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HAMILTONIAN ACTIONS AND HOMOGENEOUS LAGRANGIAN SUBMANIFOLDS

Dedicated to Professor Francesco Mercuri on his sixtieth birthday

LEONARDO BILIOTTI

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Abstract. We consider a connected symplectic manifold M acted on properly and in a Hamiltonian fashion by a connected Lie group G. Inspired by recent results, we study Lagrangian orbits of Hamiltonian actions. The dimension of the moduli space of the Lagrangian orbits is given. Also, we describe under which condition a Lagrangian orbit is isolated. If M is a compact Kähler manifold, we give a necessary and sufficient condition for an isometric action to admit a Lagrangian orbit. Then we investigate homogeneous Lagrangian submanifolds on the symplectic cut and on the symplectic reduction. As an application of our results, we exhibit new examples of homogeneous Lagrangian submanifolds on the blow-up at one point of the complex projective space and on the weighted projective spaces. Finally, applying our result which may be regarded as *Lagrangian slice theorem* for a Hamiltonian group action with a fixed point, we give new examples of homogeneous Lagrangian submanifolds on irreducible Hermitian symmetric spaces of compact or noncompact type.

1. Introduction. Let (M, ω) be a symplectic manifold. A Lagrangian submanifold of M is a submanifold of half dimension of M on which the symplectic form ω vanishes. Lagrangian submanifolds are intensively studied and have classically played an important role in symplectic geometry (see [18, 25, 26, 34]). Recently, their role has been expanded beyond that of understanding symplectic diffeomorphisms.

In [26] Oh asked for a group theoretical machinery producing Lagrangian submanifolds in Hermitian symmetric spaces and in a recent paper by Bedulli and Gori [4] the existence problem of homogeneous Lagrangian submanifolds in compact Kähler manifolds has been studied, obtaining a characterization of isometric actions admitting a Lagrangian orbit for a large class of compact Kähler manifolds, including irreducible Hermitian symmetric spaces.

In this paper we study homogeneous Lagrangian submanifolds in a symplectic manifold, and determine the dimension of the moduli space of Lagrangian orbits, describing under which condition a Lagrangian orbit of a reductive Lie group G is isolated. Our uniqueness result generalizes Theorem 2 in [4].

Our main tool will be the moment map that can be defined whenever we consider Hamiltonian action on M. More precisely, let (M, ω) be a connected symplectic manifold, G a connected Lie group of symplectic diffeomorphisms acting in a Hamiltonian fashion, and $\mathfrak{g}$

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the Lie algebra of G. This means that there exists a map $\mu : M \to \mathfrak{g}^*$, called the *moment map*, satisfying the following:

(1) For each $X \in \mathfrak{g}$, let

- $\mu^X : M \to \boldsymbol{R}$, $\mu^X(p) = \mu(p)(X)$, be the component of μ along X, and
- $X^\#$ be the vector field on M generated by the one parameter subgroup $\{\exp(tX) : t \in \boldsymbol{R}\} \subseteq G$.

Then

$$\mathrm{d}\mu^X = i_{X^\#}\omega\,,$$

i.e., μ^X is a Hamiltonian function for the vector field $X^\#$.

(2) μ is G-equivariant, i.e., $\mu(gp) = Ad^*(g)(\mu(p))$, where Ad^* is the coadjoint representation on $\mathfrak{g}^*$.

In general, the matter of existence/uniqueness of μ is delicate. However, whenever $\mathfrak{g}$ is semisimple the moment map exists and is unique ([16]). If (M, ω) is a compact Kähler manifold and G is a connected compact Lie group of holomorphic isometries, then the existence problem is solved ([20]): a moment map exists if and only if G acts trivially on the Albanese torus $\mathrm{Alb}(M)$.

In the sequel we always assume that the G-action on M is proper and for every $\alpha \in \mathfrak{g}^*$, $G\alpha$ is a locally closed coadjoint orbit of G. Observe that the condition for a coadjoint orbit to be locally closed is automatic for reductive groups and for their semidirect products with vector spaces. There exists an example of a solvable Lie group due to Mautner [33, p. 512], with non-locally closed coadjoint orbits. These assumptions are needed to apply the symplectic slice (see [3, 16, 27, 32]), and the symplectic stratification of the reduced space given in [3].

