Tohoku Math. J.
60 (2008), 219–225

SMOOTH FANO POLYTOPES CAN NOT BE INDUCTIVELY CONSTRUCTED

MIKKEL ØBRO

(Received March 26, 2007, revised September 10, 2007)

Abstract. We examine a concrete smooth Fano 5-polytope P with 8 vertices with the following properties: There does not exist a smooth Fano 5-polytope Q with 7 vertices such that P contains Q, and there does not exist a smooth Fano 5-polytope R with 9 vertices such that R contains P. As the polytope P is not pseudo-symmetric, it is a counter example to a conjecture proposed by Sato.

1. Introduction.

Many papers have been concerned about the classification of smooth Fano polytopes (among these, e.g., [2, 4, 5, 8] and references therein). These polytopes have been completely classified up to dimension 4 modulo unimodular equivalence. Recently the classification of smooth Fano 5-polytopes has been announced ([7]).

One approach is to attempt to construct smooth Fano d-polytopes inductively from simpler or already known ones by adding and removing vertices according to some rule, while staying inside the realm of smooth Fano d-polytopes for some fixed $d \geq 1$.

This idea is behind the notion of F-equivalence, due to Sato in [8]. By $\mathcal{V}(P)$ we denote the set of vertices of a polytope P.

DEFINITION 1.1 (equivalent to Definitions 1.1 and 6.1 in [8]). Two smooth Fano d-polytopes P and Q are called *F-equivalent* if there exists a sequence

$$P_0, P_1, \dots, P_{k-1}, P_k, \quad k \geq 0$$

of smooth Fano d-polytopes P_i satisfying the following:

1. P and Q are unimodular equivalent to P_0 and P_k, respectively.
2. For every $1 \leq i \leq k$ either $\mathcal{V}(P_{i-1}) = \mathcal{V}(P_i) \cup \{w\}$ or $\mathcal{V}(P_i) = \mathcal{V}(P_{i-1}) \cup \{w\}$ for some lattice point $w \neq 0$.
3. If $w \in \mathcal{V}(P_i) \setminus \mathcal{V}(P_{i-1})$, then there exists a proper face F of P_{i-1} such that $w = \sum_{v \in \mathcal{V}(F)} v$ and the set of facets of P_i containing w is equal to

$$\{\mathrm{conv}(\{w\} \cup (\mathcal{V}(F') \setminus \{v\})) \mid F' \text{ facet of } P_{i-1},\ F \subseteq F',\ v \in \mathcal{V}(F)\}.$$

If $w \in \mathcal{V}(P_{i-1}) \setminus \mathcal{V}(P_i)$, a similar condition holds.

The third requirement in the definition above is the rule of vertex adding and removal. It has an equivalent formulation in terms of the corresponding smooth toric Fano varieties: The

2000 *Mathematics Subject Classification.* Primary 52B20; Secondary 14M25.
Key words and phrases. Smooth Toric Fano varieties.

toric variety corresponding to P_i is an equivariant blow-up or blow-down of the toric variety corresponding to P_{i-1}.

Clearly, F-equivalence is an equivalence relation on the set of smooth Fano d-polytopes. The problem is now: Find a set of representatives, so that every smooth Fano d-polytope is F-equivalent to one of these representatives.

Sato proposes the following conjecture. Recall that a smooth Fano polytope P is called pseudo-symmetric if there exists a facet F of P, such that $-F$ is also a facet. The notion of pseudo-symmetry is due to Ewald and pseudo-symmetric smooth Fano d-polytopes have been classified completely for every $d \geq 1$ (see [5]).

CONJECTURE 1.2 ([8, Conjectures 1.3 and 6.3]). *Any smooth Fano d-polytope is either pseudo-symmetric or F-equivalent to the simplex*

$$T_d := \mathrm{conv}\{e_1, \dots, e_d, -e_1 - \cdots - e_d\},$$

where (e_i) *is the standard integral basis of the lattice* $\mathbf{Z}^d$.

