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COMMUTATION RELATIONS OF HECKE OPERATORS FOR ARAKAWA LIFTING

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Abstract. T. Arakawa, in his unpublished note, constructed and studied a theta lifting from elliptic cusp forms to automorphic forms on the quaternion unitary group of signature $(1, q)$. The second named author proved that such a lifting provides bounded (or cuspidal) automorphic forms generating quaternionic discrete series. In this paper, restricting ourselves to the case of $q = 1$, we reformulate Arakawa's theta lifting as a theta correspondence in the adelic setting and determine a commutation relation of Hecke operators satisfied by the lifting. As an application, we show that the theta lift of an elliptic Hecke eigenform is also a Hecke eigenform on the quaternion unitary group. We furthermore study the spinor L-function attached to the theta lift.

1. Introduction. The prototype of our study in this paper is the classical work [E] by Eichler on the commutation relation of Hecke operators for theta series associated with spherical polynomials on a definite quaternion algebra over $\boldsymbol{Q}$. This result has been generalized in various cases. For example, Yoshida [Y] constructed a theta lifting from a pair of automorphic forms on a multiplicative group of a definite quaternion algebra to holomorphic Siegel modular forms of degree two, and determined a commutation relation for his lift. Kudla [Ku] studied such a relation for a theta lifting from elliptic cusp forms to holomorphic automorphic forms on $SU(2, 1)$. Moreover, we note that Rallis [Ra] investigated in great generality a commutation relation via the Weil representation for symplectic-orthogonal dual pairs. Our concern here is a theta lifting from elliptic cusp forms to automorphic forms on the quaternion unitary group $Sp(1, q)_{\boldsymbol{R}}$ of signature $(1, q)$, originally formulated by Arakawa ([Ar-1]). We study this lifting for the case of $q = 1$ along the same line as [Y] and [Ku].

Let us recall that Arakawa formulated the theta lifting mentioned above by considering the restriction of a theta correspondence of $SL_2(\boldsymbol{R}) \times SO(4, 4q)$ to $SL_2(\boldsymbol{R}) \times Sp(1, q)_{\boldsymbol{R}}$ (cf. [Ar-1], [N-1] and [N-3]). We henceforth confine ourselves to the case of $q = 1$. It turned out that Arakawa's formulation is not appropriate for proving a commutation relation, since we do not have sufficient Hecke operators. Following T. Ikeda's suggestion, we formulate our lift as a theta correspondence between $(GL_2 \times B^{\times})$ and $GSp(1, 1)$, where B is a definite quaternion algebra over $\boldsymbol{Q}$. This amounts to the same as taking a certain *average* of Arakawa's

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original lifts over the ideal classes of B. In this setting, we have enough Hecke operators to show a good commutation relation.

On the other hand, we note that our lift can be viewed as a theta correspondence of similitude groups. For references in this direction see [Ge], [H-K], [Ro], [Sz] and [W] etc. In fact, the pair of the groups $(GL_2 \times B^\times)$ and $GSp(1,1)$ comes from the dual reductive pair $O^*(4) \times Sp(1,1)$, where $O^*(4)$ is the inner form of the orthogonal group of degree four which is realized as the quaternion unitary group associated with a skew-Hermitian form of degree two over B. In addition we remark that there is a work by Pitale [P] on a lifting from elliptic Maass forms of weight $1/2$ to some class one automorphic forms on $GSp(1,1) \simeq GSpin(1,4)$, which is inspired by Saito-Kurokawa lifting or Duke-Imamoḡlu-Ikeda lifting on holomorphic Siegel modular forms.

We now explain more precisely our reformulation of the lifting, which is given in the adelic setting. Let $\kappa > 6$ be an even integer and D a divisor of the discriminant d_B of B. We denote by $S_\kappa(D)$ the space of elliptic cusp forms on $GL_2(\boldsymbol{A})$ of weight κ and level D, and let $\mathcal{A}_\kappa$ be the space of automorphic forms on $B_{\boldsymbol{A}}^\times$ (for the definition of $S_\kappa(D)$ and $\mathcal{A}_\kappa$, see Definition 2.2). Furthermore, using a metaplectic representation of $GSp(1,1)_{\boldsymbol{A}} \times GL_2(\boldsymbol{A}) \times B_{\boldsymbol{A}}^\times$ (cf. Sections 3 and 4), we define a theta kernel θ^κ on $GSp(1,1)_{\boldsymbol{A}} \times GL_2(\boldsymbol{A}) \times B_{\boldsymbol{A}}^\times$ under a special choice of a test function (cf. (4.1)). For $(f, f') \in S_\kappa(D) \times \mathcal{A}_\kappa$, we construct an automorphic form $\mathcal{L}(f, f')$ on $GSp(1,1)_{\boldsymbol{A}}$ by integrating (f, f') against θ^κ (cf. (4.2)). Then $\mathcal{L}(f, f')$ belongs to the space $\mathcal{S}_\kappa$ of bounded (or cuspidal) automorphic forms on $GSp(1,1)_{\boldsymbol{A}}$ given in Definition 2.1 (cf. Theorem 4.1). Note that $F \in S_\kappa$ generates, at the infinite place, a quaternionic discrete series in the sense of Gross and Wallach [G-W] (cf. [N-2, Theorem 8.7]).

Our main result is a formula for Hecke eigenvalues of $\mathcal{L}(f, f')$ stated in Theorem 5.1. For all finite places of $\boldsymbol{Q}$, we provide such a formula in terms of Hecke eigenvalues of f and f'. This follows from our formula for the commutation relations of Hecke operators in Propositions 6.1 and 6.2. Then we discuss an application of this formula to the study of the spinor L-functions of $\mathcal{L}(f, f')$. We define an Euler factor of the spinor L-function at a prime $p \nmid d_B$ (resp. $p \mid d_B$), using the formula for the denominator of the Hecke series by Shimura [Shim-1, Theorem 2] (resp. Hina and Sugano [H-S, Section 4, (I)], [Su, (1-34)]). Among our formulas for the L-function, the case $D = 1$ seems to be the most interesting. Indeed, if we assume that f and f' are Hecke eigenforms, the spinor L-function $L(\mathcal{L}(f, f'), \mathrm{spin}, s)$ for that case admits the following simple decomposition (cf. Corollary 5.3)

$$L(\mathcal{L}(f, f'), \mathrm{spin}, s) = L(f, s) L^{d_B}(f', s),$$

where $L(f, s)$ (resp. $L^{d_B}(f', s)$) denotes Hecke's classical L-function for f (resp. some partial L-function for f' whose Euler factors range only over $p \nmid d_B$).

This paper is organized as follows. In Section 2 we give basic notations and define the automorphic forms we need. In Section 3 we introduce a metaplectic representation of $GSp(1,1) \times GL_2 \times B^\times$ over local fields. Then we define a global metaplectic representation of the adele group and provide the adelic reformulation of the Arakawa lifting for the case of $q =$

1 in Section 4. Section 5 is devoted to the statement of our main results, i.e., Hecke eigenvalues and spinor L-functions for the lifting. In Section 6 we state our result on the commutation relations of Hecke operators, from which our main results are deduced immediately. Finally, we prove the commutation relations at unramified finite places (resp. ramified finite places) of $\boldsymbol{Q}$ in Section 7 (resp. Section 8).

NOTATION. For an algebraic group $\mathcal{G}$ over $\boldsymbol{Q}$, $\mathcal{G}_v$ stands for the group of $\boldsymbol{Q}_v$-points of $\mathcal{G}$, where $\boldsymbol{Q}_v$ denotes the p-adic field (resp. the field of real numbers) when $v = p$ is a finite place of $\boldsymbol{Q}$ (resp. $v = \infty$). By $\mathcal{G}_{\boldsymbol{A}}$ (resp. $\mathcal{G}_{\boldsymbol{A},f}$), we denote the adelization of $\mathcal{G}$ (resp. the group of finite adeles in $\mathcal{G}_{\boldsymbol{A}}$). Let ψ be the additive character of $\boldsymbol{Q}_{\boldsymbol{A}}/\boldsymbol{Q}$ such that $\psi(x_\infty) = \mathbf{e}(x_\infty)$ for $x_\infty \in \boldsymbol{R}$, where we put $\mathbf{e}(z) = \exp(2\pi i z)$ for $z \in \boldsymbol{C}$. We denote by ψ_v the restriction of ψ to $\boldsymbol{Q}_v$ for a place v of $\boldsymbol{Q}$. We denote by $\mathrm{diag}(a_1, \ldots, a_n)$ the diagonal matrix of degree n with the i-th diagonal component a_i. For a finite dimensional vector space V over $\boldsymbol{Q}_v$, we denote by $\mathcal{S}(V)$ the space of Schwartz-Bruhat functions on V. We also let $\mathcal{S}(\boldsymbol{Q}_p^\times)$ be the space of locally constant and compactly supported functions on $\boldsymbol{Q}_p^\times$. Given a condition S, we set

$$\delta(S) := \begin{cases} 1 & \text{if } S \text{ is satisfied}, \\ 0 & \text{otherwise.} \end{cases}$$

2. Automorphic forms. Let B be a definite quaternion algebra over $\boldsymbol{Q}$. In what follows, we fix an identification between $B_\infty := B \otimes_{\boldsymbol{Q}} \boldsymbol{R}$ and the Hamilton quaternion algebra $\boldsymbol{H}$, and an embedding $\boldsymbol{H} \hookrightarrow M_2(\boldsymbol{C})$. Let $B \ni b \mapsto \bar{b} \in B$ be the main involution of B, and put $\mathrm{tr}(b) := b + \bar{b}$ and $n(b) := b\bar{b}$ for $b \in B$. Let $B^\times := B \setminus \{0\}$ be the multiplicative group of B. The center $Z(B^\times)$ of $B^\times$ is $\boldsymbol{Q}^\times \cdot 1$. Let d_B be the discriminant of B. By definition, d_B is the product of primes p such that $B_p := B \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_p$ is a division algebra.

We let $G = GSp(1, 1)$ be an algebraic group over $\boldsymbol{Q}$ defined by

$$G_{\boldsymbol{Q}} = \{g \in M_2(B) \mid {}^t\bar{g} Q g = \nu(g) Q,\ \nu(g) \in \boldsymbol{Q}^\times\},$$

where $Q = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Denote by Z_G the center of G.

The Lie group $G_\infty^1 := \{g \in G_\infty \mid \nu(g) = 1\}$ acts on the hyperbolic 4-space $\mathcal{X} := \{z \in \boldsymbol{H} \mid \mathrm{tr}(z) > 0\}$ by linear fractional transformations

$$g \cdot z := (az + b)(cz + d)^{-1}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\infty^1, \quad z \in \mathcal{X}.$$

Let $\mu : G_\infty^1 \times \mathcal{X} \to \boldsymbol{H}^\times$ be the automorphy factor given by $\mu(\left(\begin{smallmatrix} a & d \\ c & d \end{smallmatrix}\right), z) := cz + d$. The stabilizer subgroup K_∞ of $z_0 := 1 \in \mathcal{X}$ in G_∞^1 is a maximal compact subgroup of G_∞^1, which is isomorphic to $Sp^*(1) \times Sp^*(1)$, where $Sp^*(1) := \{z \in \boldsymbol{H} \mid n(z) = 1\}$.

Let κ be a positive integer. Denote by $(\sigma_\kappa, V_\kappa)$ the representation of $\boldsymbol{H}$ given as

$$\boldsymbol{H} \hookrightarrow M_2(\boldsymbol{C}) \to \mathrm{End}(V_\kappa),$$

where the second arrow indicates the κ-th symmetric power representation of $M_2(\boldsymbol{C})$. Then

$$\tau_\kappa(k_\infty) := \sigma_\kappa(\mu(k_\infty, z_0)), \quad k_\infty \in K_\infty$$

gives rise to an irreducible representation of K_∞ of dimension $\kappa + 1$.

