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INTERPOLATION AND COMPLEX SYMMETRY

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Abstract. In a separable complex Hilbert space endowed with an isometric conjugate-linear involution, we study sequences orthonormal with respect to an associated bilinear form. Properties of such sequences are measured by a positive, possibly unbounded angle operator which is formally orthogonal as a matrix. Although developed in an abstract setting, this framework is relevant to a variety of eigenvector interpolation problems arising in function theory and in the study of differential operators.

1. Introduction. Interpolation problems, such as free interpolation by bounded analytic functions, are often closely related to biorthogonal sequences of vectors in associated Hilbert spaces which are equipped with symmetric bilinear forms. Similarly, the qualitative study of eigenfunctions of special classes of operators often provides sequences of vectors which are orthogonal with respect to an auxiliary bilinear form rather than the usual sesquilinear form. As a consequence, we attempt to develop an abstract framework for systems of vectors which are orthonormal with respect to a symmetric bilinear form.

Throughout this note, $\mathcal{H}$ will denote a separable, infinite dimensional, complex Hilbert space endowed with a *conjugation* $C : \mathcal{H} \to \mathcal{H}$, a conjugate-linear operator satisfying $C^2 = I$ and $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all x, y in $\mathcal{H}$. Corresponding to each conjugation on $\mathcal{H}$, we obtain a symmetric bilinear form $[\,\cdot\,,\,\cdot\,]$ on $\mathcal{H} \times \mathcal{H}$ defined by $[x, y] = \langle x, Cy \rangle$. We say that $(u_n)_{n=1}^{\infty}$ is a complete system of C-*orthonormal* vectors if the linear span of the vectors u_n is dense in $\mathcal{H}$ and if

$$[u_j, u_k] = \delta_{jk}$$

for all j, k. Here δ_{jk} denotes the Kronecker δ-symbol. It is easy to show that if $(u_n)_{n=1}^{\infty}$ is a complete sequence of vectors in $\mathcal{H}$ such that $[u_j, u_k] = 0$ whenever $j \neq k$, then $[u_n, u_n] \neq 0$ necessarily holds for every n. Thus there is no loss of generality in insisting that $[u_n, u_n] = 1$ for all n.

Complex symmetric operators are the primary source of such systems of vectors, for the eigenvectors of certain complex symmetric operators form an immediate class of concrete examples. Indeed, the present work has its origin in the recent study of complex symmetric operators [2, 4, 5, 6, 7, 8, 10, 15]. To be more specific, we say that a bounded linear operator

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$T : \mathcal{H} \to \mathcal{H}$ is *C-symmetric* if $T = CT^*C$ and *complex symmetric* if there exists a conjugation C with respect to which T is C-symmetric. The terminology stems from the fact that T is a complex symmetric operator if and only if T is unitarily equivalent to a symmetric matrix with complex entries, regarded as an operator acting on an l^2-space of the appropriate dimension [4, Sect. 2.4].

This class of complex symmetric operators includes all normal operators, operators defined by Hankel matrices, compressed Toeplitz operators (including finite Toeplitz matrices and the compressed shift), the Volterra integration operator, and various differential operators (including certain auxiliary operators produced by the complex scaling method for Schrödinger operators [15]). We refer the reader to [6, 7] or [4] (for a more expository pace) for further details. In light of this variety, the range of C-orthonormal systems obtained from complex symmetric operators is potentially vast.

In this note, we attempt to include as many examples and applications as we can, paying particular attention to interpolation problems related to the Hardy space on the unit disk. For example, as a corollary of a general theorem on interpolation of real l^2-sequences (Theorem 6) we obtain the following (stated as Theorem 7):

THEOREM. *If φ is a nonconstant inner function, then there exists a subset $E \subset \boldsymbol{D}$ of measure zero such that for each w in $\boldsymbol{D} \backslash E$ the level set $\varphi^{-1}\{w\}$ is nonempty and*

(i) *if z_n is an enumeration of $\varphi^{-1}\{w\}$, then $\varphi'(z_n) \neq 0$ for all n, i.e., φ assumes the value w with multiplicity one at each z_n,*

(ii) *for each real sequence $(a_n)_{n=1}^{\infty}$ in l^2, there exists a function f in $H^2 \ominus \varphi H^2$ such that*

$$\operatorname{Re}\left(\frac{f(z_n)}{\sqrt{\varphi'(z_n)}}\right) = a_n$$

holds for each n.

It turns out that our main object of study is the *linear* operator (defined initially on finite sums)

$$A_0\left(\sum_{n=1}^{m} c_n u_n\right) = \sum_{n=1}^{m} c_n C u_n \, .$$

It is a densely defined, non-negative symmetric operator which inherits a complex orthogonal matrix structure, when properly interpreted (see Theorem 2). In general, the operator A_0 is unbounded and thus its selfadjoint extensions become relevant. We establish several criteria for the essential selfadjointness of A_0 (Theorem 5) in addition to studying properties of the so-called the Friedrichs extension.

2. Some examples of C-orthonormal systems.

Before proceeding further, let us first discuss a few examples of C-orthonormal systems. Although the following several examples are quite simple, the underlying conjugation behind these examples is not widely discussed. Moreover, we feel that having a number of diverse examples close at hand better motivates the abstract study of C-orthonormal systems.

EXAMPLE 1. Let $A : \mathcal{D}(A) \to \mathcal{H}$ be a selfadjoint operator with simple, discrete spectrum and let $K \geq 0$ be a compact operator belonging to a Schatten-von Neumann class $\mathcal{C}_p$ for some $p > 1$. Under these hypotheses, the dissipative operator $T = A + iK$ has a complete sequence of eigenvectors [12, p. 277]. Using this principle, one can readily produce examples of non-normal complex symmetric operators which possess complete systems of eigenvectors.

On a related note, several conditions which guarantee that a rank-one perturbation of a bounded normal operator will be complex symmetric can be found in [8]. For instance, it can be shown that if N is a normal operator on $\mathcal{H}$ and U is a unitary operator in the von Neumann algebra generated by N, then the operator $T = N + a(Uv \otimes v)$ is a complex symmetric operator for all $a \in \boldsymbol{C}$ and $v \in \mathcal{H}$.

The following simple example shows how concrete C-orthonormal systems can arise from relatively standard operators:

EXAMPLE 2. Let $w = \alpha + i\beta$ where α and β are real constants and consider $\mathcal{H} = L^2[0, 1]$, endowed with the conjugation $[Cf](x) = \overline{f(1-x)}$. A short computation shows that if w is not an integer multiple of 2π, then the vectors

$$u_n(x) = \exp[i(w + 2\pi n)(x - 1/2)] \tag{1}$$

for $n \in \boldsymbol{Z}$ are eigenfunctions of the C-symmetric operator

$$\begin{aligned} [Tf](x) &= e^{iw/2} \int_0^x f(y)dy + e^{-iw/2} \int_x^1 f(y)dy \\ &= e^{iw/2} V + e^{-iw/2} V^*, \end{aligned}$$

where V denotes the Volterra integration operator. One might also say that the u_n are the eigenfunctions of the derivative operator with boundary condition $f(1) = e^{iw} f(0)$. Expanding out the exponent in (1) and simplifying, we find that each u_n is, up to constant multiples, the image of $e^{2\pi inx}$ under the bounded and invertible operator of multiplication by e^{iwx}. In particular, the system $(u_n)_{n\in\boldsymbol{Z}}$ is complete in $L^2[0, 1]$ and a straightforward computation shows that it is also C-orthonormal (see also [5, Ex. 9] or [4, Lem. 4.3]).

