b_{Rn}})\,. \end{aligned} \tag{5}$$

The matrix Q defines a morphism $\boldsymbol{Z}^R \to \boldsymbol{Z}^{n+R}$ which is a projective resolution of $N \oplus_i \boldsymbol{Z}/r_i$, and B defines a lifting $\boldsymbol{Z}^n \to \boldsymbol{Z}^{d+R}$ of the morphism $\boldsymbol{Z}^n \to N \oplus_i \boldsymbol{Z}/r_i$, $e_i \mapsto (a_{\rho_i}, \overline{b_{1\rho_i}}, \dots, \overline{b_{R\rho_i}})$, where e_i is the i-th element of the standard basis of $\boldsymbol{Z}^n$. Following [4] we associate to this data a toric DM-stack $\mathcal{X}_\Sigma := [Z/G]$.

We now define a functor of CFGs over (Sch)

$$F : \mathcal{X}_\Sigma \to \mathcal{C}_{\boldsymbol{\Delta}}\,. \tag{6}$$

Consider an object of $\mathcal{X}_\Sigma(Y)$,

$$\begin{array}{ccc} P & \xrightarrow{t} & Z \\ {\scriptstyle\pi}\big\downarrow & & \\ Y & & \end{array}$$

Set

$$L_k := B\varphi(P) \times_{(\boldsymbol{C}^*)^n \times (\boldsymbol{C}^*)^R} \boldsymbol{C},$$

the associate line bundle with respect to the action

$$\boldsymbol{C} \times ((\boldsymbol{C}^*)^n \times (\boldsymbol{C}^*)^R) \to \boldsymbol{C}$$
$$z \cdot (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_R) \mapsto z \cdot \lambda_k\,,$$

where φ is defined in (4), and $k \in \{1, \ldots, n\}$. In the same way, for any $i \in \{1, \ldots, R\}$, set

$$M_i := B\varphi(P) \times_{(\boldsymbol{C}^*)^n \times (\boldsymbol{C}^*)^R} \boldsymbol{C}\,,$$

be the line bundle associated to the action

$$\boldsymbol{C} \times ((\boldsymbol{C}^*)^n \times (\boldsymbol{C}^*)^R) \to \boldsymbol{C}$$
$$z \cdot (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_R) \mapsto z \cdot \mu_i\,.$$

We now define the sections of $L_1, \ldots, L_n$. Recall that G acts on $\boldsymbol{C}^n$ by means of the $(\boldsymbol{C}^*)^n$-component of φ. Then the composition of $t : P \to Z$ with the inclusion $Z \to \boldsymbol{C}^n$ gives an equivariant morphism $\tilde{t} : P \to \boldsymbol{C}^n$. By Lemma 2.11 we get a section u of the vector bundle

$$P \times_G \boldsymbol{C}^n = L_1 \oplus \cdots \oplus L_n\,, \tag{7}$$

then u_k is the k-th component of u with respect to the decomposition (7). Condition (2) of Definition 2.1 follows from the fact that $\tilde{t}(P) \subset Z$ and from the definition of Z as in [4].

We next define the isomorphisms c_m required in the definition of $\boldsymbol{\Delta}$-collection. By applying the functor $(B\varphi, p_G)$ defined in Lemma 2.10, we get

$$(B\varphi, p_G)(P) = (B\varphi(P), \alpha, Y \times ((\boldsymbol{C}^*)^d \times (\boldsymbol{C}^*)^R)) \tag{8}$$

where $\alpha : B\psi(B\varphi(P)) \to Y \times ((\boldsymbol{C}^*)^d \times (\boldsymbol{C}^*)^R)$ is an isomorphism of principal $((\boldsymbol{C}^*)^d \times (\boldsymbol{C}^*)^R)$-bundles. Consider now the diagonal action of $((\boldsymbol{C}^*)^d \times (\boldsymbol{C}^*)^R)$ on $\boldsymbol{C}^{d+R}$ with weights 1. Then from (5) it follows that the associated vector bundle

$$B\psi(B\varphi(P)) \times_{(\boldsymbol{C}^*)^d \times (\boldsymbol{C}^*)^R} \boldsymbol{C}^{d+R}$$

is canonically isomorphic to

$$\oplus_{l=1}^d (\otimes_{k=1}^n L_k^{\otimes a_{lk}}) \oplus_i^R (M^{\otimes r_i} \otimes_{k=1}^n L^{\otimes b_{ik}})\,. \tag{9}$$

The isomorphism α in (8) gives an isomorphism between (9) and the trivial vector bundle, then we get isomorphisms $c_{e_1^*}, \ldots, c_{e_d^*}, d_1, \ldots, d_R$, where $e_1^*, \ldots, e_d^*$ is the dual basis of the standard basis $e_1, \ldots, e_d$ of $\boldsymbol{Z}^d$. The c_m's are now uniquely determined by condition (1) of Definition 2.1.

We have defined F on objects. The definition on morphisms and the verification of the fact that it is a functor is straightforward. Moreover, F is an equivalence of categories. This follows from Lemmas 2.10 and 2.11, and from the equivalence between the category of principal $\boldsymbol{C}^*$-bundles and the one of line bundles. The result now follows since $\mathcal{X}_\Sigma$ is a separated smooth DM-stack whose coarse moduli space is the toric variety associated to the fan Δ and the lattice N (Proposition 3.2 and Proposition 3.7 of [4]).

We next consider the case where $\{\rho \in \Delta(1)\}$ does not span $N_{\boldsymbol{Q}}$. We follow the ideas used in the proof of Theorem 1.1 in [6]. Set

$$N' := \mathrm{Span}(\{\rho \in \Delta(1)\}) \cap N\,.$$

The fan Δ can be regarded as a fan in $N'_{\boldsymbol{Q}}$, then set

$$\boldsymbol{\Delta}' = \{N', \Delta, \{a_\rho\}, r_1, \dots, r_R, \{b_{i\rho}\}\}\,.$$

From the first part of the proof we have that $\mathcal{C}_{\Delta'}$ is a separated smooth DM-stack.

N/N' is torsion free, so we can find a subgroup N'' of N such that $N = N' \oplus N''$. The projection $N \to N'$ determines an inclusion $\iota : M' \to M$ such that $M = \iota(M') \oplus N'^{\perp}$.