Define $\omega_\kappa : G^1_\infty \to \mathrm{End}(V_\kappa)$ by

$$\omega_\kappa(g) := \sigma_\kappa(D(g))^{-1} n(D(g))^{-1}, \quad g \in G^1_\infty,$$

where $D(g) := 2^{-1}(g \cdot z_0 + 1)\mu(g, z_0)$. It is known that ω_κ is a matrix coefficient of the discrete series representation with minimal K_∞-type (τ_κ, V_κ) (cf. [Ar-3, Section 2.6]). This discrete series is a quaternionic discrete series in the sense of Gross and Wallach [G-W]. We note that ω_κ is integrable if $\kappa > 4$ (cf. [Ar-2, Lemma 1.1 (ii)], [Ar-3, Lemma 2.10 (ii)]).

Throughout the paper, we fix a maximal order $\mathcal{O}$ of B. We also fix a divisor D of d_B and let $\mathfrak{A}$ be a two-sided ideal of $\mathcal{O}$ such that, for each $p < \infty$, the p-adic completion $\mathfrak{A}_p$ of $\mathfrak{A}$ is equal to $\mathcal{O}_p$ (resp. $\mathfrak{P}_p$) if $p \nmid D^{-1}d_B$ (resp. $p \mid D^{-1}d_B$), where $\mathfrak{P}_p$ denotes the maximal ideal of $\mathcal{O}_p$ for $p \mid d_B$.

Let $L := {}^t(\mathcal{O} \oplus \mathfrak{A}^{-1})$. Then L is a maximal lattice of $B^{\oplus 2}$. Namely, if L' is a lattice of $B^{\oplus 2}$ with $L \subset L'$ and $\{{}^t\bar{X}QX \mid X \in L'\} = \boldsymbol{Z}$, we have $L' = L$. For a finite place p of $\boldsymbol{Q}$, $K_p = \{k \in G_p \mid kL_p = L_p\}$ is a maximal compact subgroup of G_p, where $L_p := L \otimes_{\boldsymbol{Z}} \boldsymbol{Z}_p$. We set $K_f := \prod_{p<\infty} K_p$.

DEFINITION 2.1. For an even integer $\kappa > 4$, let $\mathcal{S}_\kappa$ be the space of smooth functions $F : G_{\boldsymbol{A}} \to V_\kappa$ satisfying the following conditions:

1. For any $(z, \gamma, g, k_f, k_\infty) \in Z_{G,\boldsymbol{A}} \times G_{\boldsymbol{Q}} \times G_{\boldsymbol{A}} \times K_f \times K_\infty$, we have
$$F(z\gamma g k_f k_\infty) = \tau_\kappa(k_\infty)^{-1} F(g).$$
2. F is bounded.
3. For any fixed $(g_f, g_\infty) \in G_{\boldsymbol{A},f} \times G_\infty$, we have
$$c_\kappa \int_{G^1_\infty} \omega_\kappa(h_\infty^{-1} g_\infty) F(g_f h_\infty) dh_\infty = F(g_f g_\infty),$$

where $c_\kappa := 2^{-4}\pi^{-2}\kappa(\kappa-1)$ and dh_∞ is the normalized invariant measure of G^1_∞ introduced in [Ar-3, 1.2].

Here we note that this automorphic form has been shown to be cuspidal (cf. [Ar-3, Proposition 3.1]) and generates a quaternionic discrete series at the infinite place (cf. [N-2, Theorem 8.7]).

Next, let H and H' be algebraic groups over $\boldsymbol{Q}$ defined by $H_{\boldsymbol{Q}} = GL_2(\boldsymbol{Q})$ and $H'_{\boldsymbol{Q}} = B^\times$, respectively, and denote by Z_H and $Z_{H'}$ the centers of H and H', respectively. We define an action of $SL_2(\boldsymbol{R})$ on the complex upper half plane $\mathfrak{h} := \{\tau \in \boldsymbol{C} \mid \mathrm{Im}(\tau) > 0\}$ as usual. Let $U_\infty := \{h \in SL_2(\boldsymbol{R}) \mid h \cdot i = i\} = SO(2)$ and $U'_\infty := \{h' \in \boldsymbol{H} \mid n(h') = 1\} = Sp^*(1)$. Moreover, we put $U_f = \prod_{p<\infty} U_p$ and $U'_f = \prod_{p<\infty} U'_p$, where $U_p := \{u = (u_{ij}) \in GL_2(\boldsymbol{Z}_p) \mid u_{21} \in D\boldsymbol{Z}_p\}$ and $U'_p := \mathcal{O}_p^\times$.

DEFINITION 2.2. (1) Let $S_\kappa(D)$ be the space of smooth functions f on $H_{\boldsymbol{A}}$ satisfying the following conditions:

1. We have $f(z\gamma h u_f u_\infty) = j(u_\infty, i)^{-\kappa} f(h)$ for any $(z, \gamma, h, u_f, u_\infty) \in Z_{H,\boldsymbol{A}} \times H_{\boldsymbol{Q}} \times H_{\boldsymbol{A}} \times U_f \times U_\infty$.

2. For any fixed $h_f \in H_{\boldsymbol{A},f}$, $\mathfrak{h} \ni h_\infty \cdot i \mapsto j(h_\infty, i)^\kappa f(h_f h_\infty)$ is holomorphic for $h_\infty \in SL_2(\boldsymbol{R})$.

3. f is bounded.

Here $j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau) := c\tau + d$ denotes the standard $\boldsymbol{C}$-valued automorphy factor of $SL_2(\boldsymbol{R}) \times \mathfrak{h}$.

(2) Let $\mathcal{A}_\kappa$ denote the space of smooth V_κ-valued functions f' on $H'_{\boldsymbol{A}}$ such that

$$f'(z'\gamma' h' u'_f u'_\infty) = \sigma_\kappa(u'_\infty)^{-1} f(h')$$

holds for any $(z', \gamma', h', u'_f, u'_\infty) \in Z_{H',\boldsymbol{A}} \times H'_{\boldsymbol{Q}} \times H'_{\boldsymbol{A}} \times U'_f \times U'_\infty$.

3. Metaplectic representation. In this section, we fix a place v of $\boldsymbol{Q}$. When $v = p$ is a finite place (resp. $v = \infty$), $|*|_v$ denotes the p-adic valuation (resp. the usual absolute value for $\boldsymbol{R}$). For $X = {}^t(x, y) \in B_v^{\oplus 2}$, we put $X^* := (\bar{x}, \bar{y})$. For a finite place p of $\boldsymbol{Q}$, let $\boldsymbol{V}_p$ be the space of functions on $B_p^{\oplus 2} \times \boldsymbol{Q}_p^\times$ generated by $\varphi_1(X)\varphi_2(t)$, where $\varphi_1 \in \mathcal{S}(B_p^{\oplus 2})$ $(= \mathcal{S}(\boldsymbol{Q}_p^8))$ and $\varphi_2 \in \mathcal{S}(\boldsymbol{Q}_p^\times)$. We also let $\boldsymbol{V}_\infty$ be the space of smooth functions $\varphi\colon B_\infty^{\oplus 2} \times \boldsymbol{Q}_\infty^\times = \boldsymbol{H}^{\oplus 2} \times \boldsymbol{R}^\times \to \mathrm{End}(V_\kappa)$ such that, for any fixed $t \in \boldsymbol{R}^\times$, $X \mapsto \varphi(X, t)$ is in $\mathcal{S}(\boldsymbol{H}^{\oplus 2}) \otimes \mathrm{End}(V_\kappa)$ $(= \mathcal{S}(\boldsymbol{R}^8) \otimes \mathrm{End}(V_\kappa))$. We define a partial Fourier transform $\mathcal{I}$ by

$$\mathcal{I}\varphi(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, t) = \int_{B_v} \psi_v(-t\,\mathrm{tr}(\bar{y}x_1))\varphi(\begin{pmatrix} y \\ x_2 \end{pmatrix}, t)dy\,, \qquad \varphi \in \boldsymbol{V}_v\,,$$

where dy is the Haar measure on B_v self-dual with respect to the pairing $B_v \times B_v \ni (x, y) \mapsto \psi_v(\mathrm{tr}(\bar{x}y))$.

For $(g, h, h') \in G_v \times H_v \times H'_v$, define a linear operator $r_v(g, h, h')$ on $\boldsymbol{V}_v$ as follows: For $\varphi \in \boldsymbol{V}_v$, $X \in B_v^{\oplus 2}$ and $t \in \boldsymbol{Q}_v^\times$,

$$r_v(g, h, h') = r_v(g, 1, 1) \circ r_v(1, h, 1) \circ r_v(1, 1, h')\,,$$

where

$$r_v(g, 1, 1)\varphi(X, t) = |\nu(g)|_v^{-3/2}\varphi(g^{-1}X, \nu(g)t)\,, \quad g \in G_v\,, \tag{3.1}$$

$$r_v(1, 1, h')\varphi(X, t) = |n(h')|_v^{3/2}\varphi(Xh', n(h')^{-1}t)\,, \quad h' \in H'_v\,, \tag{3.2}$$

$$(\mathcal{I} \circ r_v(1, h, 1)\varphi)(X, t) = |\det h|_v^{-1/2}\mathcal{I}\varphi((\det h) \cdot h^{-1}X, (\det h)^{-1}t)\,, \quad h \in H_v\,. \tag{3.3}$$

A straightforward calculation shows that

$$r(1, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, 1)\varphi(X, t) = \psi_v\left(\frac{bt}{2}\,\mathrm{tr}(X^*QX)\right)\varphi(X, t)\,, \quad b \in \boldsymbol{Q}_v\,, \tag{3.4}$$

$$r(1, \begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix}, 1)\varphi(X, t) = |a|_v^{7/2}|a'|_v^{-1/2}\varphi(aX, (aa')^{-1}t)\,, \quad a, a' \in \boldsymbol{Q}_v^\times\,, \tag{3.5}$$

$$r(1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1)\varphi(X, t) = |t|_v^4 \int_{B_v^{\oplus 2}} \psi_v(t\,\mathrm{tr}(Y^*QX))\varphi(Y, t)d_QY\,. \tag{3.6}$$

Here d_QY is the Haar measure on $B_v^{\oplus 2}$ self-dual with respect to the pairing

$$B_v^{\oplus 2} \times B_v^{\oplus 2} \ni (Y, Y') \mapsto \psi_v(\mathrm{tr}(Y^*QY'))\,.$$

This shows that the actions of G_v, H_v and H'_v mutually commute, and that r_v gives rise to a smooth representation of $G_v \times H_v \times H'_v$ on $\boldsymbol{V}_v$.

When $v = p < \infty$, we put

$$\varphi_{0,p}(X,t) := \mathrm{char}_{L_p}(X)\,\mathrm{char}_{\boldsymbol{Z}_p^\times}(t)\,,$$

where char_{L_p} (resp. $\mathrm{char}_{\boldsymbol{Z}_p^\times}$) is the characteristic function of $L_p = {}^t(\mathcal{O}_p \oplus \mathfrak{A}_p^{-1})$ (resp. $\boldsymbol{Z}_p^\times$). When $v = \infty$, we put

$$\varphi_{0,\infty}^\kappa(X,t) := \begin{cases} t^{(\kappa+3)/2}\sigma_\kappa((1,1)X)\mathbf{e}\left(\dfrac{it}{2}\,\mathrm{tr}(X^*X)\right), & t > 0\,, \\ 0\,, & t < 0\,. \end{cases}$$

The following fact is easily verified.