EXAMPLE 3. If u belongs to the uniform algebra $H^\infty + C$ (here C denotes the algebra of continuous functions on the unit circle), then Hartman's compactness criterion tells us that the corresponding infinite Hankel matrix defines a compact operator on $l^2(\boldsymbol{N})$ [14, Ch. 1, Thm. 5.5]. Since each such Hankel matrix is C-symmetric with respect to the canonical conjugation C on $l^2(\boldsymbol{N})$, one expects many C-orthonormal systems of eigenvectors to arise in this context.

Our final example (stated as Theorem 1) in this section is somewhat involved and requires a few preliminaries. Recall that each nonconstant inner function φ gives rise to a so-called *Jordan model space* $H^2 \ominus \varphi H^2$. Here H^2 denotes the Hardy space on the open unit disk $\boldsymbol{D}$. It turns out that each such model space carries a natural conjugation:

LEMMA 1. *If φ denotes a nonconstant inner function, then $Cf = \overline{fz}\varphi$ (defined in terms of boundary functions) is a conjugation on $H^2 \ominus \varphi H^2$.*

In particular, observe that a function f in H^2 belongs to $H^2 \ominus \varphi H^2$ if and only if there exists a function g in H^2 such that

$$f = \overline{gz}\varphi \tag{2}$$

a.e. on the unit circle $\partial \boldsymbol{D}$ (henceforth we will freely identify functions in H^2 with their a.e. defined boundary values). The proof of the preceding lemma and further details can be found in [4].

The following theorem indicates that an abundance of natural C-orthonormal systems arise in the context of Hardy space theory:

THEOREM 1. *If φ is a nonconstant inner function, then there exists a subset $E \subset \boldsymbol{D}$ of measure zero such that for each w in $\boldsymbol{D}\backslash E$ the following hold:*

(i) *$\varphi^{-1}\{w\} \neq \varnothing$ and if z_n is an enumeration of $\varphi^{-1}\{w\}$, then $\varphi'(z_n) \neq 0$ for all n, i.e., φ assumes the value w with multiplicity one at each z_n.*

(ii) *For any determination of the numbers $\delta_n = \sqrt{\varphi'(z_n)}$, the functions*

$$[u_n](z) = \delta_n \cdot \frac{\varphi(z) - w}{z - z_n}, \tag{3}$$

$$[Cu_n](z) = \overline{\delta_n} \cdot \frac{1 - \bar{w}\varphi(z)}{1 - \overline{z_n}z} \tag{4}$$

both form complete C-orthonormal systems in $H^2 \ominus \varphi H^2$. In particular,

$$[f, u_n] = \frac{f(z_n)}{\sqrt{\varphi'(z_n)}} \tag{5}$$

for each $f \in H^2 \ominus \varphi H^2$.

PROOF. Without loss of generality, we may assume that φ is not a finite Blaschke product. A variant of Frostman's Theorem (see [13, Thm. 3.10.2]) asserts that there exists a subset $E \subset \boldsymbol{D}$ of measure zero such that for all $w \in \boldsymbol{D}\backslash E$, the function $b_w \circ \varphi$ is a Blaschke product having simple zeros. Here b_w denotes the disk automorphism

$$b_w(z) = \frac{z - w}{1 - \bar{w}z}. \tag{6}$$

In particular, if z_n denotes an enumeration of the set $\varphi^{-1}\{w\}$, then we have $\varphi'(z_n) \neq 0$ for each n. A short calculation based on the fact that Cu_n is a constant multiple of a reproducing kernel implies (5) and shows that $[u_j, u_k] = \delta_{jk}$ for all j, k. It therefore suffices to show that the system $(u_n)_{n=1}^{\infty}$ is complete in $H^2 \ominus \varphi H^2$.

Suppose toward a contradiction that there exists a function f in $H^2 \ominus \varphi H^2$ which does not vanish identically but such that $[f, u_n] = 0$ for all n. In light of (5), this immediately implies that $f(z_n) = 0$ for all n. Let us write $f = I_f F$ where I_f is inner and F is outer. Since the sequence z_n is exactly the zero sequence for the Blaschke product $b_w \circ \varphi$ (whose zeros are simple), it follows that $b_w \circ \varphi$ divides I_f. In particular, we may assume that $I_f = b_w \circ \varphi$ since $H^2 \ominus \varphi H^2$ is stable under conjugate-analytic Toeplitz operators (i.e., inner factors of functions in $H^2 \ominus \varphi H^2$ can be removed without leaving $H^2 \ominus \varphi H^2$).

Writing $f = (b_w \circ \varphi)F$ and using (2), it follows that there exists another inner function j such that

$$(b_w \circ \varphi)F = \overline{jFz}\varphi .$$

Since j is inner, we may rewrite the preceding in the form

$$j(b_w \circ \varphi)F/\bar{F} = \bar{z}\varphi . \tag{7}$$

To simplify our notation somewhat, we at this point fix some n and henceforth denote z_n by λ and b_{z_n} by b. A short calculation reveals that

$$[(b_w \circ \varphi)/b] \cdot [k/\bar{k}] = \bar{z}\varphi , \tag{8}$$

where k denotes the reproducing kernel function

$$k_\lambda = \frac{1 - \bar{w}\varphi(z)}{1 - \bar{\lambda}z} .$$

Upon combining (7) and (8) we find that

$$\frac{j(b_w \circ \varphi)F}{\bar{F}} = \frac{(b_w \circ \varphi)k}{b\bar{k}} .$$

Using the fact that j and b are unimodular a.e. on $\partial \boldsymbol{D}$, the preceding can be rewritten in the form

$$(1 + jb)\frac{F}{k} = \overline{\left[(1 + jb)\frac{F}{k}\right]} .$$

This means that the function

$$h = (1 + jb)\frac{F}{k}$$

belongs to H^2 (since k is invertible in H^∞) and is real-valued a.e. on $\partial \boldsymbol{D}$. Since it is well-known [9, II.13.b] that this forces h to be a constant function, it immediately follows that the function

$$\frac{1}{1 + jb} = \frac{F}{hk}$$

also belongs to H^2. This implies that the nonconstant function $i(1 - jb)(1 + jb)^{-1}$ belongs to H^2 and is real valued a.e. on $\partial \boldsymbol{D}$. Since this is a contradiction, we conclude that $(u_n)_{n=1}^\infty$ is complete in $H^2 \ominus \varphi H^2$, as desired. □

The reader may recognize that we have essentially been dealing with systems of eigenvectors of certain compressed Toeplitz operators (these too are complex symmetric—see [6, Prop. 3] or [4, Sect. 5]). We should mention the recent article [18] which, in the context of complex symmetric operators, raises numerous important questions concerning compressed Toeplitz operators.

Although we have considered only scalar inner functions here (i.e., we are interested in the eigenfunctions of contractions with defect indices $1-1$), the recent article [2] indicates that many contractions with defect indices $2-2$ are also complex symmetric operators.

3. General framework. Let $\mathcal{H}$ denote a separable infinite-dimensonal Hilbert space equipped with a conjugation C and let $(u_n)_{n=1}^{\infty}$ denote a C-orthonormal system in $\mathcal{H}$. We consider here *linear* extensions of the map $u_n \mapsto Cu_n$. Since the u_n are not necessarily orthogonal with respect to the usual sesquilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$, this map does not immediately extend (as a bounded operator) further than the dense linear submanifold $\mathcal{F}$ of finitely supported vectors.