Let now $(L_\rho, u_\rho, c_m, M_i, d_i)/Y$ be a $\boldsymbol{\Delta}$-collection. For any $m \in N'^{\perp}$, we have $\langle m, a_\rho\rangle = 0$ for all ρ. Thus c_m can be identified with an element in $H^0(Y, \mathcal{O}_Y^*)$. Under this identification, the application $N'^{\perp} \to H^0(Y, \mathcal{O}_Y^*)$, $m \mapsto c_m$, is a group homomorphism, thus induces a morphism of schemes $Y \to \mathrm{Spec}(\boldsymbol{C}[N'^{\perp}])$. In this way we get a morphism

$$\mathcal{C}_{\boldsymbol{\Delta}} \to \mathrm{Spec}(\boldsymbol{C}[N'^{\perp}])\,. \tag{10}$$

On the other hand there is a morphism

$$\mathcal{C}_{\boldsymbol{\Delta}} \to \mathcal{C}_{\boldsymbol{\Delta}'} \tag{11}$$

which associates $(L_\rho, u_\rho, c_m, M_i, d_i)/Y$ to $(L_\rho, u_\rho, \{c_m \,|\, m \in \iota(M')\}, M_i, d_i)/Y$. Then the morphism $\mathcal{C}_{\boldsymbol{\Delta}} \to \mathcal{C}_{\boldsymbol{\Delta}'} \times \mathrm{Spec}(\boldsymbol{C}[N'^{\perp}])$ whose components are (10) and (11) is an isomorphism. The result now follows since $\mathcal{C}_{\boldsymbol{\Delta}'} \times \mathrm{Spec}(\boldsymbol{C}[N'^{\perp}])$ is a smooth separated DM-stack and its coarse moduli space is the toric variety associated to Δ and N. □

3. Classification of toric DM-stacks. In this Section we show that the stack $\mathcal{C}_{\boldsymbol{\Delta}}$ can be viewed as a gerbe banded by a finite abelian group. Then we study the problem of whether two combinatorial data $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}'$ define isomorphic banded gerbes. We give an answer in combinatorial terms.

Let $\boldsymbol{\Delta}$ be a combinatorial data as in the Introduction, set

$$\boldsymbol{\Delta}_{\mathrm{rig}} := \{N,\ \Delta,\ a_\rho | \rho \in \Delta(1)\}\,.$$

Consider the line bundles $\mathfrak{V}_1, \dots, \mathfrak{V}_R$ on $\mathcal{C}_{\boldsymbol{\Delta}_{\mathrm{rig}}}$ defined as follows: for any $\boldsymbol{\Delta}_{\mathrm{rig}}$-collection $(L_\rho, u_\rho, c_m)/Y$, set

$$\mathfrak{V}_i(Y) := \otimes_\rho L_\rho^{\otimes b_{i\rho}}\,, \quad i \in \{1, \dots, R\}\,;$$

for any morphism $(f, \gamma_\rho) : (L'_\rho, u'_\rho, c'_m)/Y' \to (L_\rho, u_\rho, c_m)/Y$, set

$$\mathfrak{V}_i(f, \gamma_\rho) := \otimes_\rho \gamma_\rho^{\otimes b_{i\rho}}\,.$$

Let us denote by $\sqrt[r_i]{\mathfrak{V}_i}$ the gerbe of r_i-th roots of $\mathfrak{V}_i$, for $i \in \{1, \dots, R\}$. Just to fix notation we recall its definition here and refer to [10, Ch. IV (2.5.8.1)] and [2] for the general definition and for more details. For a scheme Y over $\mathcal{C}_{\boldsymbol{\Delta}_{\mathrm{rig}}}$, an object of $\sqrt[r_i]{\mathfrak{V}_i}(Y)$ is a pair (M, d), where M is a line bundle on Y and $d : M^{\otimes r_i} \otimes \mathfrak{V}_i(Y) \to \underline{\boldsymbol{C}}$ is an isomorphism. If (M', d') is an object over $Y' \to \mathcal{C}_{\boldsymbol{\Delta}_{\mathrm{rig}}}$, then a morphism $(M', d') \to (M, d)$ over (f, γ_ρ) is given by a morphism of line bundles $\delta : M' \to M$ such that the following diagram is cartesian

$$\begin{array}{ccc} M' & \xrightarrow{\delta} & M \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{f} & Y \end{array}$$

and $d \circ \delta^{\otimes r_i} \otimes_\rho \gamma_\rho^{\otimes b_{i\rho}} = d'$.

Consider now the morphism of stacks

$$\mathcal{R} : \mathcal{C}_{\boldsymbol{\Delta}} \to \mathcal{C}_{\boldsymbol{\Delta}_{\mathrm{rig}}} \tag{12}$$

which associates to the $\boldsymbol{\Delta}$-collection $(L_\rho, u_\rho, c_m, M_i, d_i)/Y$ the $\boldsymbol{\Delta}_{\mathrm{rig}}$-collection $(L_\rho, u_\rho, c_m)/Y$, and to $(f, \gamma_\rho, \delta_i)$ the arrow (f, γ_ρ). Then we have the following.[1]

PROPOSITION 3.1. *The morphism* (12) *is a gerbe banded by* $\times_{i=1}^R \mu_{r_i}$ *isomorphic to*

$$\sqrt[r_i]{\mathfrak{V}_1} \times_{\mathcal{C}_{\boldsymbol{\Delta}_{\mathrm{rig}}}} \times \cdots \times_{\mathcal{C}_{\boldsymbol{\Delta}_{\mathrm{rig}}}} \sqrt[r_R]{\mathfrak{V}_R}\,. \tag{13}$$

PROOF. It is straightforward to verify that $\mathcal{R}$ is a gerbe. Let now $(L_\rho, u_\rho, c_m, M_i, d_i)/Y$ be a $\boldsymbol{\Delta}$-collection. The set of its automorphisms over the identity of $(L_\rho, u_\rho, c_m)/Y$ is

$$\{(\delta_1, \dots, \delta_R) \in H^0(Y, \mathcal{O}_Y^*)^R \mid \delta_i^{r_i} = 1 \quad \text{for any } i \in \{1, \dots, R\}\}\,,$$

then the canonical inclusions $\mu_{r_i} \subset \boldsymbol{C}^*$, for $i \in \{1, \dots, R\}$, give the $\mu_{r_1} \times \cdots \times \mu_{r_R}$-banding.