LEMMA 3.1. *Let $v = p < \infty$. Then we have*

$$r(k_p, u_p, u'_p)\varphi_{0,p} = \varphi_{0,p}$$

for $k_p \in K_p$, $u_p \in U_p$ and $u'_p \in U'_p$.

LEMMA 3.2. *Let $v = \infty$. Then we have*

$$r(k_\infty, u_\infty, u'_\infty)\varphi_{0,\infty}^\kappa = j(u_\infty, i)^{-\kappa}\tau_\kappa(k_\infty)^{-1}\cdot\varphi_{0,\infty}^\kappa\cdot\sigma_\kappa(u'_\infty)$$

for $k \in K_\infty$, $u_\infty \in U_\infty$ and $u' \in U'_\infty$.

PROOF. The transformation law with respect to the U_∞-action immediately follows from [N-3, Lemma 3.8] (see also [N-1, Lemma 4.3]). The other transformation laws are verified in a straightforward way. □

4. Arakawa lift. Let $\boldsymbol{V}_{\boldsymbol{A}}$ be the restricted tensor product of $\boldsymbol{V}_v$ with respect to $\{\varphi_{0,p}\}_{p<\infty}$. By $r_{\boldsymbol{A}}$ we denote a smooth representation of $G_{\boldsymbol{A}} \times H_{\boldsymbol{A}} \times H'_{\boldsymbol{A}}$ on $\boldsymbol{V}_{\boldsymbol{A}}$ given by

$$r_{\boldsymbol{A}}(g,h,h')\varphi := \bigotimes_v r_v(g_v, h_v, h'_v)\varphi_v$$

for $\varphi = \bigotimes_v \varphi_v \in V_{\boldsymbol{A}}$ and $(g = (g_v), h = (h_v), h' = (h'_v)) \in G_{\boldsymbol{A}} \times H_{\boldsymbol{A}} \times H'_{\boldsymbol{A}}$. Define a function $\varphi_0^\kappa \in \boldsymbol{V}_{\boldsymbol{A}}$ by

$$\varphi_0^\kappa(X,t) := \varphi_{0,\infty}^\kappa(X_\infty, t_\infty)\prod_{p<\infty}\varphi_{0,p}(X_p, t_p)$$

for $X = (X_v) \in B_{\boldsymbol{A}}^{\oplus 2}$ and $t = (t_v) \in \boldsymbol{Q}_{\boldsymbol{A}}^\times$.

Set

$$\theta^\kappa(g,h,h') := \sum_{(X,t)\in B^{\oplus 2}\times \boldsymbol{Q}^\times} r_{\boldsymbol{A}}(g,h,h')\varphi_0^\kappa(X,t)\,, \quad (g,h,h') \in G_{\boldsymbol{A}} \times H_{\boldsymbol{A}} \times H'_{\boldsymbol{A}}\,. \tag{4.1}$$

Let C be any compact subset of $G_{\boldsymbol{A}} \times H_{\boldsymbol{A}} \times H'_{\boldsymbol{A}}$. Then there exist positive real numbers c_1, c_2, a lattice L of $B^{\oplus 2}$ and a finite set S of $\boldsymbol{Q}^{\times}_{>0}$ such that

$$\sum_{(X,t)\in B^{\oplus 2}\times \boldsymbol{Q}^{\times}} |r_{\boldsymbol{A}}(g,h,h')\varphi_0^{\kappa}(X,t)| \leq c_1 \sum_{X\in L, t\in S} \exp(-c_2 t X^* X)$$

holds for $(g,h,h') \in C$. This implies that the series (4.1) is absolutely convergent on any compact subset of $G_{\boldsymbol{A}} \times H_{\boldsymbol{A}} \times H'_{\boldsymbol{A}}$. By the definition of r and the Poisson summation formula, we see that θ^{κ} is left $G_{\boldsymbol{Q}} \times H_{\boldsymbol{Q}} \times H'_{\boldsymbol{Q}}$-invariant. We also have, by Lemmas 3.1 and 3.2,

$$\theta^{\kappa}(gk_f k_\infty, hu_f u_\infty, h'u'_f u'_\infty) = \tau_\kappa(k_\infty)^{-1} j(u_\infty, i)^{-\kappa}\theta(g,h,h')\sigma_\kappa(u'_\infty)$$

for $(g, k_f, k_\infty) \in G_{\boldsymbol{A}} \times K_f \times K_\infty$, $(h, u_f, u_\infty) \in H_{\boldsymbol{A}} \times U_f \times U_\infty$ and $(h', u'_f, u'_\infty) \in H'_{\boldsymbol{A}} \times U'_f \times U'_\infty$. Note that $(g_\infty, h_\infty, h_\infty) \mapsto r_\infty(g_\infty, h_\infty, h'_\infty)\varphi_{0,\infty}(X_\infty, t_\infty)$ is $Z_{G,\infty} \times Z_{H,\infty} \times Z_{H',\infty}$-invariant. We then see that θ^{κ} is $Z_{G,\boldsymbol{A}} \times Z_{H,\boldsymbol{A}} \times Z_{H',\boldsymbol{A}}$-invariant, since $\boldsymbol{Q}^{\times}_{\boldsymbol{A}} = \boldsymbol{Q}^{\times} \cdot \boldsymbol{R}_{>0} \cdot \boldsymbol{Z}^{\times}_f$ with $\boldsymbol{Z}^{\times}_f := \prod_{p<\infty} \boldsymbol{Z}^{\times}_p$.

For $f \in S_\kappa(D)$ and $f' \in \mathcal{A}_\kappa$, we set

$$(4.2)\quad \mathcal{L}(f,f')(g) := \int_{Z_{H,\boldsymbol{A}}H_{\boldsymbol{Q}}\backslash H_{\boldsymbol{A}}} dh \int_{Z_{H',\boldsymbol{A}}H'_{\boldsymbol{Q}}\backslash H'_{\boldsymbol{A}}} dh'\, \theta^{\kappa}(g,h,h')\overline{f(h)}f'(h')\,, \quad g \in G_{\boldsymbol{A}}\,.$$

THEOREM 4.1 (Arakawa, Narita). *Suppose that* $\kappa > 6$.
(i) *The integral* (4.2) *is absolutely convergent.*
(ii) $\mathcal{L}(f,f')(g) \in \mathcal{S}_\kappa$.

PROOF. Since $G_{\boldsymbol{A}} = Z_{G,\boldsymbol{A}} G_{\boldsymbol{Q}} G^1_\infty K_f$ (cf. [Shim-2, Theorem 6.14]), it is sufficient to study the restriction of $\mathcal{L}(f,f')$ to G^1_∞. By a standard argument, we see that $\mathcal{L}(f,f')|_{G^1_\infty}$ is a finite linear combination of original Arakawa lift (cf. [Ar-1], [N-1, Section 4] and [N-3, Theorem 4.1]), from which the theorem follows. □

REMARK 4.2. Our result on the Archimedean component of $\mathcal{L}(f,f')$ is compatible with the work [L, Section 6] by J. S. Li, in which he studies Archimedean theta correspondences for cohomological representations (including discrete series representations).

5. Main result.

5.1. To state the main result of the paper, we need to review several facts on Hecke operators.

First we consider the case where $p \nmid d_B$. We fix an isomorphism of B_p onto $M_2(\boldsymbol{Q}_p)$ such that $\mathcal{O}_p$ maps onto $M_2(\boldsymbol{Z}_p)$ and that the main involution of B_p corresponds to an involution of $M_2(\boldsymbol{Q}_p)$ given by

$$M_2(\boldsymbol{Q}_p) \ni X \mapsto w^{-1}\,{}^t X w\,, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The reduced trace tr corresponds to the trace Tr of $M_2(\boldsymbol{Q}_p)$. We henceforth identify B_p with $M_2(\boldsymbol{Q}_p)$ using the above isomorphism. Then G_p, K_p, H'_p and U'_p are identified with $GSp(J, \boldsymbol{Q}_p)$, $GSp(J, \boldsymbol{Z}_p)$, $GL_2(\boldsymbol{Q}_p)$ and $GL_2(\boldsymbol{Z}_p)$ respectively, where $GSp(J)$ is the group of similitudes of $J = \begin{pmatrix} 0 & w \\ w & 0 \end{pmatrix}$. Note that we can identify U_p with U'_p by the isomorphism $B_p \simeq M_2(\boldsymbol{Q}_p)$ fixed above.

Define Hecke operators $\mathcal{T}_p^i$ $(i = 0, 1, 2)$ on $\mathcal{S}_\kappa$ by

$$\mathcal{T}_p^i F(g) = \int_{G_p} F(gx)\Phi_p^i(x)dx\,,$$

where Φ_p^0, Φ_p^1 and Φ_p^2 are the characteristic functions of $K_p \operatorname{diag}(p, p, p, p)K_p$, $K_p \operatorname{diag}(p, p, 1, 1)K_p$ and $K_p \operatorname{diag}(p^2, p, p, 1)K_p$, respectively. Note that $\mathcal{T}_p^0 F = F$ for any $F \in \mathcal{S}_\kappa$.

We also define Hecke operators T_p and T'_p on $S_\kappa(D)$ and $\mathcal{A}_\kappa$ by

$$T_p f(h) = \int_{H_p} f(hx)\phi_p(x)dx\,,$$

$$T'_p f'(h') = \int_{H'_p} f'(h'x')\phi'_p(x')dx'\,,$$

where $\phi_p = \phi'_p$ is the characteristic function of $GL_2(\mathbf{Z}_p) \operatorname{diag}(p, 1)GL_2(\mathbf{Z}_p)$.

5.2. We next consider the case where $p \mid d_B$, namely, B_p is a division algebra. In this case, we fix a prime element Π of B_p and put $\pi := n(\Pi)$. Then π is a prime element of $\boldsymbol{Q}_p$.

Define Hecke operators $\mathcal{T}_p^i$ $(i = 0, 1)$ on $\mathcal{S}_\kappa$ by

$$\mathcal{T}_p^i F(g) = \int_{G_p} F(gx)\Phi_p^i(x)dx\,,$$

where Φ_p^0 and Φ_p^1 are the characteristic functions of $K_p \operatorname{diag}(\Pi, \Pi)K_p$ and $K_p \operatorname{diag}(1, \pi)K_p$ respectively. Note that $\mathcal{T}_p^0 F(g) = F(g \cdot \Pi)$ and hence that $\mathcal{T}_p^0(\mathcal{T}_p^0 F)(g) = F(g \cdot \pi) = F(g)$ for $F \in \mathcal{S}_\kappa$. We also define Hecke operators T_p and T'_p on $S_\kappa(D)$ and $\mathcal{A}_\kappa$ by

$$T_p f(h) = \int_{H_p} f(hx)\phi_p(x)dx\,,$$

$$T'_p f'(h') = \int_{H'_p} f'(h'x')\phi'_p(x')dx'\,.$$

Here ϕ'_p is the characteristic function of $U'_p \Pi U'_p = \Pi U'_p$ and ϕ_p is defined as follows: If $p \mid D$, ϕ_p is the sum of the characteristic functions of $U_p \operatorname{diag}(\pi, 1)U_p$ and $U_p \operatorname{diag}(1, \pi)U_p$. If $p \nmid D$, ϕ_p is the characteristic function of $U_p \operatorname{diag}(\pi, 1)U_p$.