To be specific, we say that a vector f in $\mathcal{H}$ is *finitely supported* if it is of the form $f = \sum_{n=1}^{m_f} c_n u_n$ for some positive integer m_f. Since $[u_j, u_k] = \delta_{jk}$, it follows that the coefficients $c_n(f)$ of a vector f in $\mathcal{F}$ are given by the formula

$$c_n(f) = [f, u_n] \tag{9}$$

and hence each such f can be recovered via the *skew-Fourier expansion*

$$f = \sum_{n=1}^{m_f} [f, u_n] u_n \, . \tag{10}$$

Let $A_0 : \mathcal{F} \to \mathcal{H}$ denote the linear extension of the map $A_0 u_n = Cu_n$ to $\mathcal{F}$. In other words, A_0 is the linear operator defined on finitely supported vectors by $A_0(\sum_{n=1}^{m} c_n u_n) = \sum_{n=1}^{m} c_n C u_n$. Since $[u_j, u_k] = \delta_{jk}$, it follows that

$$\langle A_0 f, f \rangle = \sum_{n=1}^{m_f} |[f, u_n]|^2 \tag{11}$$

for any f in $\mathcal{F}$. In particular, the non-negativity of A_0 on its domain $\mathcal{D}(A_0) = \mathcal{F}$ implies that A_0 is a symmetric operator (in the sense of unbounded operators): $A_0 \subseteq A_0^*$. Since $\mathcal{F}$ is dense in $\mathcal{H}$, it also follows that if $A_0 : \mathcal{F} \to \mathcal{H}$ is bounded, then A_0 has a unique bounded, selfadjoint extension $A : \mathcal{H} \to \mathcal{H}$.

From the definition of A_0, we see that the *antilinear* operator CA_0 fixes each u_n. We are therefore lead to consider the antilinear map $J : \mathcal{F} \to \mathcal{F}$ defined by $J = CA_0$. On finitely supported vectors, we have

$$J\left(\sum_{n=1}^{m} c_n u_n \right) = \sum_{n=1}^{m} \overline{c_n} u_n \, , \tag{12}$$

and hence J is an involution of $\mathcal{F}$: $J^2 = I_{\mathcal{F}}$. Since $J = CA_0$ and C is isometric, it is clear that A_0 is bounded if and only if the conjugate-linear involution (12) is bounded on $\mathcal{F}$.

It will be useful to refer to the following theorem from [5], which contains a number of statements that are equivalent to the boundedness of A_0:

THEOREM 2. *If $(u_n)_{n=1}^{\infty}$ is a complete C-orthonormal system in $\mathcal{H}$, then the following are equivalent*:

(i) $(u_n)_{n=1}^{\infty}$ *is a Bessel sequence with Bessel bound* M *i.e.*, $\sum_{n=1}^{\infty} |\langle f, u_n\rangle|^2 \leq M\|f\|^2$ *for* $f \in \mathcal{H}$.

(ii) $(u_n)_{n=1}^{\infty}$ *is a Riesz basis with lower and upper bounds* M^{-1} *and* M, *respectively, i.e.*, $M^{-1}\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, u_n\rangle|^2 \leq M\|f\|^2$ *for* $f \in \mathcal{H}$.

(iii) A_0 *extends to a bounded linear operator* $A : \mathcal{H} \to \mathcal{H}$ *satisfying* $\|A\| \leq M$.

(iv) *There exists* $M > 0$ *satisfying*:

$$\left\| \sum_{j=1}^{n} \overline{c_j} u_j \right\| \leq M \left\| \sum_{j=1}^{n} c_j u_j \right\|,$$

for every finite sequence $c_1, c_2, \ldots, c_n$.

(v) *The Gram matrix* $(\langle u_j, u_k\rangle)_{j,k=1}^{\infty}$ *dominates its transpose*:

$$(M^2\langle u_j, u_k\rangle - \langle u_k, u_j\rangle)_{j,k=1}^{\infty} \geq 0$$

for some $M > 0$.

(vi) *The Gram matrix* $G = (\langle u_j, u_k\rangle)_{j,k=1}^{\infty}$ *is bounded on* $l^2(\boldsymbol{N})$ *and orthogonal* ($G^t G = I$ *as matrices*). *Furthermore*, $\|G\| \leq M$.

(vii) *The skew Fourier expansion* $\sum_{n=1}^{\infty} [f, u_n]u_n$ *converges in norm for each* f *in* $\mathcal{F}$ *and*

$$\frac{1}{M}\|f\|^2 \leq \sum_{n=1}^{\infty} |[f, u_n]|^2 \leq M\|f\|^2 .$$

In all cases, *the infimum over all such* M *equals the norm of* A_0.

Since a complete sequence which is a Bessel sequence need not be a Riesz basis, the implication (i) $\Rightarrow$ (ii) is false without the hypothesis of C-orthonormality. We also remark that (ii) is equivalent to saying that $(u_n)_{n=1}^{\infty}$ is the image of an orthonormal basis of $\mathcal{H}$ under a bounded, invertible linear operator [3, Prop. 3.6.4]. In fact, this is often taken as the definition of a Riesz basis.

EXAMPLE 4. For this example, we maintain the same notation as in Example 2. In this case, the map $u_n \mapsto Cu_n$ extends to a bounded operator on all of $L^2[0, 1]$. Indeed, this extension is simply the multiplication operator $[Af](x) = e^{2\beta(x-1/2)} f(x)$ whence $B = \sqrt{A}$ is given by

$$[Bf](x) = e^{\beta(x-1/2)} f(x) .$$

The system $(u_n)_{n=1}^{\infty}$ forms a Riesz basis for $L^2[0, 1]$ and is the image of the orthonormal basis $(s_n)_{n=1}^{\infty}$, defined by $s_n = Bu_n$, under the bounded and invertible operator B^{-1}. The s_n are given explicitly by

$$s_n(x) = \exp[i(\alpha + 2\pi n)(x - 1/2)] .$$

Such bases and their relationship to the Volterra integration operator and the "compressed shift" corresponding to the atomic inner function $\varphi(z) = \exp[(z + 1)/(z - 1)]$ are discussed in [4].

The existence of a constant $M > 0$ such that $\|u_n\| \leq M$ for all n is a necessary condition for $(u_n)_{n=1}^{\infty}$ to form a Riesz basis for $\mathcal{H}$. This can be seen by setting $f = u_n$ in condition (vii) of Theorem 2. Even with the additional structure introduced by the underlying conjugation, this condition is not sufficient for $(u_n)_{n=1}^{\infty}$ to be a Riesz basis:

EXAMPLE 5. Consider $\mathcal{H} = L^2[-\pi, \pi]$, equipped with normalized Lebesgue measure $dm = dt/(2\pi)$. Let h be a continuous function in $L^2[-\pi, \pi]$ which is odd, real-valued, unbounded, and such that the function $g = e^h$ also belongs to $L^2[-\pi, \pi]$. Define the conjugation $[Cf](x) = \overline{f(-x)}$ on $L^2[-\pi, \pi]$ and observe that the system of vectors $(u_n)_{n\in\mathbf{Z}}$ defined by

$$[u_n](x) = \exp[h(x) + inx] = g(x)e^{inx}$$

belongs to $L^2[-\pi, \pi]$ and satisfies $\|u_n\| = \|g\|$ for each $n \in \mathbf{Z}$. Furthermore,

$$[Cu_n](x) = \exp[-h(x) + inx] = \frac{e^{inx}}{g(x)}$$

and hence the system $(u_n)_{n\in\mathbf{Z}}$ is easily seen to be C-orthonormal.