Isomorphism classes of gerbes banded by $\mu_{r_1} \times \cdots \times \mu_{r_R}$ are in 1-to-1 correspondence with the second cohomology group $H^2(\mathcal{C}_{\boldsymbol{\Delta}_{\mathrm{rig}}}, \times_{i=1}^R \mu_{r_i})$ [10]. There is an isomorphism

$$H^2(\mathcal{C}_{\boldsymbol{\Delta}_{\mathrm{rig}}}, \times_{i=1}^R \mu_{r_i}) \to \times_{i=1}^R H^2(\mathcal{C}_{\boldsymbol{\Delta}_{\mathrm{rig}}}, \mu_{r_i}) \tag{14}$$

whose inverse associates the classes of the gerbes $\mathcal{G}_1, \dots, \mathcal{G}_R$ to the class of the fibered product $\mathcal{G}_1 \times_{\mathcal{C}_{\boldsymbol{\Delta}_{\mathrm{rig}}}} \cdots \times_{\mathcal{C}_{\boldsymbol{\Delta}_{\mathrm{rig}}}} \mathcal{G}_R$. The explicit description of (14) (as given for example in [10] IV Proposition 2.3.18) shows that the image of the class of $\mathcal{R}$ is the class of (13). This complete the proof. □

REMARK 3.2. We can also see $\mathcal{R}$ as the rigidification of $\mathcal{C}_{\boldsymbol{\Delta}}$ with respect to the constant sheaf $\times_{i=1}^R \mu_{r_i}$ [1] (this accounts for the subscript rig in the label $\boldsymbol{\Delta}_{\mathrm{rig}}$).

THEOREM 3.3. *For any toric DM-stack* $\mathcal{X}$, *there exists a combinatorial data* $\boldsymbol{\Delta} = \{N, \Delta, a_\rho, r_i, b_{i\rho}\}$ *which satisfies the condition*

$$r_1 \mid r_2 \mid \cdots \mid r_R \tag{15}$$

[1] For a different proof of Proposition 3.1, see [13] and [8].

such that $\mathcal{X} \cong \mathcal{C}_{\boldsymbol{\Delta}}$ *(here* $r_i \mid r_{i+1}$ *denotes that* r_i *divides* r_{i+1}*).*

Furthermore, if $\boldsymbol{\Delta}' = \{N, \Delta, a_\rho, r_i', b_{i\rho}'\}$ *is another combinatorial data which satisfies* (15)*, then* $\mathcal{C}_{\boldsymbol{\Delta}} \cong \mathcal{C}_{\boldsymbol{\Delta}'}$ *as banded gerbes if and only if*

$$R = R', \quad r_i = r_i' \quad \textit{for all } i,$$

and the class

$$\left[\sum_\rho (b_{i\rho} - b_{i\rho}') e_\rho^*\right] \in (\mathbf{Z}^{\Delta(1)})^*/M$$

is divisible by r_i*, for any* $i \in \{1, \dots, R\}$*. Here* M *is embedded in* $(\mathbf{Z}^{\Delta(1)})^*$ *by the dual of the morphism* $\mathbf{Z}^{\Delta(1)} \to N$, $e_\rho \mapsto a_\rho$.

PROOF. Given $\mathcal{X}$, the existence of $\boldsymbol{\Delta}$ satisfying (15) follows from Theorem 2.6, Proposition 3.1, and the classification of finite abelian groups. The condition (15) determines the isomorphism class of the generic automorphism group of $\mathcal{C}_{\boldsymbol{\Delta}}$. Hence $\mathcal{C}_{\boldsymbol{\Delta}} \cong \mathcal{C}_{\boldsymbol{\Delta}'}$ implies that $R = R'$ and $r_i = r_i'$ for all i. The same argument in the proof of Proposition 3.1 shows that $\mathcal{C}_{\boldsymbol{\Delta}} \cong \mathcal{C}_{\boldsymbol{\Delta}'}$ as $\times_i \mu_{r_i}$-gerbes if and only if $\sqrt[r_i]{\mathfrak{V}_i} \cong \sqrt[r_i]{\mathfrak{V}_i'}$ as μ_{r_i}-gerbe for any $i \in \{1, \dots, R\}$. This is equivalent to the fact that $\mathfrak{V}_i \otimes \mathfrak{V}_i'^{-1}$ is a r_i-th power of some element in $\mathrm{Pic}(\mathcal{C}_{\boldsymbol{\Delta}_{\mathrm{rig}}})$. Now the result follows from the fact that $\mathrm{Pic}(\mathcal{C}_{\boldsymbol{\Delta}_{\mathrm{rig}}})$ has the following presentation [5], [7]:

$$0 \to M \to (\mathbf{Z}^{\Delta(1)})^* \to \mathrm{Pic}(\mathcal{C}_{\boldsymbol{\Delta}_{\mathrm{rig}}}) \to 0,$$

where the morphism $M \to (\mathbf{Z}^{\Delta(1)})^*$ is the dual of $\mathbf{Z}^{\Delta(1)} \to N$, $e_\rho \mapsto a_\rho$. □

REMARK 3.4. Notice that the first part of Theorem 3.3 holds in the more general situation where $\mathcal{X}$ is a "smooth toric DM stack" in the sense of [8] (see the Appendix).

4. The torus action. Any toric variety X contains an algebraic torus T as open dense subvariety such that the action of T on itself by multiplication extends to an action on X. In this Section we show that an analogous property holds for a toric DM-stack once replacing T with a *Picard stack* $\mathcal{T}$. Picard stacks (originally called *champs de Picard*) were defined in Exposé XVIII [17]; we refer to this paper for the definition and further properties. We will denote by T the torus $\mathrm{Spec}(\boldsymbol{C}[M])$, then any morphism $g : Y \to T$ is identified with the corresponding group homomorphism $M \to H^0(Y, \mathcal{O}_Y^*)$, $m \mapsto g_m := g^*(\chi^m)$.

Let us consider

$$\mathcal{T} := \sqrt[r_1]{\underline{\boldsymbol{C}}} \times_T \cdots \times_T \sqrt[r_R]{\underline{\boldsymbol{C}}},$$

and introduce the map

$$\mathfrak{m} : \mathcal{T} \times_{(\mathrm{Sch})} \mathcal{T} \to \mathcal{T} \tag{16}$$

defined on objects by

$$\mathfrak{m}((g, N_i, e_i)/Y, (g', N_i', e_i')/Y) := (g \cdot g', N_i \otimes N_i', e_i \otimes e_i')/Y,$$

and on arrows by $\mathfrak{m}(\varepsilon_i, \varepsilon_i') := \varepsilon_i \otimes \varepsilon_i'$. The *associativity* is expressed in terms of a natural transformation

$$\sigma : \mathfrak{m} \circ (\mathfrak{m} \times \mathrm{id}_{\mathcal{T}}) \Rightarrow \mathfrak{m} \circ (\mathrm{id}_{\mathcal{T}} \times \mathfrak{m}) ,$$

and the *commutativity* with a natural transformation

$$\tau : \mathfrak{m} \Rightarrow \mathfrak{m} \circ C ,$$

where C is the functor that exchanges the factors. Here we choose the standard natural transformations σ and τ. Then $(\mathcal{T}, \mathfrak{m}, \sigma, \tau)$ is a Picard stack. Let us denote with e the neutral element of $\mathcal{T}$, which is unique up to unique isomorphism.