5.3. We say that $F \in \mathcal{S}_\kappa$ is a *Hecke eigenform* if F is a common eigenfunction of all the Hecke operators $\mathcal{T}_p^i$ for any $p < \infty$ and $i = 1, 2$. Let $F \in \mathcal{S}_\kappa$ be a Hecke eigenform with $\mathcal{T}_p^i F = \Lambda_p^i F$, $\Lambda_p^i \in \boldsymbol{C}$. We define the spinor L-function of F by

$$L(F, \text{spin}, s) = \prod_{p<\infty} L_p(F, \text{spin}, s)\,,$$

where $L_p(F, \mathrm{spin}, s) = Q_p(F, p^{-s})^{-1}$,

$$Q_p(F,t) = \begin{cases} 1 - p^{\kappa-3}\Lambda_p^1 t + p^{2\kappa-5}(\Lambda_p^2 + p^2 + 1)t^2 - p^{3\kappa-6}\Lambda_p^1 t^3 + p^{4\kappa-6}t^4\,, & p \nmid d_B\,, \\ 1 - \{p^{\kappa-3}\Lambda_p^1 - p^{\kappa-3}(p^{A_p} - 1)\Lambda_p^0\}t + p^{2\kappa-3}(\Lambda_p^0)^2 t^2\,, & p \mid d_B\,, \end{cases}$$

and

$$A_p = \begin{cases} 1\,, & p \nmid D\,, \\ 2\,, & p \mid D\,. \end{cases}$$

The Euler factor for $p \nmid d_B$ (resp. $p \mid d_B$) is given by the formula for the denominator of the Hecke series in [Shim-1, Theorem 2] (resp. [H-S, Section 4, (I)] and [Su, (1-34)]) under the normalization of the Hecke eigenvalues

$$\begin{cases} (\Lambda_p^0, \Lambda_p^1, \Lambda_p^2) \to (p^{2(\kappa-3)}\Lambda_p^0, p^{\kappa-3}\Lambda_p^1, p^{2(\kappa-3)}\Lambda_p^2)\,, & p \nmid d_B\,, \\ (\Lambda_p^0, \Lambda_p^1) \to (p^{\kappa-3}\Lambda_p^0, p^{\kappa-3}\Lambda_p^1)\,, & p \mid d_B\,. \end{cases}$$

We say that $f \in S_\kappa(D)$ (resp. $f' \in \mathcal{A}_\kappa$) is a *Hecke eigenform* if f (resp. f') is a common eigenfunction of T_p (resp. T'_p) for any $p < \infty$. For Hecke eigenforms $f \in S_\kappa(D)$ and $f' \in \mathcal{A}_\kappa$ with $T_p f = \lambda_p f$ and $T'_p f' = \lambda'_p f'$ ($\lambda_p,\ \lambda'_p \in \boldsymbol{C}$), we define L-functions

$$L^D(f,s) = \prod_{p \nmid D} (1 - \lambda_p p^{\kappa-2-s} + p^{2\kappa-3-2s})^{-1}\,,$$
$$L^{d_B}(f',s) = \prod_{p \nmid d_B} (1 - \lambda'_p p^{\kappa-2-s} + p^{2\kappa-3-2s})^{-1}\,.$$

When $D = 1$, we write $L(f,s)$ for $L^D(f,s)$, which is the usual Hecke L-function of f.

5.4. We are now able to state the main result of the paper.

THEOREM 5.1. *Let $f \in S_\kappa(D)$ and $f' \in \mathcal{A}_\kappa$, and suppose that*

$$T_p\, f = \lambda_p\, f\,, \quad T'_p\, f' = \lambda'_p\, f'$$

for every $p < \infty$. Then $F(g) := \mathcal{L}(f, f')(g)$ is a Hecke eigenform and the Hecke eigenvalues are given as follows:

(i) *If $p \nmid d_B$, we have*

$$\mathcal{T}_p^0\, F = F\,,$$
$$\mathcal{T}_p^1\, F = (p\lambda_p + p\lambda'_p)F\,,$$
$$\mathcal{T}_p^2\, F = (p\lambda_p\lambda'_p + p^2 - 1)F\,.$$

(ii) *If $p \mid d_B$, we have*

$$\mathcal{T}_p^0\, F = \lambda'_p F\,,$$
$$\mathcal{T}_p^1\, F = (p\lambda_p + (p-1)\lambda'_p)F\,.$$

REMARK 5.2. (i) We can verify that all the Hecke operators above for $S_\kappa(D)$ are self-adjoint with respect to the Petersson inner product, since the forms in $S_\kappa(D)$ are assumed to have the trivial central character. Thus λ_p is real for any $p < \infty$.

(ii) Let $p \mid D$. The *Atkin-Lehner involution* on $S_\kappa(D)$ is given by

$$W_p f(h) = f\left(hw_p\right), \quad f \in S_\kappa(D), \quad w_p = \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} \in H_p .$$

Note that W_p commutes with the Hecke operator T_p. Suppose that $W_p f = \lambda''_p f$. Then we can check that $\mathcal{T}_p^0\, \mathcal{L}(f, f') = \lambda''_p \mathcal{L}(f, f')$. This implies that $\mathcal{L}(f, f') = 0$ unless $\lambda'_p = \lambda''_p$ for every $p \mid D$.

COROLLARY 5.3. *Let f and f' be as in Theorem* 5.1. *Then we have*

$$L(\mathcal{L}(f, f'), \mathrm{spin}, s) = L^D(f, s)\, L^{d_B}(f', s) \prod_{p \mid D} (1 - \{\lambda_p + (1-p)\lambda'_p\} p^{\kappa-2-s} + p^{2\kappa-3-2s})^{-1} .$$

In particular, if $D = 1$, we have

$$L(\mathcal{L}(f, f'), \mathrm{spin}, s) = L(f, s)\, L^{d_B}(f', s) .$$

REMARK 5.4. When $p \nmid d_B$, the formula for the Hecke eigenvalues in Theorem 5.1 is essentially the same as the corresponding result for the Yoshida lifting (cf. [Y, Theorem 6.1]).

6. Commutation relations. In this section, we state the commutation relations of Hecke operators, from which Theorem 5.1 immediately follows. For a function ϕ on H_p, we put $\hat{\phi}(h) = \phi(h^{-1})$, $h \in H_p$. We define $\hat{\phi}'$ for $\phi' : H'_p \to \boldsymbol{C}$ in a similar manner.

6.1. In this subsection, suppose that $p \nmid d_B$ and let the notation be the same as in 5.1. The metaplectic representation r in this case is given as follows:

Let $\Phi \in \mathcal{S}(M_{4,2}(\boldsymbol{Q}_p)) \otimes \mathcal{S}(\boldsymbol{Q}_p^\times)$, $X \in M_{4,2}(\boldsymbol{Q}_p)$ and $t \in \boldsymbol{Q}_p^\times$. We have

$$r(g, 1, 1)\Phi(X, t) = |\nu(g)|_p^{-3/2} \Phi(g^{-1}X, \nu(g)t), \quad g \in G_p ,$$

$$r(1, 1, h')\Phi(X, t) = |\det h'|_p^{3/2} \Phi(Xh', (\det h')^{-1}t), \quad h' \in H'_p ,$$

$$r(1, \begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix}, 1)\Phi(X, t) = |a|_p^{7/2} |a'|_p^{-1/2} \Phi(aX, (aa')^{-1}t), \quad a, a' \in \boldsymbol{Q}_p^\times ,$$

$$r(1, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, 1)\Phi(X, t) = \psi_p\left(\frac{bt}{2} \mathrm{Tr}({}^tXJXw^{-1})\right) \Phi(X, t), \quad b \in \boldsymbol{Q}_p ,$$

$$r(1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1)\Phi(X, t) = |t|_p^4 \int_{M_{4,2}(\boldsymbol{Q}_p)} \psi_p(t \cdot \mathrm{Tr}({}^tYJXw^{-1}))\Phi(Y, t) d_Q Y .$$

6.2. The commutation relations are stated as follows:

PROPOSITION 6.1. *Suppose that $p \nmid d_B$. Then we have*

$$r(\Phi_p^1, 1, 1)\varphi_{0,p} = p \cdot r(1, \hat{\phi}_p, 1)\varphi_{0,p} + p \cdot r(1, 1, \hat{\phi}'_p)\varphi_{0,p} , \tag{6.1}$$

$$r(\Phi_p^2, 1, 1)\varphi_{0,p} + (1 - p^2) r(\Phi_p^0, 1, 1)\varphi_{0,p} = p \cdot r(1, \hat{\phi}_p, \hat{\phi}'_p)\varphi_{0,p} . \tag{6.2}$$

PROPOSITION 6.2. *Suppose that $p \mid d_B$. Then we have*

$$r(\Phi_p^0, 1, 1)\varphi_{0,p} = r(1, 1, \hat{\phi}'_p)\varphi_{0,p}\,, \tag{6.3}$$

$$r(\Phi_p^1, 1, 1)\varphi_{0,p} = p \cdot r(1, \hat{\phi}_p, 1)\varphi_{0,p} + (p-1)r(1, 1, \hat{\phi}'_p)\varphi_{0,p}\,. \tag{6.4}$$

7. Proof of Proposition 6.1.

7.1. In this section, we assume that $p \nmid d_B$ and prove Proposition 6.1. We keep the notation in 5.2 and Section 6. In the remaining part of the paper, we write φ_0 for $\varphi_{0,p}$. For $X \in M_2(\boldsymbol{Q}_p)$ with $wX + {}^tXw = 0$ and $Y \in GL_2(\boldsymbol{Q}_p)$, we put

$$u(X) := \begin{pmatrix} 1_2 & X \\ 0_2 & 1_2 \end{pmatrix}, \quad \bar{u}(X) := \begin{pmatrix} 1_2 & 0_2 \\ X & 1_2 \end{pmatrix}, \quad \tau(Y) := \begin{pmatrix} Y & 0_2 \\ 0_2 & w^{-1}\,{}^tY^{-1}w \end{pmatrix} \in G_p\,.$$

Let

$$\Lambda := \{(a,b,c) \in (\mathbf{Z}_p/p\mathbf{Z}_p)^3 \mid (a,b,c) \not\equiv (0,0,0) \pmod{(p\mathbf{Z}_p)^3},\ a^2+bc \equiv 0 \pmod{p\mathbf{Z}_p}\}\,.$$

Note that $\sharp(\Lambda) = p^2 - 1$.