The operator A_0 is easily seen to be multiplication by the function $\exp[h(-x) - h(x)] = \exp[-2h(x)]$, with domain equal to the linear span of the sequence $(u_n)_{n\in\mathbf{Z}}$. In particular, A_0 is essentially self-adjoint and unbounded. Thus $(u_n)_{n\in\mathbf{Z}}$ is a complete C-orthonormal system which is uniformly bounded ($\sup_{n\in\mathbf{Z}} \|u_n\| < \infty$) but which fails to be a Riesz basis by Theorem 2.

For C-symmetric contractive operators, we do have the following Riesz basis criterion:

THEOREM 3. *Let T be a contractive C-symmetric operator (i.e., $\|T\| \leq 1$ and $T = CT^*C$) with simple spectrum $(z_n)_{n=1}^{\infty}$ and a complete sequence of corresponding C-orthonormal eigenvectors $(u_n)_{n=1}^{\infty}$. If $\sup_n \|u_n\| \leq M$ holds and if the matrix*

$$\left(\frac{\sqrt{1 - |z_j|^2}\sqrt{1 - |z_k|^2}}{|1 - z_j\overline{z_k}|} \right)_{j,k=1}^{\infty} \tag{13}$$

defines a bounded linear operator on l^2, then the sequence $(u_n)_{n=1}^{\infty}$ forms a Riesz basis for $\mathcal{H}$. In particular, if the matrix (13) *is bounded above, then it is also invertible.*

PROOF. Let $D = I - T^*T$ and note that

$$\begin{aligned}
|1 - z_j\overline{z_k}||\langle u_j, u_k\rangle| &= |\langle u_j, u_k\rangle - \langle Tu_j, Tu_k\rangle| \\
&= |\langle Du_j, u_k\rangle| \\
&\leq \sqrt{\langle Du_j, u_j\rangle}\sqrt{\langle Du_k, u_k\rangle} \\
&= \|u_j\|\|u_k\|\sqrt{1 - |z_j|^2}\sqrt{1 - |z_k|^2}
\end{aligned}$$

since $\langle Dx, y\rangle$ defines a non-negative sesquilinear form on $\mathcal{H} \times \mathcal{H}$. It follows that

$$|\langle u_j, u_k\rangle| \leq M^2 \frac{\sqrt{1 - |z_j|^2}\sqrt{1 - |z_k|^2}}{|1 - z_j\overline{z_k}|}$$

for all j, k whence the desired result follows from (vi) of Theorem 2 (the orthogonality of the Gram matrix follows immediately from purely formal manipulations). □

The corresponding result for dissipative operators can be deduced in a completely analogous manner:

THEOREM 4. *Let $T : \mathcal{D} \to \mathcal{H}$ be a C-symmetric, pure dissipative operator with simple spectrum $(z_n)_{n=1}^{\infty}$ and a complete sequence of corresponding C-orthonormal eigenvectors $(u_n)_{n=1}^{\infty}$. If $\sup_n \|u_n\| \le M$ holds and if the matrix*

$$\left(\frac{\sqrt{(\operatorname{Im} z_j)(\operatorname{Im} z_k)}}{|z_j - \overline{z_k}|} \right)_{j,k=1}^{\infty} \tag{14}$$

defines a bounded linear operator on l^2, then the sequence $(u_n)_{n=1}^{\infty}$ forms a Riesz basis for $\mathcal{H}$. In particular, if the matrix (14) *is bounded above, then it is also invertible.*

This is related to a classical observation due to Glazman which gives conditions solely in terms of the (simple) spectrum of a dissipative operator for the corresponding unit eigenvectors to form a Riesz basis [11]. This idea was put into a more general context in the last chapter of the monograph [12]. We remark that Glazman's result, which did not have a complex symmetry assumption, required the finiteness of the Hilbert-Schmidt norm of the associated Gram matrix.

4. Criteria for the essential self-adjointness of A_0. In general, the operator $A_0 : \mathcal{F} \to \mathcal{H}$ is unbounded. Without further assumptions on the C-orthonormal system $(u_n)_{n=1}^{\infty}$, it may occur that $A_0 \subsetneq A_0^*$ and hence we must search for selfadjoint extensions of A_0. The main goal of this section is to establish several practical criteria to determine when the operator A_0 is essentially selfadjoint (i.e., when the closure of A_0 is selfadjoint). These are summarized in Theorem 5. We first require several preparatory remarks.

If f is finitely supported, then by (10) it follows that $A_0 f = \sum_{n=1}^{m_f} [f, u_n] C u_n$. Thus, since $(Cu_n)_{n=1}^{\infty}$ is also a complete C-orthonormal system,

$$[f, u_n] = [A_0 f, Cu_n]$$

for each n. This motivates the consideration of the linear submanifold $\Gamma \subset \mathcal{H} \oplus \mathcal{H}$ defined by

$$\Gamma = \{\ (f, g) \in \mathcal{H} \oplus \mathcal{H}\,;\ [f, u_n] = [g, Cu_n] \ \text{ for all } n\,\}\,. \tag{15}$$

It is not hard to see that Γ is a closed graph which contains the graph of A_0. Indeed, (u_n, Cu_n) belongs to Γ for every n, and hence $(f, A_0 f)$ belongs to Γ for all f in $\mathcal{F}$. That Γ is a closed subset of $\mathcal{H} \oplus \mathcal{H}$ is also clear. Moreover, if $(0, g)$ belongs to Γ, then $\langle g, u_n \rangle = [g, Cu_n] = [0, u_n] = 0$ holds for all n and hence $g = 0$ since the system $(u_n)_{n=1}^{\infty}$ is complete. It turns out that Γ can be identified with the graph of A_0^*:

LEMMA 2. *The graph of A_0^* is precisely Γ.*

PROOF. If f belongs to $\mathcal{D}(A_0^*)$, then

$$[f, u_n] = \langle f, Cu_n\rangle = \langle f, A_0 u_n\rangle = \langle A_0^* f, u_n\rangle = [A_0^* f, Cu_n]$$

holds for all n, whence $(f, A_0^* f)$ belongs to Γ. Conversely, if (f, g) belongs to Γ, then it follows that

$$\langle f, A_0 u_n\rangle = \langle f, Cu_n\rangle = [f, u_n] = [g, Cu_n] = \langle g, u_n\rangle$$

for all n. Thus $\langle f, A_0 h\rangle = \langle g, h\rangle$ for every h in $\mathcal{F}$ and $h \mapsto \langle A_0 h, f\rangle$ is a bounded linear functional on $\mathcal{F}$. In particular, f belongs to $\mathcal{D}(A_0^*)$. □

In the bounded case, the antilinear involution $J = CA_0$ on $\mathcal{F}$, given explicitly by (12), was of particular importance since it is bounded if and only if A_0 extends to a bounded linear operator on all of $\mathcal{H}$. If A_0 is unbounded, then it turns out that J extends to an involution on $\mathcal{D}(A_0^*)$:

LEMMA 3. *The antilinear operator $J = CA_0^*$ maps $\mathcal{D}(A_0^*)$ onto itself and satisfies $J^2 = I_{\mathcal{D}(A_0^*)}$.*

PROOF. Clearly $J = CA_0^*$ is a well-defined extension of $J = CA_0$ to $\mathcal{D}(A_0^*)$. If $(f, A_0^* f)$ belongs to Γ, then the computation

$$[Jf, u_n] = \langle Jf, Cu_n\rangle = \langle u_n, CJf\rangle = \langle u_n, A_0^* f\rangle = \langle Cu_n, f\rangle = [Cf, Cu_n]$$

shows that (Jf, Cf) belongs to Γ as well. Thus Jf belongs to $\mathcal{D}(A_0^*)$ and $A_0^*(Jf) = Cf$. Moreover, we also see that $J^2 f = CA_0^*(Jf) = C^2 f = f$ for all f in $\mathcal{D}(A_0^*)$ and hence $J^2 = I_{\mathcal{D}(A_0^*)}$. □

Since A_0^* is a closed operator, its domain is complete with respect to the graph norm $\|f\|_{A_0^*}^2 = \|A_0^* f\|^2 + \|f\|^2$ on $\mathcal{D}(A_0^*)$. Let $\mathcal{H}_{\boldsymbol{R}}$ denote the $\boldsymbol{R}$-linear manifold of vectors whose formal skew-Fourier coefficients are real:

$$\mathcal{H}_{\boldsymbol{R}} = \{f \in \mathcal{H};\ [f, u_n] \in \boldsymbol{R} \quad \text{for all}\ \ n\}\,.$$

We note that $\mathcal{H}_{\boldsymbol{R}} \subset \mathcal{D}(A_0^*)$, for if f belongs to $\mathcal{H}_{\boldsymbol{R}}$, then $[f, u_n] = [Cf, Cu_n]$ holds for all n whence f belongs to $\mathcal{D}(A_0^*)$ by Lemma 2.