The conjecture is known to hold for $d \leq 4$ ([8, Theorems 7.1 and 8.1]). Indeed, every smooth Fano 3-polytope is F-equivalent to the simplex T_3, and there are only 2 smooth Fano 4-polytopes not F-equivalent to the simplex T_4: They are the *del Pezzo* 4*-polytope* V^4 and the *pseudo del Pezzo* 4*-polytope* $\tilde{V}^4$, where

$$\begin{aligned} V^{2k} &= \mathrm{conv}\{\pm e_1, \dots, \pm e_{2k}, \pm(e_1 + \cdots + e_k - e_{k+1} - \cdots - e_{2k})\}, \\ \tilde{V}^{2k} &= \mathrm{conv}\{\pm e_1, \dots, \pm e_{2k}, e_1 + \cdots + e_k - e_{k+1} - \cdots - e_{2k}\}. \end{aligned}$$

Both V^4 and $\tilde{V}^4$ are alone in their F-equivalence class. However, notice that

$$V^{2k} = \mathrm{conv}(\mathcal{V}(\tilde{V}^{2k}) \cup \{-e_1 - \cdots - e_k + e_{k+1} + \cdots + e_{2k}\})$$

and

$$\tilde{V}^{2k} = \mathrm{conv}(\{\pm e_1, \dots, \pm e_{2k}\} \cup \{e_1 + \cdots + e_k - e_{k+1} - \cdots - e_{2k}\}).$$

Since $\mathrm{conv}\{\pm e_1, \dots, \pm e_{2k}\}$ is a smooth Fano $2k$-polytope F-equivalent to T_{2k} ([8, Theorem 6.7]), one might be tempted to define a new equivalence relation, say *I-equivalence* (I for inductive), by requiring only 1 and 2 in Definition 1.1, meaning that there are no restrictions on vertex adding and removal. Then by the classification of pseudo-symmetric smooth Fano polytopes ([5]) and Theorem 6.7 in [8], any pseudo-symmetric smooth Fano d-polytope is I-equivalent to the simplex T_d. Inspired by Sato's conjecture one might then suspect: *Every smooth Fano d-polytope is I-equivalent to* T_d. This would indeed hold for $d \leq 4$.

The result of this paper is that Conjecture 1.2 is not true. We show this by means of an explicit counter example. More precisely, we examine a smooth Fano 5-polytope P with 8 vertices with the following properties:

(i) P is not pseudo-symmetric.

(ii) There does not exist a smooth Fano 5-polytope Q with 7 vertices, such that $Q \subset P$ (Proposition 4.1).

(iii) There does not exist a smooth Fano 5-polytope R with 9 vertices, such that $P \subset R$ (Proposition 4.2).

Furthermore, the example shows the existence of 'isolated' smooth Fano d-polytopes: It is not possible to obtain P from another smooth Fano 5-polytope by adding or removing a vertex, no matter what rule one uses for the inductive construction.

The author would like to thank Anders Buch and Johan P. Hansen for their advice and encouragement. The author also wishes to thank the anonymous referee for corrections and useful suggestions.

2. Notation. We begin by fixing the notation and recalling some basic facts.

By conv$K$ we denote the *convex hull* of the set K. When P is any polytope, i.e. the convex hull of a finite set of points, $\mathcal{V}(P)$ denotes the set of vertices of P.

A simplicial convex lattice polytope in $\boldsymbol{R}^d$ is called a *smooth Fano d-polytope* if the origin is contained in the interior of P and the vertices $\mathcal{V}(F)$ of every facet F of P is a $\mathbf{Z}$-basis of the integral lattice $\mathbf{Z}^d \subset \boldsymbol{R}^d$. Two smooth Fano d-polytopes $P_1, P_2 \subset \boldsymbol{R}^d$ are called *unimodular equivalent*, if there exists a bijective linear map $\varphi : \boldsymbol{R}^d \to \boldsymbol{R}^d$, such that $\varphi(\mathbf{Z}^d) = \mathbf{Z}^d$ and $\varphi(P_1) = P_2$. Unimodular equivalence classes of smooth Fano d-polytopes correspond to isomorphism classes of smooth Fano toric d-folds ([2, Theorem 2.2.4]).

When P is a smooth Fano d-polytope and F is any facet of P, there exists a unique linear map $u_F : \boldsymbol{R}^d \to \boldsymbol{R}$ such that $u_F(v) = 1$ for every $v \in \mathcal{V}(F)$. Clearly, $u_F(x) \leq 1$ for any $x \in P$ with equality if and only if $x \in F$. Every vertex v of P is a $\mathbf{Z}$-linear combination of $\mathcal{V}(F)$, so $u_F(v) \in \mathbf{Z}$. In particular, $u_F(v) \leq 0$ if and only if $v \notin \mathcal{V}(F)$.

Recall that $(d-2)$-dimensional faces of a d-polytope are called *ridges*. Every ridge is the intersection of precisely two facets of the polytope.

LEMMA 2.1. *Let P be a smooth Fano d-polytope. Let F_1 and F_2 be two facets of P such that $F_1 \cap F_2$ is a ridge G of P. Let v_i, $i = 1, 2$, be the vertex of F_i which is not contained in G. Then*

$$v_1 + v_2 = \sum_{w \in \mathcal{V}(G)} a_w w\,,$$

for some integers a_w.