LEMMA 7.1. (i)

$$\begin{aligned} K_p \operatorname{diag}(p,p,1,1)K_p = {} & \operatorname{diag}(1,1,p,p)K_p \cup \bigcup_{c \in \mathbf{Z}_p/p\mathbf{Z}_p} u\left(\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}\right) \operatorname{diag}(1,p,1,p)K_p \\ & \cup \bigcup_{b,d \in \mathbf{Z}_p/p\mathbf{Z}_p} \tau\left(\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\right) u\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right) \operatorname{diag}(p,1,p,1)K_p \\ & \cup \bigcup_{a,b,c \in \mathbf{Z}_p/p\mathbf{Z}_p} u\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right) \operatorname{diag}(p,p,1,1)K_p\,. \end{aligned}$$

(ii)

$$\begin{aligned} K_p \operatorname{diag}(p^2,p,p,1)K_p = {} & \operatorname{diag}(1,p,p,p^2)K_p \cup \bigcup_{d \in \mathbf{Z}_p/p\mathbf{Z}_p} \tau\left(\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\right) \operatorname{diag}(p,1,p^2,p)K_p \\ & \cup \bigcup_{a \in \mathbf{Z}_p/p\mathbf{Z}_p,\ c \in \mathbf{Z}_p/p^2\mathbf{Z}_p} u\left(\begin{pmatrix} a & 0 \\ c & -a \end{pmatrix}\right) \operatorname{diag}(p,p^2,1,p)K_p \\ & \cup \bigcup_{(a,b,c) \in \Lambda} u\left(p^{-1}\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right) \operatorname{diag}(p,p,p,p)K_p \\ & \cup \bigcup_{a,d \in \mathbf{Z}_p/p\mathbf{Z}_p,\ b \in \mathbf{Z}_p/p^2\mathbf{Z}_p} \tau\left(\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\right) u\left(\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}\right) \operatorname{diag}(p^2,p,p,1)K_p\,. \end{aligned}$$

PROOF. We give a detailed proof for completeness. To simplify the notation, we write G and K for G_p and K_p, respectively. Let $\mathcal{B} = \{k = (k_{ij})_{1 \le i,j \le 4} \in K \mid k_{ij} \in p\mathbf{Z}_p\ (i > j)\}$

be the Iwahori subgroup of G and

$$\mathcal{N}_0 = \left\{ \tau\left(\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\right) u\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right) \,\middle|\, a, b, c, d \in \mathbf{Z}_p \right\} .$$

Then we have the Bruhat decomposition of K:

$$K = \bigcup_{\gamma \in W} \mathcal{B}\gamma\mathcal{B} ,$$

where W is the Weyl group of G. Recall that $\#(W) = 8$ and W is generated by

$$\gamma_1 = \tau\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right), \quad \gamma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Put $g_1 = \mathrm{diag}(1, 1, p, p)$. Observe that $\mathcal{B}\gamma\mathcal{B}g_1 = \mathcal{N}_0\gamma\mathcal{B}g_1 \subset \mathcal{N}_0\gamma g_1 K$ for $\gamma \in W$. Since $\gamma_1 g_1 = g_1\gamma_1 \in g_1 K$, we have

$$\begin{aligned} K\,\mathrm{diag}(p, p, 1, 1)K &= K g_1 K \\ &= \bigcup_{\gamma \in W/\langle\gamma_1\rangle} \mathcal{N}_0\gamma g_1 K \\ &= \mathcal{N}_0 g_1 K \cup \mathcal{N}_0\gamma_2 g_1 K \cup \mathcal{N}_0\gamma_1\gamma_2 g_1 K \cup \mathcal{N}_0\gamma_2\gamma_1\gamma_2 g_1 K . \end{aligned}$$

It is easily verified that

$$\begin{aligned} \mathcal{N}_0 g_1 K &= \mathrm{diag}(1, 1, p, p)K , \\ \mathcal{N}_0\gamma_2 g_1 K &= \bigcup_{c \in \mathbf{Z}_p/p\mathbf{Z}_p} u\left(\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}\right) \mathrm{diag}(1, p, 1, p)K , \\ \mathcal{N}_0\gamma_1\gamma_2 g_1 K &= \bigcup_{b,d \in \mathbf{Z}_p/p\mathbf{Z}_p} \tau\left(\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\right) u\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right) \mathrm{diag}(p, 1, p, 1)K , \\ \mathcal{N}_0\gamma_2\gamma_1\gamma_2 g_1 K &= \bigcup_{a,b,c \in \mathbf{Z}_p/p\mathbf{Z}_p} u\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right) \mathrm{diag}(p, p, 1, 1)K , \end{aligned}$$

which proves the first assertion of the lemma.

Next put $g_2 = \mathrm{diag}(1, p, p, p^2)$. By an argument similar as above, we have

$$K\,\mathrm{diag}(p^2, p, p, 1)K = \mathcal{B}g_2 K \cup \mathcal{B}\gamma_1 g_2 K \cup \mathcal{B}\gamma_2\gamma_1 g_2 K \cup \mathcal{B}\gamma_1\gamma_2\gamma_1 g_2 K .$$

First we see that

$$\begin{aligned} \mathcal{B}\gamma_2\gamma_1 g_2 K &= \mathcal{N}_0\gamma_2\gamma_1 g_2 K \\ &= \bigcup_{a \in \mathbf{Z}_p/p\mathbf{Z}_p,\ c \in \mathbf{Z}_p/p^2\mathbf{Z}_p} u\left(\begin{pmatrix} a & 0 \\ c & -a \end{pmatrix}\right) \mathrm{diag}(p, p^2, 1, p)K . \end{aligned}$$

Similarly we have

$$\mathcal{B}\gamma_1\gamma_2\gamma_1 g_2 K = \bigcup_{a,d\in \mathbf{Z}_p/p\mathbf{Z}_p,\ b\in\mathbf{Z}_p/p^2\mathbf{Z}_p} \tau\left(\begin{pmatrix}1 & d\\ 0 & 1\end{pmatrix}\right) u\left(\begin{pmatrix}a & b\\ 0 & -a\end{pmatrix}\right) \operatorname{diag}(p^2, p, p, 1)K\,.$$

Using the decomposition

$$\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix} = \begin{pmatrix}1 & c^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix}\begin{pmatrix}c & 0\\ 0 & c^{-1}\end{pmatrix} \quad (c \in \boldsymbol{Q}_p^\times)\,,$$

we obtain

$$\mathcal{B}g_2K = \bigcup_{c\in p\mathbf{Z}_p/p^2\mathbf{Z}_p} \bar{u}\left(\begin{pmatrix}0 & 0\\ c & 0\end{pmatrix}\right) g_2K = g_2K \cup C_1\,,$$

where

$$C_1 = \bigcup_{b\in(\mathbf{Z}_p - p\mathbf{Z}_p)/p\mathbf{Z}_p} u\left(p^{-1}\begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix}\right) \operatorname{diag}(p, p, p, p)K\,.$$

Similarly we have

$$\mathcal{B}\gamma_1 g_2 K = \bigcup_{d\in\mathbf{Z}_p/p\mathbf{Z}_p} \tau\left(\begin{pmatrix}1 & d\\ 0 & 1\end{pmatrix}\right) \operatorname{diag}(p, 1, p^2, p)K \cup C_2\,,$$

where

$$C_2 = \bigcup_{b\in(\mathbf{Z}_p - p\mathbf{Z}_p)/p\mathbf{Z}_p, d\in\mathbf{Z}_p/p\mathbf{Z}_p} \tau\left(p^{-1}b\begin{pmatrix}d & -d^2\\ 1 & -d\end{pmatrix}\right) \operatorname{diag}(p, p, p, p)K\,.$$

Since

$$C_1 \cup C_2 = \bigcup_{(a,b,c)\in\Lambda} u\left(p^{-1}\begin{pmatrix}a & b\\ c & -a\end{pmatrix}\right) \operatorname{diag}(p, p, p, p)K\,,$$

we are done. □

7.2. Denote by $\sigma_{m,n}$ (resp. σ') the characteristic function of $M_{m,n}(\mathbf{Z}_p)$ (resp. $\mathbf{Z}_p^\times$). Then we have $\varphi_0(X, t) = \sigma_{4,2}(X)\sigma'(t)$. From now on, we often write σ for $\sigma_{m,n}$ if there is no fear of confusion. The following elementary facts are frequently used in the later discussion.

LEMMA 7.2. *For $t, t' \in \boldsymbol{Q}_p$, we have*

$$\sum_{a\in\mathbf{Z}_p/p\mathbf{Z}_p} \sigma(p^{-1}(t+a)) = \sigma(t)\,,$$

$$\sum_{a\in\mathbf{Z}_p/p\mathbf{Z}_p} \sigma(p^{-1}(t+a))\sigma(p^{-1}(t'-a)) = \sigma(p^{-1}(t+t'))$$

and

$$\sum_{a\in\mathbf{Z}_p/p\mathbf{Z}_p} \sigma(t)\sigma(p^{-1}(at+t')) = p\sigma(p^{-1}t)\sigma(p^{-1}t') + \sigma(t)\sigma(t') - \sigma(p^{-1}t)\sigma(t')\,.$$

LEMMA 7.3. *For $x \in M_{2,2}(\boldsymbol{Q}_p)$, set*

$$\lambda(x) := \sigma\left(\begin{pmatrix}1 & 0\\ 0 & p^{-1}\end{pmatrix} x\right) + \sum_{a\in \mathbf{Z}_p/p\mathbf{Z}_p} \sigma\left(\begin{pmatrix}p^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix} x\right)$$

and

$$\rho(x) := \sigma\left(x\begin{pmatrix}p^{-1} & 0\\ 0 & 1\end{pmatrix}\right) + \sum_{a\in \mathbf{Z}_p/p\mathbf{Z}_p} \sigma\left(x\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & p^{-1}\end{pmatrix}\right).$$

Then

$$\lambda(x) = \rho(x).$$

For the rest of this section, we put

$$[i_1, i_2, i_3, i_4](x) = \sigma_{2,2}\left(\begin{pmatrix}p^{-i_1}x_1 & p^{-i_2}x_2\\ p^{-i_3}x_3 & p^{-i_4}x_4\end{pmatrix}\right)$$
$$\text{for } (i_1, i_2, i_3, i_4) \in \boldsymbol{Z}^4, \quad x = \begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix} \in M_{2,2}(\boldsymbol{Q}_p).$$

7.3. Proof of (6.1). Let $X = \binom{x}{y} \in M_{4,2}(\boldsymbol{Q}_p)$ and $t \in \boldsymbol{Q}_p^{\times}$. In view of 6.2 and Lemma 7.1, we obtain

$$r(\Phi_p^1, 1, 1)\varphi_0(X, t) = p^{3/2}\sigma'(pt)I(X),$$

where

$$\begin{aligned} I(X) = &\sum_{a,b,c\in \mathbf{Z}_p/p\mathbf{Z}_p} \sigma\begin{pmatrix} p^{-1}\left\{x + \begin{pmatrix}a & b\\ c & -a\end{pmatrix} y\right\} \\ y \end{pmatrix} \\ &+ \sum_{b,d\in \mathbf{Z}_p/p\mathbf{Z}_p} \sigma\begin{pmatrix} \begin{pmatrix}p^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & d\\ 0 & 1\end{pmatrix}\left\{x + \begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix} y\right\} \\ \begin{pmatrix}p^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & d\\ 0 & 1\end{pmatrix} y \end{pmatrix} \\ &+ \sum_{c\in \mathbf{Z}_p/p\mathbf{Z}_p} \sigma\begin{pmatrix} \begin{pmatrix}1 & 0\\ 0 & p^{-1}\end{pmatrix}\left\{x + \begin{pmatrix}0 & 0\\ c & 0\end{pmatrix} y\right\} \\ \begin{pmatrix}1 & 0\\ 0 & p^{-1}\end{pmatrix} y \end{pmatrix} + \sigma\begin{pmatrix} x \\ p^{-1}y \end{pmatrix}. \end{aligned}$$

On the other hand, since

$$U_p\begin{pmatrix}p^{-1} & 0\\ 0 & 1\end{pmatrix}U_p = \bigcup_{a\in \mathbf{Z}_p/p\mathbf{Z}_p}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & p^{-1}\end{pmatrix}U_p \cup \begin{pmatrix}p^{-1} & 0\\ 0 & 1\end{pmatrix}U_p,$$

we obtain

$$p\{r(1, \hat{\phi}_p, 1)\varphi_0(X, t) + r(1, 1, \hat{\phi}'_p)\varphi_0(X, t)\} = p^{3/2}\sigma'(pt)I'(X),$$

where

$$I'(X) = \delta(\mathrm{Tr}(xw^{-1t}yw) \in p\boldsymbol{Z}_p)\sigma(X) + p^3\sigma(p^{-1}X)$$
$$+ p \sum_{a \in \boldsymbol{Z}_p/p\boldsymbol{Z}_p} \sigma\left(X \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}\right) + p\sigma\left(X \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right).$$

The proof of (6.1) is reduced to proving the following formula:

(7.1) $$I(X) = I'(X).$$

Without loss of generality, we may assume that $y = \mathrm{diag}(p^\lambda, p^\mu)$ with $\lambda \geq \mu \geq 0$ in view of the elementary divisor theorem. First suppose that $\mu > 0$. Then

$$I(X) = p^3\sigma(p^{-1}x) + p\lambda(x) + \sigma(x)$$

and

$$I'(X) = \sigma(x) + p^3\sigma(p^{-1}x) + p\rho(x).$$

Equality (7.1) immediately follows from Lemma 7.3.