The action of a group on a stack has been defined in [15]. The extension to the case of a group stack is straightforward, the only difference here is that the action must be compatible with the associativity of $\mathcal{T}$ which is expressed in terms of σ. We follow the approach of [15], for a different one we refer to [8].

The Picard stack $\mathcal{T}$ acts on $\mathcal{C}_{\boldsymbol{\Delta}}$. The action is given by the functor

$$\mathfrak{a} : \mathcal{C}_{\boldsymbol{\Delta}} \times_{(\mathrm{Sch})} \mathcal{T} \to \mathcal{C}_{\boldsymbol{\Delta}}$$

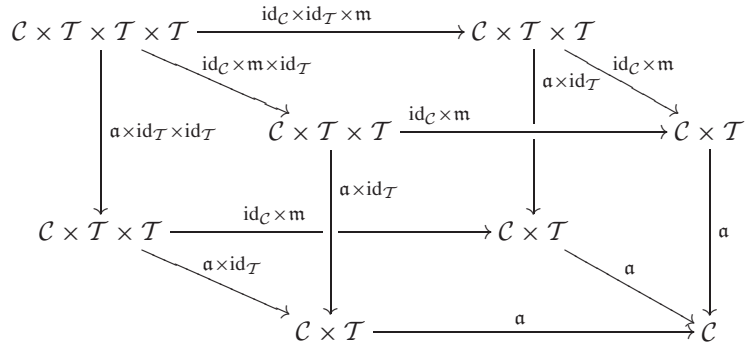
which is defined on objects as

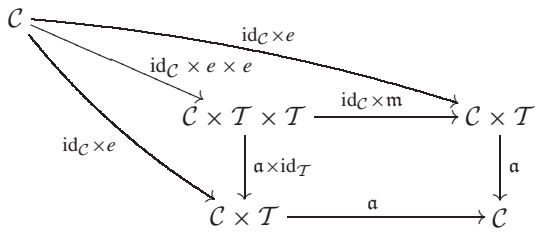
$$\mathfrak{a}((L_\rho, u_\rho, c_m, M_i, d_i), (g_m, N_i, e_i))/Y := (L_\rho, u_\rho, c_m \cdot g_m, M_i \otimes N_i, d_i \otimes e_i)/Y ,$$

and on arrows as

$$\mathfrak{a}((f, \gamma_\rho, \delta_i), \varepsilon_i) := (f, \gamma_\rho, \delta_i \otimes \varepsilon_i) ,$$

and natural transformations $\alpha : \mathfrak{a} \circ (\mathrm{id}_{\mathcal{C}_{\boldsymbol{\Delta}}} \times \mathfrak{m}) \Rightarrow \mathfrak{a} \circ (\mathfrak{a} \times \mathrm{id}_{\mathcal{T}})$ and $\beta : \mathrm{id}_{\mathcal{C}_{\boldsymbol{\Delta}}} \Rightarrow \mathfrak{a} \circ (\mathrm{id}_{\mathcal{C}_{\boldsymbol{\Delta}}} \times e)$ such that the following diagrams are 2-commutative (we put in each square (resp. triangle) the appropriate natural transformation):





where $\mathcal{C}$ denotes $\mathcal{C}_{\boldsymbol{\Delta}}$. There is a standard choice for α and β which satisfy these conditions, we adopt this one.

PROPOSITION 4.1. *There is a morphism*

$$\mathcal{T} \to \mathcal{C}_{\boldsymbol{\Delta}}$$

whose image is open and dense with respect to the small étale site. The restriction of $\mathfrak{a}$ *to* $\mathcal{T}$ *with respect to the above morphism is isomorphic to* $\mathfrak{m}$.

PROOF. Let $Y \to \mathcal{T}$ be a morphism given by $g : Y \to T$, N_i and $e_i : N_i^{\otimes r_i} \to \underline{\boldsymbol{C}}$, for $i \in \{1, \dots, R\}$. Set $L_\rho := \underline{\boldsymbol{C}}$ and $u_\rho := 1$ for all ρ, $c_m := g_m$, $m \in M$. Then $(L_\rho, u_\rho, c_m, N_i, e_i)$ is a $\boldsymbol{\Delta}$-collection. This defines the morphism on objects, on arrows it sends (f, ε_i) to $(f, \mathrm{id}, \varepsilon_i)$.

The morphism is open with respect to the small étale site. Indeed, let $\pi : U \to \mathcal{C}_{\boldsymbol{\Delta}}$ be an étale cover. It corresponds to a $\boldsymbol{\Delta}$-family $(L_\rho, u_\rho, c_m, M_i, d_i)/U$. Set $U' := \{x \in U \,|\, u_\rho(x) \neq 0 \text{ for all } \rho\}$. The restriction of $(L_\rho, u_\rho, c_m, M_i, d_i)/U$ to U' gives a morphism $U' \to \mathcal{T}$ such that the following diagram is 2-Cartesian

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathcal{T} & \longrightarrow & \mathcal{C}_{\boldsymbol{\Delta}}\,. \end{array}$$

The compatibility between $\mathfrak{a}$ and $\mathfrak{m}$ follows directly from the definition. This completes the proof. □

We proceed by observing that using the definition of [4] it is possible to give another description of the torus action. Indeed let $\phi : G \to (\boldsymbol{C}^*)^n$ be the composition of φ in (4) with the projection to $(\boldsymbol{C}^*)^n$. Recall that $\mathcal{X}_\Sigma = [Z/G]$, where the G-action is induced by ϕ. Let us consider the Picard stack $\mathcal{G} := [(\boldsymbol{C}^*)^n/G]$ (see 1.4.11 of Exposé XVIII in [17]). Let $\mathcal{G}^{\mathrm{pre}}$ and $\mathcal{X}_\Sigma^{\mathrm{pre}}$ be the pre-stacks associated to the groupoids $((\boldsymbol{C}^*)^n \times G \rightrightarrows (\boldsymbol{C}^*)^n)$ and $(Z \times G \rightrightarrows Z)$ respectively. $\mathcal{G}^{\mathrm{pre}}$ acts on $\mathcal{X}_\Sigma^{\mathrm{pre}}$ in the obvious way and the stackification of this action gives an action of $\mathcal{G}$ on $\mathcal{X}_\Sigma$. We denote by $\mathfrak{a}_{\mathcal{X}}$ this action.