In light of (12), one suspects that $\mathcal{H}_{\boldsymbol{R}}$ is fixed by J. Indeed, this is true since

$$[Jf, u_n] = \langle CA_0^* f, Cu_n\rangle = \langle u_n, A_0^* f\rangle = \langle Cu_n, f\rangle = \overline{[f, u_n]}$$

holds for all n. Thus J extends to all of $\mathcal{D}(A_0^*)$ and fixes those vectors f whose formal skew-Fourier coefficients $[f, u_n]$ are real. This is despite the fact that vectors in $\mathcal{D}(A_0^*)$ do not necessarily enjoy norm convergent skew-Fourier expansions. Moreover, we also note that $\|A_0^* x\| = \|x\|$ for any x in $\mathcal{H}_{\boldsymbol{R}}$ since $A_0^* = CJ$ and C is isometric.

Our next several lemmas concern the relationship between the conjugate-linear involution J and the structure of $\mathcal{D}(A_0^*)$ as an $\boldsymbol{R}$-linear space.

LEMMA 4. *The orthogonal decomposition*

$$\mathcal{D}(A_0^*) = \mathcal{H}_{\boldsymbol{R}} \oplus_{\boldsymbol{R}} i\mathcal{H}_{\boldsymbol{R}}$$

holds, where the orthogonal direct sum is taken with respect to the real part $\mathrm{Re}\langle\,\cdot\,,\,\cdot\,\rangle_{A_0^*}$ *of the inner product associated to the graph norm of* $\mathcal{D}(A_0^*)$*. Moreover, the antilinear involution* J *restricts to the identity* $I_{\mathcal{H}_{\boldsymbol{R}}}$ *on* $\mathcal{H}_{\boldsymbol{R}}$*.*

PROOF. The last statement of the lemma has already been proved. Now note that if f is in $\mathcal{D}(A_0^*)$, then we may write $f = (1/2)(f + Jf) + i(1/2i)(f - Jf)$. Using the fact that $J = CA_0^*$, a routine calculation shows that both terms $(1/2)(f + Jf)$ and $(1/2i)(f - Jf)$ belong to $\mathcal{H}_{\boldsymbol{R}}$. For every pair of vectors x, y in $\mathcal{H}_{\boldsymbol{R}}$, we have

$$\begin{aligned}\langle x, iy\rangle_{A_0^*} &= \langle A_0^*x, iA_0^*y\rangle + \langle x, iy\rangle\\ &= -i\{\langle CJx, CJy\rangle + \langle x, y\rangle\}\\ &= -i\{\langle Cx, Cy\rangle + \langle x, y\rangle\}\\ &= -i\{\langle x, y\rangle + \overline{\langle x, y\rangle}\}\end{aligned}$$

and hence $\mathrm{Re}\langle x, iy\rangle_{A_0^*} = 0$. □

It follows from the preceding lemma that each f in $\mathcal{D}(A_0^*)$ can be written in the form $f = x + iy$ with x, y in $\mathcal{H}_{\boldsymbol{R}}$. With respect to this decomposition, the involution $J = CA_0^*$ assumes the simple form

$$J(x + iy) = x - iy\,, \qquad x, y \in \mathcal{H}_{\boldsymbol{R}}\,. \tag{16}$$

Of course, if J is bounded (that is, if any of the equivalent conditions listed in Theorem 2 hold), then the vectors x and y can be developed into norm convergent skew-Fourier series whose coefficients $[x, u_n]$ and $[y, u_n]$ are real for every n.

If J is unbounded, the skew-Cartesian decomposition (16) still holds, despite the fact that convenient series developments for x, y and $x + iy$ are no longer at hand. Although J acts isometrically on $\mathcal{H}_{\boldsymbol{R}}$, the quantities $\|x\|$ and $\|y\|$ cannot be estimated from $\|x + iy\|$ in a uniform manner and J is far from isometric on $\mathcal{H}$ itself.

On the other hand, J acts isometrically on $\mathcal{D}(A_0^*)$ if we instead consider the graph norm on $\mathcal{D}(A_0^*)$:

LEMMA 5. *J is isometric with respect to the graph norm of $\mathcal{D}(A_0^*)$.*

PROOF. A short calculation implies that

$$\|Cx \pm iCy\|^2 + \|x \pm iy\|^2 = 2(\|x\|^2 + \|y\|^2)$$

for any x, y in $\mathcal{H}$. Moreover, for each pair x, y in $\mathcal{H}_{\boldsymbol{R}}$ we have

$$A_0^*(x + iy) = A_0^*x + iA_0^*y = CJx + iCJy = Cx + iCy\,.$$

Putting the preceding equations together yields

$$\|A_0^*(x + iy)\|^2 + \|x + iy\|^2 = \|A_0^*(x - iy)\|^2 + \|x - iy\|^2\,,$$

which is equivalent to saying that $\|Jf\|_{A_0^*} = \|f\|_{A_0^*}$. □

We wish now to consider the density of the $\boldsymbol{R}$-linear manifold $\mathcal{H}_{\boldsymbol{R}} \cap \mathcal{F}$. As we will see, this is intimately connected with the question of whether A_0 is essentially selfadjoint. To this end, let $\ominus_{\boldsymbol{R}}$ denote the orthogonal complement with respect to the real inner product $\operatorname{Re}\langle\cdot\,,\,\cdot\rangle$ on $\mathcal{H}$ (as opposed to $\operatorname{Re}\langle\cdot\,,\,\cdot\rangle_{A_0^*}$ on $\mathcal{D}(A_0^*)$) and consider the following lemma:

LEMMA 6. $\mathcal{H} \ominus_{\boldsymbol{R}} (\mathcal{H}_{\boldsymbol{R}} \cap \mathcal{F}) = iC\mathcal{H}_{\boldsymbol{R}}$.