Next, suppose that $\lambda = \mu = 0$. Then

$$I(X) = \sum_{a,b,c \in \boldsymbol{Z}_p/p\boldsymbol{Z}_p} \sigma\left(p^{-1}\left(x + \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right)\right)$$

and

$$I'(X) = \delta(\mathrm{Tr}(x) \in p\boldsymbol{Z}_p)\sigma(x).$$

Let $x = \left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right)$. If $x \notin M_2(\boldsymbol{Z}_p)$, we have $I(X) = I'(X) = 0$. Assume that $x \in M_2(\boldsymbol{Z}_p)$. If $\mathrm{Tr}(x) = x_1 + x_4 \in p\boldsymbol{Z}_p$ (resp. $\in \boldsymbol{Z}_p^\times$), we have $I(X) = I'(X) = 1$ (resp. $= 0$) by Lemma 7.2, which proves (7.1).

Finally, suppose that $\lambda > 0$ and $\mu = 0$. We then have

$$\begin{aligned} I(X) &= p \sum_{a,b \in \boldsymbol{Z}_p/p\boldsymbol{Z}_p} \sigma\left(p^{-1}x + \begin{pmatrix} 0 & p^{-1}b \\ 0 & p^{-1}a \end{pmatrix}\right) \\ &\quad + \sum_{b \in \boldsymbol{Z}_p/p\boldsymbol{Z}_p} \sigma\left(\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}\left\{x + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right\}\right) \\ &= p[1,0,1,0](x) + [1,0,0,0](x) \end{aligned}$$

and

$$I'(X) = \delta(x_1 \in p\boldsymbol{Z}_p)\sigma(x) + p[1,0,1,0](x),$$

which implies that $I(X) = I'(X)$. This completes the proof of (6.1).

7.4. Proof of (6.2). Let $X = \binom{x}{y} \in M_{4,2}(\boldsymbol{Q}_p)$ and $t \in \boldsymbol{Q}_p^\times$. Fisrt observe that

$$r(\Phi_p^2, 1, 1)\varphi_0(X, t) = p^3\sigma'(p^2t)J(X),$$

where

$$J(X)=\sigma\begin{pmatrix}\begin{pmatrix}1&0\\0&p^{-1}\end{pmatrix}x\\\begin{pmatrix}p^{-1}&0\\0&p^{-2}\end{pmatrix}y\end{pmatrix}+\sum_{d\in\mathbf{Z}_p/p\mathbf{Z}_p}\sigma\begin{pmatrix}\begin{pmatrix}p^{-1}&p^{-1}d\\0&1\end{pmatrix}x\\\begin{pmatrix}p^{-2}&p^{-2}d\\0&p^{-1}\end{pmatrix}y\end{pmatrix}$$

$$+\sum_{\substack{a\in\mathbf{Z}_p/p\mathbf{Z}_p\\c\in\mathbf{Z}_p/p^2\mathbf{Z}_p}}\sigma\begin{pmatrix}\begin{pmatrix}p^{-1}&0\\0&p^{-2}\end{pmatrix}x+\begin{pmatrix}p^{-1}a&0\\p^{-2}c&-p^{-2}a\end{pmatrix}y\\\begin{pmatrix}1&0\\0&p^{-1}\end{pmatrix}y\end{pmatrix}$$

$$+\sum_{(a,b,c)\in\Lambda}\sigma\begin{pmatrix}p^{-1}x+p^{-2}\begin{pmatrix}a&b\\c&-a\end{pmatrix}y\\p^{-1}y\end{pmatrix}$$

$$+\sum_{\substack{a,d\in\mathbf{Z}_p/p\mathbf{Z}_p\\b\in\mathbf{Z}_p/p^2\mathbf{Z}_p}}\sigma\begin{pmatrix}\begin{pmatrix}p^{-2}&p^{-2}d\\0&p^{-1}\end{pmatrix}x+\begin{pmatrix}p^{-2}a&p^{-2}(b+ad)\\0&-p^{-1}a\end{pmatrix}y\\\begin{pmatrix}p^{-1}&p^{-1}d\\0&1\end{pmatrix}y\end{pmatrix}.$$

We also have

$$r(\Phi_p^0,1,1)\varphi_0(X,t)=p^3\sigma'(p^2t)J'(X),$$

where

$$J'(X):=\sigma\begin{pmatrix}p^{-1}x\\p^{-1}y\end{pmatrix}.$$

On the other hand, we obtain

$$r(1,\hat{\phi}_p,\hat{\phi}_p)\varphi_0(X,t)=p^2\sigma'(p^2t)J''(X),$$

where

$$J''(X):=\delta(\mathrm{Tr}(xw^{-1\,t}yw)\in p^2\mathbf{Z}_p)\sum_{b\in\mathbf{Z}_p/p\mathbf{Z}_p}\sigma\left(\begin{pmatrix}x\\y\end{pmatrix}\begin{pmatrix}1&b\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&p^{-1}\end{pmatrix}\right)$$

$$+\delta(\mathrm{Tr}(xw^{-1\,t}yw)\in p^2\mathbf{Z}_p)\sigma\left(\begin{pmatrix}x\\y\end{pmatrix}\begin{pmatrix}p^{-1}&0\\0&1\end{pmatrix}\right)$$

$$+p^3\sum_{b\in\mathbf{Z}_p/p\mathbf{Z}_p}\sigma\left(\begin{pmatrix}x\\y\end{pmatrix}\begin{pmatrix}1&b\\0&1\end{pmatrix}\begin{pmatrix}p^{-1}&0\\0&p^{-2}\end{pmatrix}\right)+p^3\sigma\left(\begin{pmatrix}x\\y\end{pmatrix}\begin{pmatrix}p^{-2}&0\\0&p^{-1}\end{pmatrix}\right).$$

To show (6.2), it remains to verify

$$J(X)+(1-p^2)J'(X)=J''(X). \tag{7.2}$$

As in 7.3, we may assume that $y=\mathrm{diag}(p^\alpha,p^\beta)$, where $\alpha\geq\beta\geq 0$. Let $x=\begin{pmatrix}x_1&x_2\\x_3&x_4\end{pmatrix}$. We divide the proof into the following five cases:

(a) $\beta\geq 2$, (b) $\alpha\geq 2$, $\beta=1$, (c) $\alpha=\beta=1$, (d) $\alpha\geq 1$, $\beta=0$, (e) $\alpha=\beta=0$.

(a) In this case, we have

$$\begin{aligned} J(X)+(1-p^2)J'(X) &= \sigma\left(\begin{pmatrix}1 & 0\\ 0 & p^{-1}\end{pmatrix}x\right)+\sum_{d\in\mathbf{Z}_p/p\mathbf{Z}_p}\sigma\left(\begin{pmatrix}p^{-1} & p^{-1}d\\ 0 & 1\end{pmatrix}x\right)\\ &\quad + p^3\sigma\left(\begin{pmatrix}p^{-1} & 0\\ 0 & p^{-2}\end{pmatrix}x\right)+p^3\sum_{d\in\mathbf{Z}_p/p\mathbf{Z}_p}\sigma\left(\begin{pmatrix}p^{-2} & p^{-2}d\\ 0 & p^{-1}\end{pmatrix}x\right)\\ &= \lambda(x)+p^3\lambda(p^{-1}x) \end{aligned}$$

and

$$J''(X)=\rho(x)+p^3\rho(p^{-1}x)\,.$$

Equality (7.2) follows from Lemma 7.3.

(b) In this case, we have

$$\begin{aligned} J(X) &= [1,1,0,0](x)+p^2\sum_{a\in\mathbf{Z}_p/p\mathbf{Z}_p}\sigma\left(\begin{pmatrix}p^{-1}x_1 & p^{-1}x_2\\ p^{-2}x_3 & p^{-2}x_4-p^{-1}a\end{pmatrix}\right)\\ &\quad+\sum_{(a,b,c)\in\Lambda}\sigma\left(\begin{pmatrix}p^{-1}x_1 & p^{-1}(x_2+b)\\ p^{-1}x_3 & p^{-1}(x_4-a)\end{pmatrix}\right)\\ &\quad+p\sum_{\substack{d\in\mathbf{Z}_p/p\mathbf{Z}_p\\ b\in\mathbf{Z}_p/p^2\mathbf{Z}_p}}\sigma\left(\begin{pmatrix}p^{-2}(x_1+dx_3) & p^{-2}(x_2+dx_4)+p^{-1}b\\ p^{-1}x_3 & p^{-1}x_4\end{pmatrix}\right)\\ &= [1,1,0,0](x)+p^2[1,1,2,1](x)\\ &\quad+\sum_{\substack{a\equiv 0 \pmod p\\ bc\equiv 0 \pmod p,\ (b,c)\not\equiv(0,0) \pmod p}}\sigma\left(\begin{pmatrix}p^{-1}x_1 & p^{-1}(x_2+b)\\ p^{-1}x_3 & p^{-1}x_4\end{pmatrix}\right)\\ &\quad+\sum_{\substack{a\not\equiv 0 \pmod p\\ b\not\equiv 0 \pmod p}}\sigma\left(\begin{pmatrix}p^{-1}x_1 & p^{-1}(x_2+b)\\ p^{-1}x_3 & p^{-1}(x_4-a)\end{pmatrix}\right)\\ &\quad+p^2\sum_{d\in\mathbf{Z}_p/p\mathbf{Z}_p}\sigma\left(\begin{pmatrix}p^{-2}(x_1+dx_3) & p^{-1}(x_2+dx_4)\\ p^{-1}x_3 & p^{-1}x_4\end{pmatrix}\right)\\ &= [1,1,0,0](x)+p^2[1,1,2,1](x)+(p-1)[1,1,1,1](x)\\ &\quad+[1,0,1,0](x)-[1,1,1,0](x)\\ &\quad+p^2\sigma(p^{-1}x_2)\sigma(p^{-1}x_4)\sum_{d\in\mathbf{Z}_p/p\mathbf{Z}_p}\sigma(p^{-1}x_3)\sigma(p^{-1}(p^{-1}x_3+p^{-1}x_1))\\ &= [1,1,0,0](x)+p^2[1,1,2,1](x)+(p-1)[1,1,1,1](x)\\ &\quad+[1,0,1,0](x)-[1,1,1,0](x)\\ &\quad+p^2\{p[2,1,2,1](x)+[1,1,1,1](x)-[1,1,2,1](x)\}\,, \end{aligned}$$

and hence

$$J(X)+(1-p^2)J'(X)=[1,1,0,0](x)+p[1,1,1,1](x)+[1,0,1,0](x)\\ -[1,1,1,0](x)+p^3[2,1,2,1](x)\,.$$

On the other hand,

$$\begin{aligned}J''(X)&=\delta(x_1\in p\boldsymbol{Z}_p)\sum_{b\in\boldsymbol{Z}_p/p\boldsymbol{Z}_p}\sigma\left(\begin{pmatrix}x_1 & p^{-1}(bx_1+x_2)\\ x_3 & p^{-1}(bx_3+x_4)\end{pmatrix}\right)\\ &\quad+\delta(x_1\in p\boldsymbol{Z}_p)\sigma\left(\begin{pmatrix}p^{-1}x_1 & x_2\\ p^{-1}x_3 & x_4\end{pmatrix}\right)+p^3\sigma\left(\begin{pmatrix}p^{-2}x_1 & p^{-1}x_2\\ p^{-2}x_3 & p^{-1}x_4\end{pmatrix}\right)\\ &=\sigma(p^{-1}x_1)\sigma(p^{-1}x_2)\sum_{b\in\boldsymbol{Z}_p/p\boldsymbol{Z}_p}\sigma(x_3)\sigma(p^{-1}(bx_3+x_4))\\ &\quad+[1,0,1,0](x)+p^3[2,1,2,1](x)\\ &=p[1,1,1,1](x)+[1,1,0,0](x)-[1,1,1,0](x)\\ &\quad+[1,0,1,0](x)+p^3[2,1,2,1](x)\,.\end{aligned}$$

Thus Equality (7.2) for this case immediately follows.