Before we state our first main result, we fix our notation for later use. Let $Gx = G/G_x$ be a G-orbit. Since G_x is compact, we may split $\mathfrak{g} = \mathfrak{g}_x \oplus \mathbf{m}$, as G_x-modules. We denote by $\mathfrak{n}(\mathfrak{g}_x)$ the Lie algebra of $N(G_x)$, i.e., the normalizer of G_x in G. Let $v = v_x + v_m \in \mathfrak{g}_x \oplus \mathbf{m}$ be an element of $\mathfrak{n}(\mathfrak{g}_x)$. Then $[v_m, \mathfrak{g}_x] \subset \mathfrak{g}_x$, i.e., $v_m \in \mathfrak{n}(\mathfrak{g}_x)$, which implies that $[v_m, \mathfrak{g}_x] = 0$, since $\mathbf{m}$ is G_x-invariant. This means

$$\mathfrak{n}(\mathfrak{g}_x) = \mathfrak{g}_x \oplus \{v \in \mathbf{m}\,;\, [v, \mathfrak{g}_x] = 0\} = \mathfrak{g}_x \oplus \mathbf{s}\,. \tag{1}$$

Let $\mathfrak{z}(\mathfrak{g})$ be the Lie algebra of the center of G. Clearly, $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{n}(\mathfrak{g}_x)$ and the projection $\pi : \mathfrak{n}(\mathfrak{g}_x) \to \mathfrak{g}_x$ maps $\mathfrak{z}(\mathfrak{g})$ to $\mathfrak{z}(\mathfrak{g}_x)$.

THEOREM 1.1. *Let (M, ω) be a connected symplectic G-Hamiltonian manifold with moment map $\mu : M \to \mathfrak{g}^*$. Assume that $G/G_x = Gx$ is a Lagrangian orbit. Then the dimension of the moduli space of the Lagrangian orbits containing Gx is* $\dim \mathfrak{z}(\mathfrak{g}) \cap \mathbf{s}$. *Therefore the dimension of the moduli space of the Lagrangian orbits is equal or less than* $\dim N(G_x)/G_x$ *and Gx is an isolated Lagrangian orbit if and only if the projection $\pi : \mathfrak{n}(\mathfrak{g}_x) \to \mathfrak{g}_x$ maps* $\mathfrak{z}(\mathfrak{g})$ *one-to-one to* $\mathfrak{z}(\mathfrak{g}_x)$. *Hence, if G is a semisimple Lie group, a Lagrangian orbit, if exists, is isolated. Moreover, in any level set $\mu^{-1}(c)$, a Lagrangian orbit, if exists, is isolated.*

Note that Theorem 1.1 answers under which condition an action have infinitely many Lagrangian orbits. The next result characterizes the actions having isolated Lagrangian orbit, generalizing Theorem 2 in [4].

THEOREM 1.2. *Let G be a connected reductive Lie group acting properly and in a Hamiltonian fashion on a connected symplectic manifold M. A Lagrangian G-orbit, Gp, is isolated if and only if there exists a semisimple connected closed Lie subgroup G' of G such that $Gp = G'p$.*

We may also characterize isometric actions on compact Kähler manifolds admitting a Lagrangian orbit, applying a result of Kirwan [21] together with the symplectic stratification of the reduced space given in [3].

THEOREM 1.3. *Let G be a compact connected Lie group acting isometrically and in a Hamiltonian fashion on a compact connected Kähler manifold M. Let μ denote the corresponding moment map. Then G admits a Lagrangian orbit if and only if there exists $c \in \mathfrak{z}(\mathfrak{g})$ such that $\mu^{-1}(c)$ is a Lagrangian submanifold.*

As an immediate corollary we have the following.