PROOF. If f belongs to $\mathcal{H} \ominus_{\boldsymbol{R}} (\mathcal{H}_{\boldsymbol{R}} \cap \mathcal{F})$, then $\operatorname{Re}\langle f, u_n\rangle = 0$ for every n since the vectors u_n clearly belong to $\mathcal{H}_{\boldsymbol{R}} \cap \mathcal{F}$. This implies $\operatorname{Re}[Cf, u_n] = 0$ for all n and hence iCf belongs to $\mathcal{H}_{\boldsymbol{R}}$, whence f belongs to $iC\mathcal{H}_{\boldsymbol{R}}$. Conversely, suppose that $g = iCf$ for some f in $\mathcal{H}_{\boldsymbol{R}}$ and note that $\operatorname{Re}\langle u_n, g\rangle = -\operatorname{Re} i\langle u_n, Cf\rangle = -\operatorname{Re} i[f, u_n] = 0$ for every n. This implies that $\operatorname{Re}\langle g, h\rangle = 0$ for any h in $\mathcal{H}_{\boldsymbol{R}} \cap \mathcal{F}$, which concludes the proof. □

We will henceforth let $A = A_0^*$, so that A is the closed extension of A_0 possessing the graph Γ as defined in (15). Let us consider also the closure $\mathcal{H}_{\boldsymbol{R}}^0$ of $\mathcal{H}_{\boldsymbol{R}} \cap \mathcal{F}$ in $\mathcal{H}$, so that $\mathcal{H}_{\boldsymbol{R}}^0$ is the closed subspace spanned by the linear manifold of finitely supported vectors whose (formal) skew-Fourier coefficients are real. Finally, let us denote by $\operatorname{cl}_{A_0}\mathcal{F}$ the closure of $\mathcal{F}$ in the graph norm of $\operatorname{cl} A_0$.

Since $\|Ax\| = \|x\|$ for $x \in \mathcal{H}_{\boldsymbol{R}} \cap \mathcal{F}$, it follows that $\mathcal{H}_{\boldsymbol{R}}^0$ must be contained in $\operatorname{cl}_{A_0}\mathcal{F}$. We therefore deduce that

$$\mathcal{H}_{\boldsymbol{R}}^0 + i\mathcal{H}_{\boldsymbol{R}}^0 \subseteq \operatorname{cl}_{A_0}\mathcal{F} \subseteq \mathcal{D}(A)\,. \tag{17}$$

Putting together the observations above we can state the following selfadjointness criteria:

THEOREM 5. *With the notation introduced above, the following are equivalent:*

(i) $A(=A_0^*)$ *is self-adjoint.*

(ii) *There are no nontrivial solutions to the equations* $Cx = \pm ix$, *where* x *belongs to* $\mathcal{H}_{\boldsymbol{R}}$.

(iii) *The space* $\mathcal{H}_{\boldsymbol{R}} \cap \mathcal{F}$ *is norm dense in* $\mathcal{H}_{\boldsymbol{R}}$.

(iv) *The sequence* $(u_n + Cu_n)_{n=1}^\infty$ *is complete in* $\mathcal{H}$.

(v) $A_0^* = \operatorname{cl}(A_0)$.

(vi) $\mathcal{F}$ *is dense in the graph norm of* $\mathcal{D}(A)$.

(vii) $\mathcal{H}_{\boldsymbol{R}}^0 + i\mathcal{H}_{\boldsymbol{R}}^0 = \mathcal{D}(A)$.

PROOF. (i) ⇒ (ii) Since $Cx = CJx = A_0^*x$ for any x in $\mathcal{H}_{\boldsymbol{R}}$, it follows that the equations $Cx = \pm ix$ have no nontrivial solutions since A_0^* is selfadjoint.

(ii) ⇒ (i) Suppose that the equations $Cx = \pm ix$ admits no nontrivial solutions in $\mathcal{H}_{\boldsymbol{R}}$. To show that A_0^* is selfadjoint, we will show that the equations $A_0^*f = \pm if$ have no nontrivial solutions in $\mathcal{D}(A_0^*)$. If $A_0^*f = if$ holds, then writing $f = x + iy$ with x, y in $\mathcal{H}_{\boldsymbol{R}}$, it follows that $Cx + iCy = ix - y$, or equivalently

$$(C - i)x = (C + i)(iy)\,. \tag{18}$$

Simple algebra yields the equations

$$(C - i)(C + i) = -2i(C + i)\,, \tag{19}$$

$$(C + i)(C - i) = 2i(C - i)\,, \tag{20}$$

$$(C - i)^2 = 0\,, \tag{21}$$

$$(C + i)^2 = 0\,. \tag{22}$$

Using (20), (18) and (22) we find that $2i(C-i)x = (C+i)^2(iy) = 0$. This implies that $Cx = ix$ whence $x = 0$. Similarly, using (19), (18) and (21) we find that and $-2i(C + i)(iy) = (C - i)^2 x = 0$. This reveals that $Cy = iy$ whence $y = 0$. Therefore the equation $A_0^* f = if$ admits no nontrivial solutions in $\mathcal{D}(A_0^*)$. A similar argument applies to $A_0^* f = -if$ and thus A_0^* is selfadjoint.

(iii) $\Rightarrow$ (ii) Suppose that $\mathcal{H}_{\boldsymbol{R}} \cap \mathcal{F}$ is norm dense in $\mathcal{H}_{\boldsymbol{R}}$. If x belongs to $\mathcal{H}_{\boldsymbol{R}} \cap \mathcal{F}$, then x is of the form $x = \sum_{n=1}^m a_n u_n$ where each a_n is real. Since $\langle x, Cx\rangle = [x, x] = \sum_{n=1}^m a_n^2 \geq 0$, it follows that $\langle x, Cx\rangle \geq 0$ holds on $\mathcal{H}_{\boldsymbol{R}}$ by continuity. This inequality clearly precludes the possibility of either of the equations $Cx = \pm ix$ holding for a nonzero vector x in $\mathcal{H}_{\boldsymbol{R}}$.

(i) $\Rightarrow$ (iii) We prove the contrapositive of this implication. If $\mathcal{H}_{\boldsymbol{R}} \cap \mathcal{F}$ is not norm dense in $\mathcal{H}_{\boldsymbol{R}}$, then there exists a nonzero vector $x \in \mathcal{H}_{\boldsymbol{R}} \ominus_{\boldsymbol{R}} (\mathcal{H}_{\boldsymbol{R}} \cap \mathcal{F})$. In view of Lemma 6, x must belong to $iC\mathcal{H}_{\boldsymbol{R}}$. In other words, $x = iCy = iAy$ for some y in $\mathcal{H}_{\boldsymbol{R}}$. Therefore

$$Ax = Cx = C(iCy) = -iy\,. \tag{23}$$

Taken together, the equations $x = iAy$ and $Ax = -iy$ imply that $(A - i)x = (A - i)y$. It follows that either the kernel of $(A - i)$ is nontrivial (hence A is not selfadjoint) or $x = y$. In the latter case, (23) implies that $(A + i)x = 0$, whence A is not selfadjoint.

(i) $\Rightarrow$ (iv) Since A is selfadjoint i.e., A is the closure of A_0, $\mathcal{F}$ is dense in $\mathcal{D}(A)$ with respect to the graph norm of A. Since $A \geq 0$, the map $(I + A) : \mathcal{D}(A) \to \mathcal{H}$ is bijective and continuous (from the graph norm on $\mathcal{D}(A)$ to the norm topology on $\mathcal{H}$). Therefore $(I + A)\mathcal{F}$ is dense in $\mathcal{H}$ and thus the sequence $(u_n + Cu_n)_{n=1}^\infty$ is complete.

(iv) $\Rightarrow$ (i) Conversely, if the sequence $(I + A)u_n = u_n + Cu_n$ is complete in $\mathcal{H}$, then $\mathcal{F}$ is dense in $\mathcal{D}(A)$ with respect to the graph norm of A. It therefore follows that $\operatorname{cl} A_0 = A$ and that A is selfadjoint.

(v) $\Leftrightarrow$ (vi) These are simply restatements of each other.

(i) $\Leftrightarrow$ (vi) If $\mathcal{F}$ is dense in the graph norm of A, then $A = \operatorname{cl} A_0$. As before $A^* = A$ would imply $\operatorname{cl} A_0 = A_0^*$. On the other hand, $\mathcal{F}$ is dense in the graph norm of $\operatorname{cl} A_0$, by the very definition of A_0.