(c) In this case, we have

$$\begin{aligned}J(X)=&\sum_{\substack{a\in\boldsymbol{Z}_p/p\boldsymbol{Z}_p\\ c\in\boldsymbol{Z}_p/p^2\boldsymbol{Z}_p}}\sigma\left(\begin{pmatrix}p^{-1}x_1 & p^{-1}x_2\\ p^{-2}x_2+p^{-1}c & p^{-2}x_4-p^{-1}a\end{pmatrix}\right)\\ &+\sum_{(a,b,c)\in\Lambda}\sigma\left(p^{-1}\left(x+\begin{pmatrix}a & b\\ c & -a\end{pmatrix}\right)\right)\\ &+\sum_{\substack{a,d\in\boldsymbol{Z}_p/p\boldsymbol{Z}_p\\ b\in\boldsymbol{Z}_p/p^2\boldsymbol{Z}_p}}\sigma\left(\begin{pmatrix}p^{-2}(x_1+dx_3)+p^{-1}a & p^{-2}(x_2+dx_4)+p^{-1}(b+ad)\\ p^{-1}x_3 & p^{-1}x_4-a\end{pmatrix}\right)\\ =&\ p\sigma(p^{-1}x)+\delta(x_1+x_4\in p\boldsymbol{Z}_p,\ x_1^2+x_2x_3\in p\boldsymbol{Z}_p)(\sigma(x)-\sigma(p^{-1}x))\\ &+p^2\sigma(p^{-1}x)\\ =&\ (p^2+p-1)\sigma(p^{-1}x)+\delta(x_1+x_4\in p\boldsymbol{Z}_p,\ x_1^2+x_2x_4\in p\boldsymbol{Z}_p)\sigma(x)\,.\end{aligned}$$

On the other hand, it is easily seen that

$$J'(X)=\sigma(p^{-1}x)$$

and

$$J''(X)=\delta(x_1+x_4\in p\boldsymbol{Z}_p)\left\{\sigma\left(\begin{pmatrix}p^{-1}x_1 & x_2\\ p^{-1}x_3 & x_4\end{pmatrix}\right)+\sum_{b\in\boldsymbol{Z}_p/p\boldsymbol{Z}_p}\sigma\left(\begin{pmatrix}x_1 & p^{-1}(bx_1+x_2)\\ x_3 & p^{-1}(bx_3+x_4)\end{pmatrix}\right)\right\}.$$

To prove (7.2), it is sufficient to show that

$$p\sigma(p^{-1}x)+\delta(x_1+x_4\in p\mathbf{Z}_p)\Bigg\{\delta(x_1+x_2x_3\in p\mathbf{Z}_p)\sigma(x) - \sum_{b\in\mathbf{Z}_p/p\mathbf{Z}_p}\sigma\left(\begin{pmatrix} x_1 & p^{-1}(bx_1+x_2) \\ x_3 & p^{-1}(bx_3+x_4)\end{pmatrix}\right)-[1,0,1,0](x)\Bigg\}$$

vanishes. This is proved by a tedious but straightforward calculation and we omit the detail.
(d) In this case, we have

$$J(X)=\sum_{\substack{a\in\mathbf{Z}_p/p\mathbf{Z}_p\\ b\in\mathbf{Z}_p/p^2\mathbf{Z}_p}}\sigma\left(\begin{pmatrix} p^{-2} & 0 \\ 0 & p^{-1}\end{pmatrix}x+\begin{pmatrix} p^{\alpha-2}a & p^{-2}b \\ 0 & -p^{-1}a\end{pmatrix}\right),$$

$$J'(X)=0\,,$$

$$J''(X)=\delta(x_1+p^{\alpha}x_4\in p^2\mathbf{Z}_p)\sigma\left(\begin{pmatrix} p^{-1}x_1 & x_2 \\ p^{-1}x_3 & x_4\end{pmatrix}\right).$$

Since

$$J(X)=\sum_{a\in\mathbf{Z}_p/p\mathbf{Z}_p}\sigma\left(\begin{pmatrix} p^{-2}x_1+p^{\alpha-2}a & x_2 \\ p^{-1}x_3 & p^{-1}(x_4-a)\end{pmatrix}\right)=J''(X)\,,$$

we are done.
(e) In this remaining case, we have

$$J(X)=J'(X)=J''(X)=0$$

and the proof of (7.2) has been completed.

8. Proof of Proposition 6.2. In this section, we assume that $p\mid d_B$ and prove Proposition 6.2. The proof of (6.3) is straightforward. To prove (6.4), we need some preparation. By $\sigma_{m,n}$, we denote the characteristic function of $M_{m,n}(\mathcal{O}_p)$. As in Section 7, we often write σ for $\sigma_{m,n}$. For a subset A of B_p, we put $A^- := \{a\in A\mid \mathrm{tr}(a)=0\}$. Recall that $\mathcal{O}_p$ is the maximal order of B_p, Π is a (fixed) prime element of B_p and $\pi=n(\Pi)$. We put $\mathfrak{P}_p=\Pi\mathcal{O}_p$.

8.1. We now collect several facts on the arithmetic of B_p used in the later discussion.

LEMMA 8.1. *We have*

$$\sharp(\mathcal{O}_p/\pi\mathcal{O}_p)=p^4\,,\ \ \sharp(\mathcal{O}_p/\Pi\mathcal{O}_p)=p^2\,,\ \ \sharp(\mathcal{O}_p^-/\pi\mathcal{O}_p^-)=p^3\,,\ \ \sharp((\Pi^{-1}\mathcal{O}_p)^-/\mathcal{O}_p^-)=p^2\,.$$

LEMMA 8.2. *For $x \in B_p$, we have*

$$\sum_{b\in\mathcal{O}_p^-/\pi\mathcal{O}_p^-} \sigma(\pi^{-1}(x+b)) = \delta(\mathrm{tr}(x)\in p\mathbf{Z}_p)\sigma(x)\,,$$
$$\sum_{b\in\mathcal{O}_p^-/\pi\mathcal{O}_p^-} \sigma(\Pi^{-1}(x+b)) = p^2\delta(\mathrm{tr}(x)\in p\mathbf{Z}_p)\sigma(x)\,,$$
$$\sum_{b\in(\Pi^{-1}\mathcal{O}_p)^-/\mathcal{O}_p^-} \sigma(x+b) = \sigma(\Pi x)\,,$$
$$\sum_{b\in\mathfrak{P}_p^-/\pi\mathfrak{P}_p^-} \sigma(\pi^{-1}(x+b)) = p\sigma(\Pi^{-1}x)\,,$$
$$\sum_{b\in\mathcal{O}_p^-/\mathfrak{P}_p^-} \sigma(\Pi^{-1}(x+b)) = \delta(\mathrm{tr}(x)\in p\mathbf{Z}_p)\sigma(x)\,,$$
$$\sum_{b\in\mathfrak{P}_p^-/\pi\mathfrak{P}_p^-} \sigma(\pi^{-1}\Pi^{-1}x+b) = \delta(\mathrm{tr}(x)\in p^2\mathbf{Z}_p)\sigma(\Pi^{-1}x)\,.$$

8.2. We first consider the case $p \mid D$. We then have $\hat{\phi}_p = \phi_p^+ + \phi_p^-$, where ϕ_p^+ (resp. ϕ_p^-) is the characteristic function of $U_p\,\mathrm{diag}(1, p^{-1})U_p$ (resp. $U_p\,\mathrm{diag}(p^{-1}, 1)U_p$).

LEMMA 8.3. *We have*

$$r(\Phi_p^1, 1, 1)\varphi_0(X, t) = p^{3/2}\sigma'(\pi t)I(X)\,,$$

where

$$I\begin{pmatrix} x \\ y \end{pmatrix} = \sum_{b\in\mathcal{O}_p^-/\pi\mathcal{O}_p^-} \sigma\begin{pmatrix} \pi^{-1}(x+by) \\ y \end{pmatrix} + \sum_{c\in(\Pi^{-1}\mathcal{O}_p-\mathcal{O}_p)^-/\mathcal{O}_p^-} \sigma\begin{pmatrix} \Pi^{-1}(x+cy) \\ \Pi^{-1}y \end{pmatrix} + \sigma\begin{pmatrix} x \\ \pi^{-1}y \end{pmatrix}.$$

PROOF. This follows from the definition of r and the coset decomposition

$$K_p\begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}K_p = \bigcup_{b\in\mathcal{O}_p^-/\pi\mathcal{O}_p^-} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}K_p \cup \bigcup_{c\in(\Pi^{-1}\mathcal{O}_p-\mathcal{O}_p)^-/\mathcal{O}_p^-} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Pi & 0 \\ 0 & \Pi \end{pmatrix}K_p \cup \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}K_p\,. \qquad \square$$

LEMMA 8.4. (i) *If $y \in \mathcal{O}_p^\times$, we have*

$$I\begin{pmatrix} x \\ y \end{pmatrix} = \delta(\mathrm{tr}(xy^{-1})\in p\mathbf{Z}_p)\sigma(x)\,.$$

(ii) *If $y \in \Pi\mathcal{O}_p^\times$, we have*

$$I\begin{pmatrix} x \\ y \end{pmatrix} = p^2\delta(\mathrm{tr}(xy^{-1}) \in p\mathbf{Z}_p)\sigma(\Pi^{-1}x) - \sigma(\Pi^{-1}x) + \sigma(x)\,.$$

(iii) *If $y \in \pi\mathcal{O}_p$, we have*

$$I\begin{pmatrix} x \\ y \end{pmatrix} = p^3\sigma(\pi^{-1}x) + (p^2-1)\sigma(\Pi^{-1}x) + \sigma(x)\,.$$

PROOF. When $y \in \mathcal{O}_p^\times$, we have

$$I\begin{pmatrix} x \\ y \end{pmatrix} = \sum_{b\in\mathcal{O}_p^-/\pi\mathcal{O}_p^-} \sigma(\pi^{-1}(xy^{-1}+b)) = \delta(\mathrm{tr}(xy^{-1}) \in p\mathbf{Z}_p)\sigma(x)\,.$$

When $y \in \Pi\mathcal{O}_p^\times$, we have

$$\begin{aligned} I\begin{pmatrix} x \\ y \end{pmatrix} &= \sum_{b\in\mathcal{O}_p^-/\pi\mathcal{O}_p^-} \sigma(\Pi^{-1}(xy^{-1}+b)) + \sum_{c\in(\Pi^{-1}\mathcal{O}_p)^-/\mathcal{O}_p^-} \sigma(xy^{-1}+c) - \sigma(xy^{-1}) \\ &= p^2\delta(\mathrm{tr}(xy^{-1}) \in p\mathbf{Z}_p)\sigma(\Pi^{-1}x) + \sigma(x) - \sigma(\Pi^{-1}x)\,. \end{aligned}$$