COROLLARY 1.4. *If G is a compact connected Lie group acting isometrically in a Hamiltonian fashion on compact Kähler manifold. Then at any level set $\mu^{-1}(c)$ there exists at most one Lagrangian orbit. Moreover, if G is a compact connected semisimple Lie group, then G admits a Lagrangian orbit if and only if $\mu^{-1}(0)$ is a Lagrangian submanifold of M.*

If G is compact, it is standard to fix an $Ad(G)$-invariant scalar product $\langle \cdot, \cdot \rangle$ and identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by means of $\langle \cdot, \cdot \rangle$, regarding μ as a $\mathfrak{g}$-valued map. It is also natural to study the squared moment map $\| \mu \|^2$ and its critical set. This function has been intensively studied in [21], obtaining strong information on the topology of M. In [6] we prove that if a point x realizes a local maximum of $\|\mu\|^2$, then the G-orbit through x is symplectic. It is natural to study the "dual" problem, i.e., the points realizing the minimum of $\|\mu\|^2$. Note that a Lagrangian orbit could describe this set by Theorem 1.3, whenever M is a compact Kähler manifold.

Next, we use Hamiltonian actions to construct non-standard homogeneous Lagrangian submanifolds in Kähler manifolds. Non-standard examples on the complex projective space are given in [2, 13, 14, 15]. Recently in [4], the classification of isometric actions of simple Lie groups admitting a Lagrangian orbit on complex projective spaces is given.

We study homogeneous Lagrangian submanifolds on the symplectic reduction ([11]) as well as on the symplectic cut ([10, 24]) by making use of a connection between Lagrangian orbits on M and those on symplectic reduction and on symplectic cut, see Proposition 3.1 and Proposition 3.5. These results are interesting, since the complex projective space is the reduced space of the standard S^1-action on $\boldsymbol{C}^n$ and the symplectic cut can be obtained by blowing-up M along a symplectic submanifold ([10, 24]).

TABLE 1.

G	M	M^*
$\mathrm{G}_2 \subseteq \mathrm{SO}(7) \subseteq \mathrm{SO}(8)$	$\mathrm{SO}(8)/\mathrm{U}(4)$	
$\mathrm{SO}(2) \times \mathrm{SO}(n),\ n \geq 3$	$\mathrm{SO}(n+2)/\mathrm{SO}(2) \times \mathrm{SO}(n)$	$\mathrm{SO}_o(2,n)/\mathrm{SO}(2) \times \mathrm{SO}(n)$
$Z(\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))) \times \mathrm{SO}(n)$	$\boldsymbol{CP}^n$	$\mathrm{SU}(1,n)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))$
$Z(\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2n))) \times \mathrm{Sp}(n),\ n \geq 2$	$\mathrm{SU}(2n+2)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2n))$	$\mathrm{SU}(2,2n)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2n))$
$\mathrm{U}(2n)$	$\mathrm{SO}(4n)/\mathrm{U}(2n)$	$\mathrm{SO}^*(4n)/\mathrm{U}(2n)$
$\mathrm{U}(2n+1) \subset \mathrm{U}(2n+2),\ n \geq 2$	$\mathrm{SO}(4n+4)/\mathrm{U}(2n+2)$	$\mathrm{SO}^*(4n+4)/\mathrm{U}(2n+2)$
$\mathrm{U}(n)$	$\mathrm{Sp}(n)/\mathrm{U}(n)$	$\mathrm{Sp}(n,\boldsymbol{R})/\mathrm{U}(n)$
$\mathrm{T}^1 \cdot \mathrm{E}_6$	$\mathrm{E}_7/\mathrm{T}^1 \cdot \mathrm{E}_6$	$\mathrm{E}_7^{-25}/\mathrm{T}^1 \cdot \mathrm{E}_6$
$\mathrm{T}^1 \cdot \mathrm{Spin}(9) \subseteq \mathrm{T}^1 \cdot \mathrm{Spin}(10)$	$\mathrm{E}_6/\mathrm{T}^1 \cdot \mathrm{Spin}(10)$	$\mathrm{E}_6^{-14}/\mathrm{T}^1 \cdot \mathrm{Spin}(10)$
$\mathrm{Spin}(7) \subset \mathrm{SO}(8) \subset \mathrm{SU}(8)$	$\mathrm{SU}(8)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(6))$	

We use these results to construct non-standard homogeneous Lagrangian submanifolds on the blow-up at one point of the complex projective space (Example 3.6 and Section 5). We also deduce, using the classification given in [4], the classification of the simple compact Lie groups K acting isometrically on $\boldsymbol{C}^n$ such that $\mathrm{S}^1 \times K$ admits a Lagrangian orbit on $\boldsymbol{C}^n$, (see Corollary 3.3 and Remark 3.4).