(i) $\Leftrightarrow$ (vii) By (17), $\mathcal{H}_{\boldsymbol{R}}^0 + i\mathcal{H}_{\boldsymbol{R}}^0 = \mathcal{D}(A)$ if and only if A_0^* is selfadjoint. Indeed, in one direction, if $\operatorname{cl}_{A_0} \mathcal{F} = \mathcal{D}(A)$, then we know that $\operatorname{cl} A_0 = A = A_0^*$. Conversely, if $A_0^* = \operatorname{cl} A_0$, then we know that $\mathcal{H}_{\boldsymbol{R}}^0 = \mathcal{H}_{\boldsymbol{R}}$ and $\mathcal{D}(A_0^*) = \mathcal{H}_{\boldsymbol{R}} + i\mathcal{H}_{\boldsymbol{R}}$. □

Before moving on, let us now briefly summarize a few points. In part, Theorem 2 and Theorem 5 assert:

(a) A_0 is *bounded* if and only if every vector f in $\mathcal{H}$ can be developed in a norm-convergent skew Fourier expansion: $f = \sum_{n=1}^{\infty}[f, u_n]u_n$. This is also equivalent to $(u_n)_{n=1}^{\infty}$ being a Riesz basis for $\mathcal{H}$.

(b) A_0 is *essentially selfadjoint* if and only if every vector f in $\mathcal{H}_{\boldsymbol{R}}$ (i.e., so that $[f, u_n] \in \boldsymbol{R}$ for all n) can be approximated by finite sums $\sum_{n=1}^{m} a_n u_n$ where the a_n are real. A simple algebraic criterion for this is that the equation $Cx = ix$ (or $Cx = -ix$) has no nonzero solutions x in $\mathcal{H}_{\boldsymbol{R}}$.

5. The Friedrichs extension and real interpolation. Since the symmetric operator A_0 is non-negative, it admits non-negative selfadjoint extensions (see [16, Thm. X.1] and its corollaries). Recall that the *Friedrichs extension* of A_0, which we will henceforth denote by A, is defined on a domain contained in the closure $\mathcal{H}_{\mathfrak{F}}$ of $\mathcal{D}(A_0) = \mathcal{F}$ with respect to the norm $\|f\|_{\mathfrak{F}}^2 = \langle A_0 f, f\rangle + \|f\|^2$ on $\mathcal{F}$. More precisely, A represents the preceding sesquilinear form in the sense that

$$\|f\|_{\mathfrak{F}}^2 = \langle Af, f\rangle + \|f\|^2 \tag{24}$$

for any f in $\mathcal{D}(A)$. In particular, $\|f\|_{\mathfrak{F}}$ coincides with the graph norm of the selfadjoint operator $\sqrt{A}$ and $\mathcal{H}_{\mathfrak{F}} = \mathcal{D}(\sqrt{A})$ [16, Sec. X.3]. For every finitely supported vector $f = \sum_{n=1}^{m} c_n u_n$, we therefore see that

$$\|f\|_{\mathfrak{F}}^2 = \|f\|^2 + \sum_{n=1}^{m} |c_n(f)|^2 .$$

Since $\mathcal{H}_{\mathfrak{F}}$ is the completion of $\mathcal{F}$ with respect to this norm, we deduce the following lemma:

LEMMA 7. *A vector f in $\mathcal{H}$ belongs to $\mathcal{D}(\sqrt{A})$ if and only if the corresponding sequence $c_n(f) = [f, u_n]$ of skew Fourier coefficients is square summable. Moreover,*

$$\|f\|_{\mathfrak{F}}^2 = \|f\|^2 + \sum_{n=1}^{\infty} |[f, u_n]|^2 .$$

Surprisingly, even if A_0 is unbounded (in other words $(u_n)_{n=1}^{\infty}$ is not a Riesz basis—see Theorem 2), it is possible to interpolate *real* sequences in l^2:

THEOREM 6. *Let $(u_n)_{n=1}^{\infty}$ be a complete C-orthonormal sequence in $\mathcal{H}$. For each sequence of real numbers $(c_n)_{n=1}^{\infty}$ in l^2, there exists a vector f in $\mathcal{H}$ such that* $\operatorname{Re}[f, u_n] = c_n$ *for all n.*

PROOF. Let $L : \mathcal{D}(\sqrt{A}) \to l^2$ be the coefficient map $Lf = ([f, u_n])_{n=1}^{\infty}$ where we consider $\mathcal{D}(\sqrt{A})$ with respect to the norm $\|\cdot\|_{\mathfrak{F}}$. By Lemma 7, $\|Lf\| = \sum_{n=1}^{\infty} |[f, u_n]|^2$ is finite for each f in $\mathcal{D}(\sqrt{A})$ and hence L is bounded by the Uniform Boundeness Principle. Let $c = (c_n)_{n=1}^{\infty}$ be any real sequence in l^2 and consider the element L^*c in $\mathcal{D}(\sqrt{A})$. Since $\langle L^*c, u_n\rangle_{\mathfrak{F}} = \langle c, L(u_n)\rangle_{l^2} = c_n$ holds for all n, (24) tells us that

$$\langle L^*c, Cu_n\rangle_{\mathcal{H}} + \langle L^*c, u_n\rangle_{\mathcal{H}} = \langle L^*c, u_n\rangle_{\mathfrak{F}} = c_n ,$$

which is equivalent to

$$[L^*c, u_n] + \overline{[CL^*c, u_n]} = c_n .$$

A straightforward calculation then shows that

$$\operatorname{Re}[L^*c + CL^*c, u_n] = \operatorname{Re} c_n = c_n$$

and hence the vector $f = (L^*c + CL^*c)$ satisfies $\operatorname{Re}[f, u_n] = c_n$ for all n. □

In general, we cannot interpolate complex l^2-sequences. Moreover, Theorem 6 does not assert any relationship between the l^2-norm of the real sequence $(c_n)_{n=1}^{\infty}$ and the $\mathcal{H}$-norm of the interpolating vector f. In particular, one does not expect such a relationship to hold when A_0 is unbounded.

The following theorem is a simple application of Theorem 6 (and Theorem 1). In particular, note that it applies to *any* nonconstant inner function—we do not have to restrict ourselves to interpolating Blaschke products or any other such subclass.

THEOREM 7. *If φ is a nonconstant inner function, then there exists a subset $E \subset \boldsymbol{D}$ of measure zero such that for each w in $\boldsymbol{D}\backslash E$ the level set $\varphi^{-1}\{w\}$ is nonempty and*

(i) *if z_n is an enumeration of $\varphi^{-1}\{w\}$, then $\varphi'(z_n) \neq 0$ for all n (i.e., φ assumes the value w with multiplicity one at each z_n),*

(ii) *for each real sequence $(a_n)_{n=1}^{\infty}$ in l^2, there exists a function f in $H^2 \ominus \varphi H^2$ such that*

$$\operatorname{Re}\left(\frac{f(z_n)}{\sqrt{\varphi'(z_n)}}\right) = a_n$$

holds for each n.

Recall that the antilinear involution $J = CA_0^*$ leaves invariant both domains of $\mathcal{D}(A_0) = \mathcal{F}$ and $\mathcal{D}(A_0^*)$ and that J is continuous in their respective graph norms. Unfortunately, this is not always the case for the Friedrichs space $\mathcal{H}_{\mathfrak{F}}$:

THEOREM 8. *The antilinear involution J leaves the space $\mathcal{H}_{\mathfrak{F}}$ invariant and is continuous in the norm of $\mathcal{H}_{\mathfrak{F}}$ if and only if A_0 is bounded.*

PROOF. If A_0 is bounded, then J is defined on all of $\mathcal{H}$ and bounded there by $\|A_0\|$ by Theorem 2. This implies that J is bounded by $1 + \|A_0\|$ with respect to the Friedrichs norm.