When $y \in \pi\mathcal{O}_p$, we have

$$\begin{aligned} I\begin{pmatrix} x \\ y \end{pmatrix} &= \sharp(\mathcal{O}_p^-/\pi\mathcal{O}_p^-)\sigma(\pi^{-1}x) + \sharp((\Pi^{-1}\mathcal{O}_p - \mathcal{O}_p)^-/\mathcal{O}_p^-)\sigma(\Pi^- x) + \sigma(x) \\ &= p^3\sigma(\pi^{-1}x) + (p^2-1)\sigma(\Pi^{-1}x) + \sigma(x)\,. \end{aligned}$$

□

LEMMA 8.5. *We have*

$$r(1, \phi_p^+, 1)\varphi_0(X,t) = p^{1/2}\sigma'(pt)J^+(X)\,,$$

where

$$J^+\begin{pmatrix} x \\ y \end{pmatrix} = \sigma(y) \times \begin{cases} \delta(\mathrm{tr}(xy^{-1}) \in p\mathbf{Z}_p)\sigma(x)\,, & y \in \mathcal{O}_p^\times\,, \\ \sigma(x)\,, & y \in \Pi\mathcal{O}_p\,. \end{cases}$$

PROOF. Since

$$U_p\,\mathrm{diag}(1, p^{-1})U_p = \bigcup_{b\in\mathbf{Z}_p/p\mathbf{Z}_p} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} U_p\,,$$

we have

$$\begin{aligned} r(1, \phi_p^+, 1)\varphi_0(X,t) =&\ p^{-1/2}\sigma'(pt) \sum_{b\in\mathbf{Z}_p/p\mathbf{Z}_p} \psi\left(\frac{bt}{2}\,\mathrm{tr}(X^*QX)\right)\sigma(X) \\ &= p^{-1/2}\sigma'(pt)\cdot p\cdot\delta(\mathrm{tr}(x^\sigma y) \in \mathbf{Z}_p)\sigma\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) \\ &= p^{1/2}\sigma'(pt)\sigma(x) \times \begin{cases} \delta(\mathrm{tr}(xy^{-1}) \in p\mathbf{Z}_p)\,, & y \in \mathcal{O}_p^\times\,, \\ 1\,, & y \in \Pi\mathcal{O}_p\,, \end{cases} \end{aligned}$$

which proves the lemma.

LEMMA 8.6. *We have*

$$r(1, \phi_p^-, 1)\varphi_0(X, t) = p^{5/2}\sigma'(pt)J^-(X)\,,$$

where

$$J^-\begin{pmatrix} x \\ y \end{pmatrix} = \sigma(y) \times \begin{cases} 0\,, & y \in \mathcal{O}_p^\times\,, \\ (\delta(\mathrm{tr}(xy^{-1}) \in p\boldsymbol{Z}_p) - p^{-1})\sigma(\Pi^{-1}x)\,, & y \in \Pi\mathcal{O}_p^\times\,, \\ p\sigma(\pi^{-1}x) + (1 - p^{-1})\sigma(\Pi^{-1}x)\,, & y \in \pi\mathcal{O}_p\,. \end{cases}$$

PROOF. To prove the lemma, we recall that, for $h \in GL_2(\boldsymbol{Q}_p)$ and $\varphi \in \boldsymbol{V}_p$,

$$(\mathcal{I} \circ r(1, h, 1)\varphi)(X, t) = |\det h|^{-1/2}\mathcal{I}\varphi(\det(h) \cdot h^{-1}X, \det(h^{-1}) \cdot t)\,,$$

where

$$\mathcal{I}\varphi\left(\begin{pmatrix} x \\ y \end{pmatrix}, t\right) = \int_{B_p} \psi(-t\,\mathrm{tr}(u^\sigma x))\varphi\left(\begin{pmatrix} u \\ y \end{pmatrix}, t\right)du\,.$$

Here the measure du on B_p is normalized by $\mathrm{vol}(\mathcal{O}_p) = p^{-1}$. It is easily verified that

$$\mathcal{I}\varphi_0\left(\begin{pmatrix} x \\ y \end{pmatrix}, t\right) = p^{-1}\sigma(\Pi x)\sigma(y)\sigma'(t)$$

and

$$\mathcal{I}^{-1}\varphi\left(\begin{pmatrix} x \\ y \end{pmatrix}, t\right) = |t|^4 \int_{B_p} \psi(t\,\mathrm{tr}(u^\sigma x))\varphi\left(\begin{pmatrix} u \\ y \end{pmatrix}, t\right)du\,.$$

It follows that

$$\begin{aligned}(\mathcal{I} \circ r(1, \phi_p^-, 1)\varphi_0)(X, t) &= p^{-1/2} \sum_{b \in \boldsymbol{Z}_p/p\boldsymbol{Z}_p} \mathcal{I}\varphi_0\left(\begin{pmatrix} 1 & 0 \\ b & p^{-1} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, pt\right) \\ &= p^{-3/2}\sigma(\Pi x)\sigma'(pt) \sum_{b \in \boldsymbol{Z}_p/p\boldsymbol{Z}_p} \sigma(bx + p^{-1}y)\,.\end{aligned}$$

We thus have

$$\begin{aligned}r(1, \phi_p^-, 1)\varphi_0(X, t) &= |t|^4 \int_{B_p} \psi(t\,\mathrm{tr}(u^\sigma x))p^{-3/2}\sigma(\Pi u)\sigma'(pt) \sum_{b \in \boldsymbol{Z}_p/p\boldsymbol{Z}_p} \sigma(bu + p^{-1}y)du \\ &= p^{5/2}\sigma'(pt)K\begin{pmatrix} x \\ y \end{pmatrix},\end{aligned}$$

where

$$K\begin{pmatrix} x \\ y \end{pmatrix} = \int_{B_p} \psi(p^{-1}\,\mathrm{tr}(u^\sigma x))\sigma(\Pi x) \sum_{b \in \boldsymbol{Z}_p/p\boldsymbol{Z}_p} \sigma(bu + p^{-1}y)du.$$

First observe that $K\binom{x}{y} = 0$ if $y \notin \Pi\mathcal{O}_p$. Assume that $y \in \Pi\mathcal{O}_p$. Then

$$\begin{aligned}
K\begin{pmatrix} x \\ y \end{pmatrix} &= \sigma(p^{-1}y)\int_{\Pi^{-1}\mathcal{O}_p} \psi(p^{-1}\operatorname{tr}(u^\sigma x))du \\
&\quad + \sum_{b\in(\mathbf{Z}_p - p\mathbf{Z}_p)/p\mathbf{Z}_p} \int_{\Pi^{-1}\mathcal{O}_p} \psi(p^{-1}\operatorname{tr}(u^\sigma x))\sigma(b^{-1}u + p^{-1}y)du \\
&= \operatorname{vol}(\Pi^{-1}\mathcal{O}_p)\sigma(p^{-1}y)\sigma(p^{-1}x) \\
&\quad + \sum_{b\in(\mathbf{Z}_p - p\mathbf{Z}_p)/p\mathbf{Z}_p} \int_{\Pi^{-1}\mathcal{O}_p} \psi(p^{-1}\operatorname{tr}((u - bp^{-1}y)^\sigma x))\sigma(b^{-1}u)du \\
&= p\sigma(\pi^{-1}x)\sigma(\pi^{-1}y) \\
&\quad + \sum_{b\in(\mathbf{Z}_p - p\mathbf{Z}_p)/p\mathbf{Z}_p} \psi(p^{-2}b\operatorname{tr}(y^\sigma x))\int_{\Pi^{-1}\mathcal{O}_p} \psi(p^{-1}\operatorname{tr}(u^\sigma x))\sigma(u)du \\
&= p\sigma(p^{-1}x)\sigma(\pi^{-1}y) \\
&\quad + \{p\delta(\operatorname{tr}(y^\sigma x) \in p^2\mathbf{Z}_p) - 1\}\operatorname{vol}(\mathcal{O}_p)\sigma(\Pi^{-1}x)\,.
\end{aligned}$$

The last term is equal to $\{\delta(\operatorname{tr}(xy^{-1}) \in p\mathbf{Z}_p) - p^{-1}\}\sigma(\Pi^{-1}x)$ if $y \in \Pi\mathcal{O}_p^\times$, and $(1 - p^{-1})\sigma(\Pi^{-1}x)$ if $y \in \pi\mathcal{O}_p$. This proves the lemma.

The following fact is clear.

LEMMA 8.7. *We have*

$$r(1, 1, \phi'_p)\varphi_0(X, t) = p^{3/2}\sigma'(pt)J'(X)\,,$$

where

$$J'\begin{pmatrix} x \\ y \end{pmatrix} = \sigma(\Pi^{-1}x)\sigma(\Pi^{-1}y)\,.$$

A straightforward calculation shows the following formula, which completes the proof of (6.4) in the case where $p \mid D$.

PROPOSITION 8.8. *We have*

$$J(X) - J^+(X) - p^2J^-(X) + (1 - p)J'(X) = 0\,.$$

8.3. In this subsection, we suppose that $p \nmid D$. We only give a sketch of the proof of (6.4) in this case, since the proof is similar to that in 8.2. First observe that

$$\begin{aligned}
K_p\begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}K_p &= \bigcup_{b\in\mathfrak{P}_p^-/\pi\mathfrak{P}_p^-} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}K_p \\
&\qquad \cup \bigcup_{c\in(\mathcal{O}_p - \mathfrak{P}_p)^-/\mathfrak{P}_p^-} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Pi & 0 \\ 0 & \Pi \end{pmatrix}K_p \cup \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}K_p\,.
\end{aligned}$$

LEMMA 8.9. *We have*

$$r(\Phi_p^1, 1, 1)\varphi_0(X,t) = p^{3/2}\delta'(\pi t)I(X)\,,$$
$$r(1, \hat{\phi}_p, 1)\varphi_0(X,t) = p^{1/2}\delta'(\pi t)J(X)\,,$$
$$r(1, 1, \hat{\phi}_p)\varphi_0(X,t) = p^{3/2}\delta'(\pi t)J'(X)\,,$$

where

$$I(X) = \sum_{b\in\mathfrak{P}_p^-/\pi\mathfrak{P}_p^-} \sigma\begin{pmatrix} \pi^{-1}(x+by) \\ \Pi y \end{pmatrix} + \sum_{c\in(\mathcal{O}_p-\mathfrak{P}_p)^-/\mathfrak{P}_p^-} \sigma\begin{pmatrix} \Pi^{-1}(x+cy) \\ y \end{pmatrix} + \sigma\begin{pmatrix} x \\ \Pi^{-1}y \end{pmatrix},$$

$$J(X) = \delta(\mathrm{tr}(x^\sigma y) \in p\mathbf{Z}_p)\sigma\begin{pmatrix} x \\ \Pi y \end{pmatrix} + p^3\sigma\begin{pmatrix} \pi^{-1} \\ \Pi^{-1}y \end{pmatrix},$$

$$J'(X) = \sigma\begin{pmatrix} \Pi^{-1}x \\ y \end{pmatrix}.$$

Using Lemmas 8.2 and 8.9, we obtain the following formula, from which (6.4) immediately follows.

PROPOSITION 8.10.

$$I(X) = J(X) + (p-1)J'(X)\,.$$

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