Since our result also holds when the symplectic reduction is an orbifold (see [11] for more detail about orbifolds), in Section 4 we give new examples of homogeneous Lagrangian submanifolds on weighted projective spaces.

Finally, applying Proposition 3.7 which deals with *Lagrangian slice* for a group acting with a fixed point, we give new examples of homogeneous Lagrangian submanifolds on irreducible Hermitian symmetric spaces of compact or noncompact type.

PROPOSITION 1.5. *Let G be a Lie group which appears in Table* 1. *Then G admits a Lagrangian orbit.*

We note that in Table 1, $Z(G)$ denotes the center of G.

2. Existence and uniqueness. Here we follow the notation as in [3] and in Introduction. The first easy remark is the following: if Gx is a Lagrangian submanifold of M, then $\mu(x) \in \mathfrak{z}(\mathfrak{g}^*)$, where

$$\mathfrak{z}(\mathfrak{g}^*) = \{\alpha \in \mathfrak{g}^* \,;\, \mathrm{ad}_X^*(\alpha) = 0 \ \text{ for any } X \in \mathfrak{g}\}\,.$$

Indeed, since $\operatorname{Ker} \mathrm{d}\mu_x = (T_x Gx)^{\perp_\omega}$ we have $\mu_{|Gx} = c \in \mathfrak{g}^*$, which implies that $c \in \mathfrak{z}(\mathfrak{g}^*)$.

PROOF OF THEOREM 1.1. Suppose that Gx is a Lagrangian orbit. We may assume that $\mu(x) = 0$. It follows from the symplectic slice that there exists a G-invariant neighborhood which is symplectomorphic to a neighborhood of the zero section of $Y = G \times_{G_x} (\mathfrak{g}/\mathfrak{g}_x)^*$ and the moment map is given by

$$\mu([g, v]) = Ad^*(g)(j(v)).$$

Recall that we may split $\mathfrak{g} = \mathfrak{g}_x \oplus \mathbf{m}$ as G_x-modules and j is induced by the above decomposition (see [3]). It is well-known, shrinking the neighborhood of the zero section of Y if necessary, that $\dim G[g, v] \geq \dim Gx$. Hence $G[e, v]$ is Lagrangian if and only if $\mu([e, v]) \in \mathfrak{z}(\mathfrak{g}^*)$ so that if and only if $j(v) \in \mathbf{m}^* \cap \mathfrak{z}(\mathfrak{g}^*)$, and hence if and only if $j(v) \in \mathfrak{z}(\mathfrak{g}^*) \cap \mathbf{s}^*$ by (1). This proves that the dimension of the moduli space of the Lagrangian orbits containing Gx is $\dim(\mathfrak{z}(\mathfrak{g}) \cap \mathbf{s}) \leq \dim N(G_x)/G_x$. We also deduce that Gx is isolated if and only if $\mathfrak{z}(\mathfrak{g}) \cap \mathbf{s} = \{0\}$ so that if and only if the projection of $\mathfrak{g}$ onto $\mathfrak{g}_x$ maps $\mathfrak{z}(\mathfrak{g})$ one-to-one to $\mathfrak{z}(\mathfrak{g}_x)$. Finally, since in a G-invariant neighborhood of a Lagrangian orbit the moment map is given by $\mu([g, m]) = Ad^*(g)(j(m))$, the moment map locally separates G-orbits, concluding the proof.