If J maps $\mathcal{H}_{\mathfrak{F}}$ onto $\mathcal{H}_{\mathfrak{F}}$ and is continuous in the norm of $\mathcal{H}_{\mathfrak{F}}$, then there exists a constant $\gamma > 0$ such that $\|Jf\|_{\mathfrak{F}} \leq \gamma\|f\|_{\mathfrak{F}}$ for every f in $\mathcal{H}_{\mathfrak{F}}$. Since $Ju_n = u_n$ for all n, we see that $[f, u_n] = \overline{[Jf, u_n]}$ for any finitely supported vector f. Noting that $\mathcal{F} \subset \mathcal{H}_{\mathfrak{F}}$, we see that

$$\begin{aligned}
\|A_0 f\|^2 &\leq \sum_{n=1}^{m_f} |[f, u_n]|^2 + \|A_0 f\|^2 = \sum_{n=1}^{m_f} |[Jf, u_n]|^2 + \|CA_0 f\|^2 \\
&= \langle A_0 Jf, Jf\rangle + \|Jf\|^2 = \|Jf\|_{\mathfrak{F}}^2 \leq \gamma^2 \|f\|_{\mathfrak{F}}^2 \\
&= \gamma^2(\langle A_0 f, f\rangle + \|f\|^2) \leq \gamma^2(\|A_0 f\|\|f\| + \|f\|^2)
\end{aligned}$$

for every f in $\mathcal{F}$. If A_0 were unbounded, then there would exist a sequence of unit vectors f_n in $\mathcal{F}$ such that $\|A_0 f_n\| \to \infty$. For sufficiently large n, this would violate the inequality $\|A_0 f_n\|^2 \leq \gamma^2(1 + \|A_0 f_n\|)$. □

Although the original sequence of vectors $(u_n)_{n=1}^{\infty}$ is not even a Bessel sequence in general, the construction of the Friedrichs extension of A_0 provides a canonical orthonormal basis of $\mathcal{H}$:

THEOREM 9. *The system* $(\sqrt{A}u_n)_{n=1}^{\infty}$ *is orthonormal and complete in* $\mathcal{H}$.

PROOF. Since each u_n belongs to $\mathcal{D}(A)$, we have

$$\langle \sqrt{A}u_j, \sqrt{A}u_k \rangle = \langle u_j, Au_k \rangle = \langle u_j, Cu_k \rangle = \delta_{jk}$$

and hence the vectors $(\sqrt{A}u_n)_{n=1}^{\infty}$ are orthonormal. We now show that they are complete. Suppose that g belongs to $\mathcal{H}$ and is orthogonal to each $\sqrt{A}u_n$ (and hence to all of $\mathcal{F}$). Since $\mathcal{F}$ is dense in the Friedrichs space $\mathcal{H}_{\mathfrak{F}} = \mathcal{D}(\sqrt{A})$, it follows that for any f in $\mathcal{H}_{\mathfrak{F}}$, there exists a sequence f_n in $\mathcal{F}$ such that $\|f - f_n\|_{\mathfrak{F}}$ tends to zero. Thus

$$\begin{aligned} |\langle \sqrt{A}f, g \rangle| &\leq |\langle \sqrt{A}f_n, g \rangle| + |\langle \sqrt{A}(f - f_n), g \rangle| \\ &\leq 0 + \|\sqrt{A}(f - f_n)\| \|g\| \\ &\leq \|f - f_n\|_{\mathfrak{F}} \|g\| \end{aligned}$$

and hence $\langle \sqrt{A}f, g \rangle = 0$ for every f in $\mathcal{H}_{\mathfrak{F}}$. This implies that g belongs to $\mathcal{D}(\sqrt{A})$ and hence $\sqrt{A}g = 0$. In particular, we see that $Ag = 0$ and therefore

$$0 = \langle Ag, u_n \rangle = \langle g, Au_n \rangle = \langle g, Cu_n \rangle = [g, u_n]$$

for all n. Since the system $(u_n)_{n=1}^{\infty}$ is complete, it follows that $g = 0$ and hence the system $(\sqrt{A}u_n)_{n=1}^{\infty}$ is also complete. □

We next reverse the preceding computations, giving a general method for producing complete C-orthonormal systems:

THEOREM 10. *Let B be an injective, non-negative selfadjoint operator with dense domain $\mathcal{D}(B)$ in a separable complex Hilbert space $\mathcal{H}$. If $(e_n)_{n=0}^{\infty}$ is an orthonormal basis of $\mathcal{H}$ which satisfies*

(i) $(e_n)_{n=1}^{\infty}$ *is contained into* $\mathcal{D}(B) \cap \mathcal{D}(B^{-1})$,
(ii) *for every finitely supported sequence* $(c_n)_{n=1}^{m}$ *of complex numbers we have*

$$\left\| \sum_{n=1}^{m} \overline{c_n} B^{-1} e_n \right\| = \left\| \sum_{n=1}^{m} c_n B e_n \right\|,$$

then $C(Be_n) = B^{-1}e_n$, for $n = 1, 2, \ldots$, extends by conjugate-linearity to an isometric involution of $\mathcal{H}$, i.e., C is a conjugation on $\mathcal{H}$. Moreover $u_n = Ce_n$ is a complete C-orthonormal system of vectors in $\mathcal{H}$.

PROOF. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be finitely supported sequences of complex numbers. By the definition of C we have

$$C\Big(\sum_{n=1}^{\infty} a_n B e_n\Big) = \sum_{n=1}^{\infty} \overline{a_n} B^{-1} e_n \,.$$

Condition (ii) ensures that C is well-defined, isometric as an $\boldsymbol{R}$-linear map, and can be extended to all of $\mathcal{H}$. Let $[f,g] = \langle f, Cg\rangle$ denote the associated bilinear form. We will show that this form is symmetric. If $f = \sum_{n=1}^{\infty} a_n B e_n$ and $g = \sum_{n=1}^{\infty} b_n B_n e_n$, then

$$[f,g] = \langle f, Cg\rangle = \sum_{j,k=1}^{\infty} a_j b_k \langle Be_j, B^{-1}e_k\rangle = \sum_{j,k=1}^{\infty} a_j b_k \delta_{jk} = \sum_{j=1}^{\infty} a_j b_j \,,$$

which a similar computation reveals also equals $[g,f]$.

Next we observe that both systems $(Be_n)_{n=1}^{\infty}$ and $(B^{-1}e_n)_{n=1}^{\infty}$ are complete in $\mathcal{H}$ due to the fact that B is selfadjoint and injective. For arbitrary vectors x and y we therefore have

$$\langle x, Cy\rangle = \langle y, Cx\rangle \,, \quad \langle Cx, Cy\rangle = \langle y, x\rangle \,.$$

These identities imply $C^2 = I$ and the proof is complete. $\square$

In light of the preceding material, we see that there is a bijective correspondence between

(i) pairs $(C, (u_n)_{n=1}^{\infty})$ consisting of a conjugation C and a complete C-orthonormal system $(u_n)_{n=1}^{\infty}$

(ii) pairs $(B, (e_n)_{n=1}^{\infty})$ consisting of an invertible, non-negative selfadjoint operator B, an orthonormal basis $(e_n)_{n=1}^{\infty}$ satisfying conditions (i) and (ii) which is dense in $\mathcal{D}(B)$ with respect to the graph norm.

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