We want to compare the torus actions previously defined on $\mathcal{X}_\Sigma$ and on $\mathcal{C}_{\boldsymbol{\Delta}}$. Note that the restriction of (6) to $\mathcal{G}$ gives an isomorphism

$$F_{|\mathcal{G}} : \mathcal{G} \to \mathcal{T}\,. \tag{17}$$

Moreover there is a natural transformation

$$\nu : \mathfrak{a} \circ (F \times F_{|\mathcal{G}}) \Rightarrow F \circ \mathfrak{a}_{\mathcal{X}} \tag{18}$$

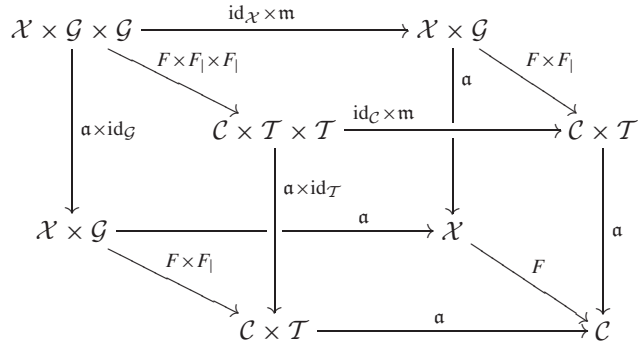
defined as follows. Consider the diagram

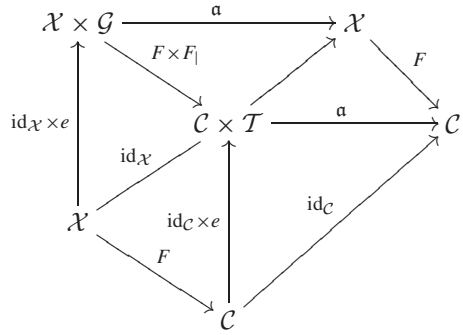
$$\begin{array}{ccc} \mathcal{X}^{\mathrm{pre}} \times \mathcal{G}^{\mathrm{pre}} & \longrightarrow & \mathcal{C}_{\boldsymbol{\Delta}} \times \mathcal{T} \\ {\scriptstyle \mathfrak{a}_{\mathcal{X}^{\mathrm{pre}}}}\downarrow & & \downarrow{\scriptstyle \mathfrak{a}} \\ \mathcal{X}^{\mathrm{pre}} & \longrightarrow & \mathcal{C}_{\boldsymbol{\Delta}} \end{array} \tag{19}$$

where each row is the composition of $(F \times F_{|\mathcal{G}})$ (F resp.) with the corresponding stackification morphism. There is a canonical natural transformation ν^{pre} which makes (19) 2-commutative. Then we define ν in (18) to be the unique natural transformation induced by ν^{pre}. Finally we have the following.

PROPOSITION 4.2. *The isomorphism* (6) *together with* ν *is* $F_{|\mathcal{G}}$*-equivariant.*

PROOF. Following [15], we have to prove that the diagrams below are 2-commutative with respect to the natural transformations previously defined:





With abuse of notation, we have denoted with the same $\mathfrak{m}$ the two multiplications of the Picard stacks and with the same $\mathfrak{a}$ the two actions, $\mathcal{C}$ denotes $\mathcal{C}_{\boldsymbol{\Delta}}$. Notice that it is enough to prove the 2-commutativity of the above diagrams with $\mathcal{X}$ and $\mathcal{G}$ being replaced by $\mathcal{X}^{\mathrm{pre}}$ and $\mathcal{G}^{\mathrm{pre}}$

respectively. A direct computation shows that with these replacements the above diagrams are indeed 1-commutative, hence the result follows. □

5. Morphisms between toric stacks. In this Section we give a description of morphisms between toric DM-stacks parallel to the one given in Theorem 3.2 of [6] in the context of toric varieties. We need some introductory notations. Let $\mathcal{Y} := \mathcal{C}_{\boldsymbol{\Delta}'}$ be the toric DM-stack defined by $\boldsymbol{\Delta}' := \{N', \Delta', a'_\rho\}$ (notice that here we set $R' = 0$). We assume that the coarse moduli space Y of $\mathcal{Y}$ is a complete variety. Let $\mathcal{X} := \mathcal{C}_{\boldsymbol{\Delta}}$ be a toric DM-stack such that the 1-dimensional rays generate $N_{\boldsymbol{Q}}$. Let us fix the presentations $\mathcal{Y} = [Z'/G']$ and $\mathcal{X} = [Z/G]$ as in [4].

The Picard group of $\mathcal{Y}$ is isomorphic to the group of characters of G' [5], [7]. The isomorphism associates to the character χ the isomorphism class $[\mathcal{L}(\chi)] \in \mathrm{Pic}(\mathcal{Y})$ of the trivial line bundle on Z' with the G'-linearization given by χ. We use this isomorphism to identify the two groups.

Considered $\mathcal{Y}$, we recall that there are distinguished elements $[\mathcal{L}_\rho] \in \mathrm{Pic}(\mathcal{Y})$, for any $\rho \in \Delta'(1)$. Let $\phi' : G' \to (\boldsymbol{C}^*)^{\Delta'(1)}$ be the $(\boldsymbol{C}^*)^{\Delta'(1)}$-component of φ' in (4), the components $(\phi')_\rho$ of ϕ' are characters of G', then the $[\mathcal{L}_\rho]$'s are the corresponding isomorphism classes of line bundles.