EXAMPLE 2.1. We consider $G = \mathrm{T}^1 \times \mathrm{SU}(2)$ acting on $\boldsymbol{C}^2 \oplus \boldsymbol{C}^2$ by

$$(t, A)(v, w) = (At^{-1}v, Aw).$$

This action is Hamiltonian with moment map

$$\mu((z_1, z_2), (w_1, w_2)) = \frac{i}{2} \begin{pmatrix} \|w_1\|^2 - \|w_2\|^2 & z_1\overline{z_2} + w_1\overline{w_2} \\ z_2\overline{z_1} + w_2\overline{w_1} & \|w_2\|^2 - \|w_1\|^2 \end{pmatrix} + \frac{1}{2} \|(z_1, z_2)\|^2 .$$

The orbit through $((1, 1), (1, -1))$ is Lagrangian and the T^1-orbit through $((1, 1), (1, -1))$ induces a curve in the moduli space of the Lagrangian orbits.

PROOF OF THEOREM 1.2. In the sequel we denote by G_{ss} the closed Lie semisimple connected subgroup of G whose Lie algebra is $[\mathfrak{g}, \mathfrak{g}]$.

Assume that there exists a closed semisimple Lie subgroup G' of G such that $Gp = G'p$ is Lagrangian. It is well-known (see [29]) that there exists a G-invariant neighborhood Y of Gp such that $\dim Gq \geq \dim Gp$ for every $q \in Y$. Since $G' \subset G_{\mathrm{ss}}$ we have $G_{\mathrm{ss}}p = Gp$. If we denote by $\bar{\mu}$ the moment map corresponding to the G_{ss}-action on M, we see that $\bar{\mu}(q) = 0$ if and only if $\mu(q) \in \mathfrak{z}(\mathfrak{g}^*)$, where μ is the moment map for the G-action on M, since the moment map of the G_{ss}-action is the projection on $\mathfrak{g}_{\mathrm{ss}}$ of μ. Therefore, shrinking Y if necessary, given $q \in Y$ we see that Gq is Lagrangian if and only if $G_{\mathrm{ss}}q$ is also so, proving that Gp is isolated, since G_{ss} is semisimple.

Vice-versa, let Gp be an isolated Lagrangian orbit. We may assume $\mu(p) = 0$. Since Gp is isolated, there exists a G-invariant neighborhood Y of Gp such that $\mu(Y) \cap \mathfrak{z}(\mathfrak{g}^*) = \{0\}$. Hence, given $x \in Y$ we have $\bar{\mu}(x) = 0$ if and only if $\mu(x) \in \mathfrak{z}(\mathfrak{g}^*)$ so that if and only if $\mu(x) = 0$. This proves that $Gp = \mu^{-1}(0) \cap Y = \bar{\mu}^{-1}(0) \cap Y$.

Let H be a principal isotropy for the G_{ss}-action on Gp. It is well-known (see [3, 32]) that the intersection of the stratum $M^{(H)}$ of orbit type (H) with the zero level set of the moment map $\bar{\mu}$ is a submanifold of M of constant rank, and the orbit space

$$(M_o)^{(H)} = (\bar{\mu}^{-1}(0) \cap M^{(H)})/G_{\mathrm{ss}}$$

has a natural symplectic structure $(\omega_o)_{(H)}$ whose pullback to $\bar{\mu}^{-1}(0) \cap M^{(H)}$ coincides with the restriction to $\bar{\mu}^{-1}(0) \cap M^{(H)}$ of the symplectic form on M. Since $\bar{\mu}^{-1}(0) \cap Y$ is Lagrangian, we get $(\omega_o)_{(H)} = 0$, which proves that $(M_o)^{(H)}$ is discrete. This implies that G_{ss} acts transitively on Gp, concluding the proof.

PROOF OF THE THEOREM 1.3. Assume that G admits a Lagrangian orbit. We may suppose that Gp is Lagrangian and $\mu(p) = 0$, where μ is the corresponding moment map. It is easy to check that $G^{\boldsymbol{C}} p$ is open in M. Therefore, M is projective algebraic and there exists a $G^{\boldsymbol{C}}$-equivariant embedding into some complex projective space ([20]). In particular, since the $G^{\boldsymbol{C}}$-action on M is algebraic, $G^{\boldsymbol{C}} p$ is Zariski open in M.