Every point $x \in \mathbf{Z}^d$ is a unique $\mathbf{Z}$-linear combination of the vertices of F_1

$$x = \sum_{w \in \mathcal{V}(F_1)} b_w w\,, \quad b_w \in \mathbf{Z}$$

and

$$u_{F_2}(x) = u_{F_1}(x) + b_{v_1}(u_{F_1}(v_2) - 1)\,.$$

If $x \in \mathcal{V}(P)$, $x \neq v_2$ and $b_{v_1} < 0$, then $u_{F_1}(v_2) > u_{F_1}(x)$.

PROOF. Both $\mathcal{V}(F_1)$ and $\mathcal{V}(F_2)$ are lattice bases of $\mathbf{Z}^d$ and the first assertion follows. The second statement is clear for all $x \in \mathcal{V}(F_2)$, and then for all $x \in \mathbf{Z}^d$. Suppose $x \in$

$\mathcal{V}(P) \setminus (\mathcal{V}(F_1) \cup \mathcal{V}(F_2))$ and $b_{v_1} < 0$. Then $u_{F_2}(x) \leq 0$ and

$$u_{F_1}(x) \leq b_{v_1}(1 - u_{F_1}(v_2)) < u_{F_1}(v_2)\,,$$

which proves the last inequality. □

3. Primitive relations. Now we recall the concepts of primitive collections and relations, which are due to Batyrev in [1]. These are excellent tools for representation and classification of smooth Fano polytopes (see [2, 4, 8]).

Let $C = \{v_1, \dots, v_k\}$ be a subset of $\mathcal{V}(P)$, where P is a smooth Fano polytope. The set C is called a *primitive collection* if $\mathrm{conv}(C)$ is not a face of P, but $\mathrm{conv}(C \setminus \{v_i\})$ is a face of P for every $1 \leq i \leq k$. Consider the lattice point $x = v_1 + \cdots + v_k$. There exists a unique face $\sigma(C) \neq P$ of P, called the *focus* of C, such that x is a positive $\mathbf{Z}$-linear combination of vertices of $\sigma(C)$, that is

$$x = a_1 w_1 + \cdots + a_m w_m\,, \quad a_i \in \mathbf{Z}_+\,,$$

where $\{w_1, \dots, w_m\} = \mathcal{V}(\sigma(C))$. The linear relation

$$v_1 + \cdots + v_k = a_1 w_1 + \cdots + a_m w_m \tag{1}$$

is called a *primitive relation*. The integer $k - a_1 - \cdots - a_m$ is called the *degree* of the primitive relation (1) and is always positive ([2, Proposition 2.1.10]).

LEMMA 3.1 ([3, Corollary 4.4]). *Let*

$$v_1 + \cdots + v_k = a_1 w_1 + \cdots + a_m w_m \tag{2}$$

be a linear relation of vertices of a smooth Fano polytope P such that $a_i \in \mathbf{Z}_+$ and $\{v_1, \dots, v_k\} \cap \{w_1, \dots, w_m\} = \emptyset$. Suppose $k - a_1 - \cdots - a_m = 1$ and that $\mathrm{conv}\{w_1, \dots, w_m\}$ *is a face of P. Then* (2) *is a primitive relation, and whenever $\{w_1, \dots, w_m\}$ is contained in a face F, $(F \cup \{v_1, \dots, v_k\}) \setminus \{v_i\}$ is a face of P for every $1 \leq i \leq k$.*

We recall the well-known classification of smooth Fano d-polytopes with $d + 2$ vertices.

THEOREM 3.2 ([6, Theorem 1]). *Let P be a smooth Fano d-polytope with $d + 2$ vertices, $\mathcal{V}(P) = \{v_1, \dots, v_{d+2}\}$. Then the primitive relations of P are* (*up to renumeration of the vertices*)

$$v_1 + \cdots + v_k = 0\,, \quad 2 \leq k \leq d$$

and

$$v_{k+1} + \cdots + v_{d+2} = a_1 v_1 + \cdots + a_k v_k\,, \quad a_1, \dots, a_k \geq 0\,, \ a_1 + \cdots + a_k < d + 2 - k\,.$$

4. A counter example to Conjecture 1.2. Let $e_1, \dots, e_5$ be the standard basis of the integral lattice $\boldsymbol{Z}^5 \subset \boldsymbol{R}^5$. Consider the smooth Fano 5-polytope P with 8 vertices, $\mathcal{V}(P) = \{v_1, \dots, v_8\}$.