The homogeneous coordinate ring of $\mathcal{Y}$ is defined in [7], it is the polynomial ring $S^{\mathcal{Y}} := \boldsymbol{C}[z_\rho]$ in the variables z_ρ, where $\rho \in \Delta'(1)$. It is endowed with a $\mathrm{Pic}(\mathcal{Y})$-grading: the monomial $\prod_\rho z_\rho^{l_\rho}$ has degree $\prod_\rho [\mathcal{L}_\rho]^{l_\rho} \in \mathrm{Pic}(\mathcal{Y})$. For any $\chi \in \mathrm{Pic}(\mathcal{Y})$, let us denote by $S^{\mathcal{Y}}_\chi$ the subset of $S^{\mathcal{Y}}$ consisting of homogeneous polynomials of degree χ. There is an isomorphism of complex vector spaces [7]

$$H^0(\mathcal{Y}, \mathcal{L}(\chi)) \cong S^{\mathcal{Y}}_\chi$$

which we use to identify them. With these notations we can now state the following

THEOREM 5.1. *Let $P_\rho \in S^{\mathcal{Y}}$ be homogeneous polynomials indexed by $\rho \in \Delta(1)$, and let $\chi_i \in \mathrm{Pic}(\mathcal{Y})$ for $i \in \{1, \dots, R\}$ such that*

(a) *If $P_\rho \in S^{\mathcal{Y}}_{\chi_\rho}$, then $\sum_\rho \chi_\rho \otimes a_\rho = 0$ in $\mathrm{Pic}(\mathcal{Y}) \otimes_{\boldsymbol{Z}} N$, and $\prod_\rho \chi_\rho^{b_{i\rho}} \cdot \chi_i^{r_i} = 1$ in $\mathrm{Pic}(\mathcal{Y})$ for any i,*

(b) *$(P_\rho(z)) \in Z$ whenever $z \in Z'$.*

Let $\tilde{f}(z) := (P_\rho(z)) \in \boldsymbol{C}^{\Delta(1)}$. Then there is a morphism $f : \mathcal{Y} \to \mathcal{X}$ such that the diagram

$$\begin{array}{ccc} Z' & \xrightarrow{\tilde{f}} & Z \\ \downarrow & & \downarrow \\ \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \end{array}$$

is 2-commutative, where the vertical arrows are the quotient maps. Furthermore:

(i) *Let $\chi_i \in \mathrm{Pic}(\mathcal{Y})$ fixed for $i \in \{1, \dots, R\}$. Then two sets of polynomials $\{P_\rho\}$ and $\{P'_\rho\}$ determine 2-isomorphic morphisms if and only if there exists $g \in G$ such that $P'_\rho = \varphi_\rho(g) P_\rho$ for any $\rho \in \Delta(1)$, where φ_ρ is the ρ-th component of φ defined in* (4).

(ii) *All morphisms $f : \mathcal{Y} \to \mathcal{X}$ arise in this way up to* 2*-isomorphisms.*

PROOF. Let us choose a representative $\mathcal{L}(\chi)$ of the class $\chi \in \mathrm{Pic}(\mathcal{Y})$ for any χ. Note that there are canonical isomorphisms $\mathcal{L}(\chi_1) \otimes \mathcal{L}(\chi_2) \cong \mathcal{L}(\chi_1 \cdot \chi_2)$.

Let $\{P_\rho \mid \rho \in \Delta(1)\}$ and $\{\chi_i \mid i \in \{1, \dots, R\}\}$ satisfying (a) and (b). Then $\prod_\rho \chi_\rho^{\langle m, a_\rho \rangle} = 1$, for any $m \in M$. As a consequence there are canonical isomorphisms

$$c_m^{\mathrm{can}} : \otimes_\rho \mathcal{L}(\chi_\rho)^{\otimes \langle m, a_\rho \rangle} \to \underline{\boldsymbol{C}}, \quad m \in M,$$

and

$$d_i^{\mathrm{can}} : \otimes_\rho \mathcal{L}(\chi_\rho)^{\otimes b_{i\rho}} \otimes \mathcal{L}(\chi_i)^{\otimes r_i} \to \underline{\boldsymbol{C}}, \quad i \in \{1, \dots, R\}.$$

It follows that $(\mathcal{L}(\chi_\rho), P_\rho, c_m^{\mathrm{can}}, \mathcal{L}(\chi_i), d_i^{\mathrm{can}})/\mathcal{Y}$ is a $\boldsymbol{\Delta}$-collection, therefore it corresponds to a morphism $f : \mathcal{Y} \to \mathcal{X}$. The commutativity of the diagram follows easily.

Let now $\{P_\rho\}$ and $\{P'_\rho\}$ be two sets of polynomials defining 2-isomorphic morphisms. Then there are isomorphisms

$$\gamma_\rho : \mathcal{L}(\chi_\rho) \to \mathcal{L}(\chi_\rho) \quad \text{and} \quad \delta_i : \mathcal{L}(\chi_i) \to \mathcal{L}(\chi_i),$$

such that

$$\gamma_\rho(P_\rho) = P'_\rho, \quad \otimes_\rho \gamma_\rho^{\otimes \langle m, a_\rho \rangle} = \mathrm{id}, \quad \otimes_\rho \gamma_\rho^{\otimes b_{i\rho}} \otimes \delta_i^{\otimes r_i} = \mathrm{id}$$

for any ρ, m and i. The γ_ρ's and δ_i's are multiplications by non-zero complex numbers which we denote by the same symbols. The previous conditions means that $((\gamma_\rho)_\rho, (\delta_i)_i) \in \ker(\psi)$, where ψ is defined in (4). So there exists $g \in G$ such that $\varphi(g) = ((\gamma_\rho)_\rho, (\delta_i)_i)$.

To conclude the proof, let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism, and let $(\mathcal{L}_\rho, u_\rho, c_m, M_i, d_i)/\mathcal{Y}$ be the corresponding $\boldsymbol{\Delta}$-collection (Corollary 2.7). Then there is a morphism

$$(\mathcal{L}_\rho, u_\rho, c_m, M_i, d_i) \to (\mathcal{L}(\chi_\rho), P_\rho, \tilde{c}_m, \mathcal{L}(\chi_i), \tilde{d}_i)$$

for some $\chi_\rho, \chi_i \in \mathrm{Pic}(\mathcal{Y})$. Clearly the P_ρ's, χ_ρ's and χ_i's satisfy conditions (a) and (b). Let us now consider the automorphisms $(c_m^{\mathrm{can}})^{-1} \circ \tilde{c}_m$ and $(d_i^{\mathrm{can}})^{-1} \circ \tilde{d}_i$ of $\otimes_\rho \mathcal{L}(\chi_\rho)^{\otimes \langle m, a_\rho \rangle}$ and $\otimes_\rho \mathcal{L}(\chi_\rho)^{b_{i\rho}} \otimes \mathcal{L}(\chi_i)^{r_i}$ respectively. They correspond to an element $\big(((c_m^{\mathrm{can}})^{-1} \circ \tilde{c}_m)_m, ((d_i^{\mathrm{can}})^{-1} \circ \tilde{d}_i)_i\big) \in (\boldsymbol{C}^*)^d \times (\boldsymbol{C}^*)^R$. Let now $((\gamma_\rho)_\rho, (\delta_i)_i) \in (\boldsymbol{C}^*)^n \times (\boldsymbol{C}^*)^R$ such that $\psi((\gamma_\rho)_\rho, (\delta_i)_i) = \big(((c_m^{\mathrm{can}})^{-1} \circ \tilde{c}_m)_m, ((d_i^{\mathrm{can}})^{-1} \circ \tilde{d}_i)_i\big)$. Then (5) implies that

$$(\gamma_\rho, \delta_i) : (\mathcal{L}(\chi_\rho), P_\rho, \tilde{c}_m, \mathcal{L}(\chi_i), \tilde{d}_i) \to (\mathcal{L}(\chi_\rho), P_\rho, c_m^{\mathrm{can}}, \mathcal{L}(\chi_i), d_i^{\mathrm{can}})$$

is a morphism of $\boldsymbol{\Delta}$-collections. From Corollary 2.7 it follows that the morphism associated to $(\mathcal{L}(\chi_\rho), P_\rho, c_m^{\mathrm{can}}, \mathcal{L}(\chi_i), d_i^{\mathrm{can}})$ is 2-isomorphic to f. □

6. Relations with logarithmic geometry. In [11] toric stacks have been defined in a different way than in [4], in this Section we briefly compare the two approaches. We work over the field of complex numbers $\boldsymbol{C}$, keeping in mind that the construction of [11] works over a general field. Over a general field toric stacks in the sense of [11] are Artin stacks of finite type with finite diagonal.

Let $\boldsymbol{\Delta} = \{N, \Delta, a_\rho\}$ be a combinatorial data as in the Introduction, note that here we assume $R = 0$. In [11], for any scheme Y, the author considers triples $(\pi : \mathcal{S} \to \mathcal{O}_Y, \alpha : \mathcal{M} \to \mathcal{O}_Y, \eta : \mathcal{S} \to \mathcal{M})$, where

(1) $\mathcal{S}$ is an étale sheaf of submonoids of the constant sheaf M on Y such that for any point $y \in Y$ the Zariski stalk $\mathcal{S}_y$ is isomorphic to the étale stalk $\mathcal{S}_{\bar{y}}$,
(2) $\pi : \mathcal{S} \to \mathcal{O}_Y$ is a morphism of monoids, where $\mathcal{O}_Y$ is a monoid with respect to the multiplication,
(3) for any $s \in \mathcal{S}$, $\pi(s)$ is invertible if and only if s is invertible,
(4) for any point $y \in Y$, there exists some $\sigma \in \Delta$ such that $\mathcal{S}_{\bar{y}} = \sigma^\vee \cap M$,
(5) $\alpha : \mathcal{M} \to \mathcal{O}_Y$ is a fine logarithmic structure on Y in the sense of [14],
(6) $\eta : \mathcal{S} \to \mathcal{M}$ is a homomorphism of sheaves of monoids such that $\pi = \alpha \circ \eta$, and for each generic point $\bar{y}$, $\bar{\eta} : (\mathcal{S}/\mathcal{O}_Y^*)_{\bar{y}} \to (\mathcal{M}/\mathcal{O}_Y^*)_{\bar{y}}$ is isomorphic to the $\boldsymbol{\Delta}$-*free resolution* at $\bar{y}$.

The notions of *minimal free resolution* and of $\boldsymbol{\Delta}$-*free resolution* are the most important notions in the definition of [11], we refer to [11] for more details.

One can define morphisms between two such triples. Let $\mathcal{L}_{\boldsymbol{\Delta}}$ be the resulting category. The functor

$$\mathcal{L}_{\boldsymbol{\Delta}} \to (\mathrm{Sch}) \tag{20}$$

which forget the data $\pi : \mathcal{S} \to \mathcal{O}_Y$, $\alpha : \mathcal{M} \to \mathcal{O}_Y$ and $\eta : \mathcal{S} \to \mathcal{M}$ is a CFG. Then, Theorem 2.7 in [11] states that (20) is a smooth DM-stack of finite type and separated, with coarse moduli space the toric variety associated to N and Δ.

Let Σ be the stacky fan in the sense of [4] associated to $\boldsymbol{\Delta}$, and let $\mathcal{X}_\Sigma$ be the toric stack as defined in [4]. The Proposition below follows from our Theorem 2.6 together with Theorem 1.4 in [12].

PROPOSITION 6.1. *Under the previous hypothesis, there is an isomorphism of stacks*

$$\mathcal{L}_{\boldsymbol{\Delta}} \cong \mathcal{X}_\Sigma \,.$$

7. Appendix. This appendix is aimed to give a correspondence between the stacks $\mathcal{C}_{\boldsymbol{\Delta}}$ defined in this paper and the "smooth toric DM stacks" defined in [8]. Let us recall from [8] the following

DEFINITION 7.1. A "smooth toric DM stack" is a smooth separated DM-stack $\mathcal{X}$ together with an open immersion of a Deligne-Mumford torus $\mathcal{T} \to \mathcal{X}$ with dense image such that the action of $\mathcal{T}$ on itself extends to an action on $\mathcal{X}$.

We have the following result that can be seen as a generalization of the functor of a smooth toric variety [6].

THEOREM 7.2. *For any combinatorial data $\boldsymbol{\Delta}$, $\mathcal{C}_{\boldsymbol{\Delta}}$ is a "smooth toric DM stack". Conversely, for any "smooth toric DM stack" $\mathcal{X}$ there exists a combinatorial data $\boldsymbol{\Delta}$ and an isomorphism $\mathcal{X} \cong \mathcal{C}_{\boldsymbol{\Delta}}$.*

PROOF. $\mathcal{C}_{\boldsymbol{\Delta}}$ is a smooth and separated DM-stack (Theorem 2.6), moreover there is an open immersion of a Picard stack $\mathcal{T} \to \mathcal{C}_{\boldsymbol{\Delta}}$ with dense image and such that the multiplication

of $\mathcal{T}$ extends to an action on $\mathcal{C}_{\boldsymbol{\Delta}}$ (Proposition 4.1). As our $\mathcal{T}$ is in fact a Deligne-Mumford torus, the first part of the theorem follows.

We divide the proof of the second part in three steps.