$$v_1 = e_1\,, \quad v_2 = e_2\,, \quad v_3 = e_3\,, \quad v_6 = e_4\,, \quad v_7 = e_5\,,$$
$$v_4 = -e_1 - e_2 - e_3 - 3e_4\,, \quad v_5 = -e_4\,, \quad v_8 = -e_1 - e_2 - 2e_4 - e_5\,.$$

The primitive relations are given by

$$v_1 + v_2 + v_3 + v_4 = 3v_5\,, \tag{3}$$

$$v_5 + v_7 + v_8 = v_3 + v_4\,, \tag{4}$$

$$v_3 + v_4 + v_6 = v_7 + v_8\,, \tag{5}$$

$$v_5 + v_6 = 0\,, \tag{6}$$

$$v_1 + v_2 + v_7 + v_8 = 2v_5\,. \tag{7}$$

When F is a face of P, $\mathcal{V}(F)$ is a subset of $\mathcal{V}(P) = \{v_1, \dots, v_8\}$. For simplicity we write $\{i_1, \dots, i_k\}$ to denote the polytope $\operatorname{conv}\{v_{i_1}, \dots, v_{i_k}\}$. In this notation the facets of P are

$$\begin{gathered}\{1,2,3,5,7\}\,,\quad \{1,2,3,5,8\}\,,\quad \{1,2,4,5,7\}\,,\quad \{1,2,4,5,8\}\,,\quad \{1,3,4,5,7\}\,,\\ \{1,3,4,5,8\}\,,\quad \{1,3,4,7,8\}\,,\quad \{1,3,6,7,8\}\,,\quad \{1,4,6,7,8\}\,,\quad \{2,3,4,5,7\}\,,\\ \{2,3,4,5,8\}\,,\quad \{2,3,4,7,8\}\,,\quad \{2,3,6,7,8\}\,,\quad \{2,4,6,7,8\}\,,\quad \{1,2,3,6,7\}\,,\\ \{1,2,3,6,8\}\,,\quad \{1,2,4,6,7\}\,,\quad \{1,2,4,6,8\}\,.\end{gathered}$$

We will now show that it is not possible to add or remove a lattice point from the vertex set $\mathcal{V}(P)$ and obtain another smooth Fano 5-polytope. As P is not pseudo-symmetric, it is a counter example to Conjecture 1.2.

PROPOSITION 4.1. *There does not exist a smooth Fano 5-polytope Q with 7 vertices such that $Q \subset P$.*

PROOF. Suppose there does exist a smooth Fano 5-polytope Q with 7 vertices such that $\mathcal{V}(P) = \mathcal{V}(Q) \cup \{v_i\}$ for some i, $1 \le i \le 8$. By the existing classification (Theorem 3.2) we know that Q has exactly two primitive relations of positive degree

$$v_{i_1} + \cdots + v_{i_k} = 0,\ v_{j_1} + \cdots + v_{j_{d-k}} = c_1 v_{i_1} + \cdots + c_k v_{i_k}\,.$$

There are two possibilities: Either $i \in \{5, 6\}$ or $i \in \{1, 2, 3, 4, 7, 8\}$.

Let $i \in \{5, 6\}$. Then there must be a primitive collection of vertices of Q with empty focus. But for both possible i, no non-empty subset of $\mathcal{V}(P) \setminus \{v_i\}$ add to 0.

Let $i \in \{1, 2, 3, 4, 7, 8\}$. Then $v_5 + v_6 = 0$ is a primitive relation of Q, and the other primitive collection is $C = \{v_1, v_2, v_3, v_4, v_7, v_8\} \setminus \{v_i\}$. The vertices in C must add up to cv_5, where $|c| \le 4$. It is now easy to check for every possible i that this is not the case.

Hence we are done. □

PROPOSITION 4.2. *There does not exist a smooth Fano 5-polytope R with 9 vertices, such that $P \subset R$.*

PROOF. Suppose there does exist a smooth Fano 5-polytope R with 9 vertices such that $\mathcal{V}(R) = \mathcal{V}(P) \cup \{v_9\}$ for some lattice point v_9.

As v_5 is a vertex of R, Relation (3) is a primitive relation of R (Lemma 3.1). Then $\{3, 4\}$ is a face of R. Relation (4) ensures that $\{7, 8\}$ is also a face of R. This means that Relations (3)–(5) are primitive relations of R.

As Relations (3)–(5) all have degree one, we can deduce a lot of the combinatorial structure of R: The set $\{3,4\}$ is a face of R. Thus

$$\{3,4,5,7\}\,, \quad \{3,4,5,8\}\,, \quad \{3,4,7,8\}$$

are faces of R (Relation (4)). Relation (3) implies that

$$\begin{aligned}&\{1,2,3,5,7\}\,, \quad \{1,2,4,5,7\}\,, \quad \{1,3,4,5,7\}\,, \quad \{2,3,4,5,7\}\,,\\ &\{1,2,3,5,8\}\,, \quad \{1,2,4,5,8\}\,, \quad \{1,3,4,5,8\}\,, \quad \{2,3,4,5,8\}\end{aligned}$$

are facets of R. By using Relation (4), we get 2 facets of R:

$$\{1,3,4,7,8\}\,, \quad \{2,3,4,7,8\}\,.$$

Relation (5) gives us 4 more facets of R such as

$$\{1,3,6,7,8\}\,, \quad \{1,4,6,7,8\}\,, \quad \{2,3,6,7,8\}\,, \quad \{2,4,6,7,8\}\,.$$

Among the original 18 facets of P, 14 are also facets of R. The remaining 4 facets are:

$$\{1,2,3,6,7\}\,, \quad \{1,2,3,6,8\}\,, \quad \{1,2,4,6,7\}\,, \quad \{1,2,4,6,8\}.$$

So v_9 is in a cone over one of these four facets of P, i.e., v_9 is a non-negative $\boldsymbol{Z}$-linear combination of vertices of one of the four facets. Without loss of generality we can assume that $v_9 \in \operatorname{cone}(v_1, v_2, v_3, v_6, v_7)$ (if this is not the case, apply an appropriate renumbering of the vertices of P, which fixes the primitive relations):

$$v_9 = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_6 v_6 + a_7 v_7\,, \quad a_i \geq 0 \text{ for all } i \in \{1,2,3,6,7\}\,.$$

Then $\{1,2,3,6,7\}$ is not a facet of R. But $F = \{1,2,3,5,7\}$ is a facet of R, so on the other side of the ridge $\{1,2,3,7\}$, there must be the facet $F' = \{1,2,3,7,9\}$. By Lemma 2.1 and Relation (6), $a_6 = 1$ and $1 > u_F(v_9) > u_F(v_6) = u_F(-v_5) = -1$. So $0 = u_F(v_9) = a_1 + a_2 + a_3 - 1 + a_7$.

Since $\{1,3,6,7,8\}$ and $\{2,3,6,7,8\}$ are facets of R, we must have $\{1,3,6,7,9\}$ and $\{2,3,6,7,9\}$ among the facets of R. This implies that

$$v_8 + v_9 \in \operatorname{span}\{v_1, v_3, v_6, v_7\} \cap \operatorname{span}\{v_2, v_3, v_6, v_7\} = \{0\} \times \{0\} \times \boldsymbol{R} \times \boldsymbol{R} \times \boldsymbol{R}\,.$$

As $v_8 + v_9 = (a_1 - 1)v_1 + (a_2 - 1)v_2 + a_3 v_3 + (a_6 - 2)v_6 + (a_7 - 1)v_7$, we must have $a_1 = a_2 = 1$.

Since $a_1 + a_2 + a_3 - 1 + a_7 = 0$, we must have that $a_3 < 0$ or $a_7 < 0$, which is a contradiction. We conclude that the smooth Fano 5-polytope R does not exist. □

REFERENCES

[1] V. BATYREV, On the classification of smooth projective toric varieties, Tohoku Math. J. 43 (1991), 569–585.

[2] V. BATYREV, On the classification of toric Fano 4-folds, J. Math. Sci. (New York) 94 (1999), 1021–1050.

[3] C. CASAGRANDE, Contractible classes in toric varieties, Math. Z. 243 (2003), 99–126.

[4] C. CASAGRANDE, Centrally symmetric generators in toric Fano varieties, Manuscripta Math. 111 (2003), 471–485.

[5] G. EWALD, On the classification of Toric Fano varieties, Discrete Comput. Geom. 3 (1988), 49–54.

[6] P. KLEINSCHMIDT, A classification of toric varieties with few generators, Aequationes Math. 35 (1988), 254–266.

[7] M. KREUZER AND B. NILL, Classification of toric Fano 5-folds, Preprint, math.AG/0702890.

[8] H. SATO, Toward the classification of higher-dimensional Toric Fano varieties, Tohoku Math. J. 52 (2000), 383–413.

TUSBYVEJ 9
2650 HVIDOVRE
DENMARK

E-mail address: mbro@danskebank.dk