Step 1. Let X be the coarse moduli space of $\mathcal{X}$. Since X is a simplicial toric variety, it is determined by a free abelian group N and a fan $\Delta \subset N_{\boldsymbol{Q}}$. Moreover, X has quotient singularities by finite groups, hence it is the coarse moduli space of a "canonical" smooth DM-stack which we denote with X^{can} (we refer to [8], Def. 4.4, for the definition of "canonical" stack). Let now $\mathcal{C}_{\{N,\Delta,n_\rho\}}$ be the stack associated to the combinatorial data $\{N, \Delta, n_\rho\}$, where n_ρ is the generator of the semigroup $\rho \cap N$ for any $\rho \in \Delta(1)$. We have that $\mathcal{C}_{\{N,\Delta,n_\rho\}} \cong X^{\mathrm{can}}$. Indeed, $\mathcal{C}_{\{N,\Delta,n_\rho\}} \cong \mathcal{X}_\Sigma \times (\boldsymbol{C}^*)^k$, where $\mathcal{X}_\Sigma$ is a toric DM-stack associated to $\{N, \Delta, n_\rho\}$ and k is a natural number (Theorem 2.6); moreover $\mathcal{X}_\Sigma \times (\boldsymbol{C}^*)^k$ is a "canonical" stack over X (this can be seen using an orbifold atlas for $\mathcal{X}_\Sigma$ as given for example by Prop. 4.3 in [4]). The universal property of "canonical" stacks implies that $X^{\mathrm{can}} \cong \mathcal{C}_{\{N,\Delta,n_\rho\}}$.

Step 2. Let now $\mathcal{X}_{\mathrm{rig}}$ be the rigidification of $\mathcal{X}$ by the generic stabilizer. Then, $\mathcal{X}_{\mathrm{rig}}$ is a toric orbifold obtained from X^{can} by roots of effective Cartier divisors ([8], Thm. 5.2). More precisely, let $\rho_1, \ldots, \rho_n$ be the 1-dimensional cones of Δ, and let $D_{\rho_1}, \ldots, D_{\rho_n} \subset X^{\mathrm{can}}$ be the corresponding torus invariant divisors. There are natural numbers $\alpha_{\rho_1}, \ldots, \alpha_{\rho_n}$ such that

$$\mathcal{X}_{\mathrm{rig}} \cong \sqrt[\alpha_{\rho_1}]{D_{\rho_1}/X^{\mathrm{can}}} \times_{X^{\mathrm{can}}} \cdots \times_{X^{\mathrm{can}}} \sqrt[\alpha_{\rho_1}]{D_{\rho_1}/X^{\mathrm{can}}},$$

where $\sqrt[\alpha]{D/\mathcal{X}}$ denotes the α-th root of the effective Cartier divisor D in the smooth algebraic stack $\mathcal{X}$. Set $a_\rho := \alpha_\rho \cdot n_\rho \in N$, for any $\rho \in \Delta(1)$. Then, under the identification of X^{can} with $\mathcal{C}_{\{N,\Delta,n_\rho\}}$, we obtain an isomorphism

$$\mathcal{X}_{\mathrm{rig}} \cong \mathcal{C}_{\{N,\Delta,a_\rho\}}.$$

Step 3. $\mathcal{X}$ is a gerbe over $\mathcal{X}_{\mathrm{rig}}$ isomorphic to

$$\sqrt[r_1]{\mathcal{L}_1/\mathcal{X}_{\mathrm{rig}}} \times_{\mathcal{X}_{\mathrm{rig}}} \cdots \times_{\mathcal{X}_{\mathrm{rig}}} \sqrt[r_R]{\mathcal{L}_R/\mathcal{X}_{\mathrm{rig}}},$$

for some line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_R$ over $\mathcal{X}_{\mathrm{rig}}$ and for some natural numbers $r_1, \ldots, r_R$ ([8], Cor. 6.27). The Picard group of $\mathcal{X}_{\mathrm{rig}}$ is generated by the classes of line bundles $\mathcal{O}_{\mathcal{X}_{\mathrm{rig}}}(\mathcal{D}_\rho)$ associated to the torus invariant divisors $\mathcal{D}_\rho \subset \mathcal{X}_{\mathrm{rig}}$ for $\rho \in \Delta(1)$. Hence, for any $i \in \{1, \ldots, R\}$, there are integers $b_{i\rho}$, $\rho \in \Delta(1)$, with $\mathcal{L}_i \cong \mathcal{O}(\sum_\rho b_{i\rho}\mathcal{D}_\rho)$. It follows that

$$\mathcal{X} \cong \mathcal{C}_{\boldsymbol{\Delta}}$$

with $\boldsymbol{\Delta} = \{N, \Delta, a_\rho, b_{i\rho}, r_1, \ldots, r_R\}$. □

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REFERENCES

[1] D. ABRAMOVICH, A. CORTI AND A. VISTOLI, Twisted bundles and admissible covers, Comm. Algebra 31 (2003), 3547–3618.

[2] D. ABRAMOVICH, T. GRABER AND A. VISTOLI, Gromov-Witten theory of Deligne-Mumford stacks, arXiv.org:math/0603151.

[3] K. BEHREND, B. CONRAD, D. EDIDIN, W. FULTON, B. FANTECHI, L. GÖTTSCHE AND A. KRESCH, Algebraic Stacks, in preparation.

[4] L. A. BORISOV, L. CHEN AND G. SMITH, The orbifold Chow ring of toric Deligne-Mumford stacks, J. Amer. Math. Soc. 18 (2005), 193–215.

[5] L. A. BORISOV AND R. P. HORJA, On the K-theory of smooth toric DM stacks, Contemp. Math. 401 (2006), 21–42.

[6] D. A. COX, The functor of a smooth toric variety, Tohoku Math. J. 47 (1995), 251–262.

[7] M. DÜRR AND M. HALIC, On toric stacks, preprint.

[8] B. FANTECHI, E. MANN AND F. NIRONI, Smooth toric DM stacks, arXiv:0708.1254.

[9] W. FULTON, Introduction to toric varieties, Ann. of Math. Stud. 131, Princeton University Press, Princeton, 1993.

[10] J. GIRAUD, Cohomologie non abélienne, Springer-Verlag, Berlin, 1971.

[11] I. IWANARI, The category of toric stacks, arXiv:math/0610548.

[12] I. IWANARI, Generalization of Cox functor, arXiv:0705.3524, version 1.

[13] Y. JIANG AND H.-H. TSENG, The integral (orbifold) Chow ring of toric Deligne-Mumford stacks, arXiv:0707.2972.

[14] K. KATO, Logarithmic structures of Fontaine-Illusie, in Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), 191–224, Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[15] M. ROMAGNY, Group actions on stacks and applications, Michigan Math. J. 53 (2005), 209–236.

[16] J. P. SERRE, Espaces fibrés algébriques (d'après André Weil), Séminaire Bourbaki 2 (1995), 305–311.

[17] Théorie des topos et cohomologie étale des schémas. Tome 3, (SGA 4), Lecture Notes in Mathematics 305, Springer-Verlag, Berlin, 1973.

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