If $x \in \mu^{-1}(0)$ lies in a different G-orbit, then there exist, due to a results of Kirwan (see [21]), two $G^{\boldsymbol{C}}$-invariant disjoint neighborhoods U_p and U_x of p and x, respectively. Since U_p contains Gp, the neighborhood U_p must meet U_x, which is an absurd. This claims that there exists at most one Lagrangian orbit at any level set $\mu^{-1}(c)$.

Vice-versa, assume that $\mu^{-1}(c)$ is Lagrangian. Let H be an isotropy subgroup for the G-action on M. Since $M^{(H)} \cap \mu^{-1}(c)/G$ has a natural symplectic structure $(\omega)_H$ whose pullback to $M^{(H)} \cap \mu^{-1}(c)$ coincides with the restriction to $M^{(H)} \cap \mu^{-1}(c)$ of the symplectic form on M, we obtain that $M^{(H)} \cap \mu^{-1}(c)/G$ is discrete. This proves that given $p \in \mu^{-1}(c)$ we have $\mu^{-1}(c) = Gp$, since $\mu^{-1}(c)$ is connected (see [22]), concluding the proof.

REMARK 2.2. In the sequel we always assume that G acts properly and coisotropically on M. This means that there exists an open dense subset U such that for every $x \in U$, Gx is a coisotropic submanifold of M with respect to ω, i.e., $(T_x Gx)^{\perp_\omega} \subset T_x Gx$. Coisotropic actions are intensively studied in [17, 20], and in the articles [7, 8, 30] the complete classification of compact connected Lie groups acting isometrically, in a Hamiltonian fashion and coisotropically on irreducible compact Hermitian symmetric spaces has been given. In a forthcoming paper [9], we shall study coisotropic actions of Lie groups acting properly and in a Hamiltonian fashion on a symplectic manifold M. An equivalent condition for a connected Lie group G to act coisotropically on M is that for every $\alpha \in \mathfrak{g}^*$ the set $G\mu^{-1}(\alpha)/G$ is discrete (see [9]). In particular, if the fibers of the moment map $\mu : M \to \mathfrak{g}^*$ are connected, which is the case if M and G are compact ([22]), then G admits a Lagrangian orbit if and only if $\mu^{-1}(z)$ is a Lagrangian submanifold for some $z \in \mathfrak{z}(\mathfrak{g}^*)$.

Now, assume that a principal orbit is Lagrangian. Then there exists an abelian closed Lie group T which have the same orbits of the G-action on M. Indeed, let H be a principal isotropy. Since $\mu(M^{(H)}) \subset \mathfrak{z}(\mathfrak{g}^*)$, M is mapped by μ to $\mathfrak{z}(\mathfrak{g}^*)$. Therefore, given $x \in M^H$, it follows from symplectic slice that the H-submodule $\mathbf{m}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathbf{m}$ is abelian. Let T be the closure of the torus whose Lie algebra is $\mathbf{m}$. Note that $Tx = Gx$. Therefore, the set of regular points of T is contained in the set of G-regular points, which implies that T and G have the same orbits, since both the T-action and the G-action are proper actions.

3. Symplectic cut, symplectic reduction and homogeneous Lagrangian submanifolds. In this section we shall investigate how the existence of homogeneous Lagrangian submanifolds on M implies the existence of homogeneous Lagrangian submanifolds on the symplectic cut and on the symplectic reduction. For the sake of completeness we describe briefly these constructions and refer the reader to see [10, 11, 24] for a good exposition of these subjects.

Let M be a connected symplectic manifold and G a compact connected Lie group acting in a Hamiltonian fashion on M. Let K be a semisimple compact connected Lie subgroup of G and T^k a k-dimensional torus which centralizes K in G, i.e., $\mathrm{T}^k \subset C_G(K)$. In the sequel we denote by

$$\phi : M \to \mathfrak{k} \oplus \mathfrak{t}^k ,$$

where $\mathfrak{t}^k$ is the Lie algebra of T^k, the moment map of the $\mathrm{T}^k \cdot K$-action on M. We also denote by μ, respectively by ψ, the moment map of the K-action on M, and a moment map of the T^k-action on M.

Let $\lambda \in \mathfrak{t}^k$ be such that T^k acts freely on $\psi^{-1}(\lambda)$. Then the symplectic reduction

$$M_\lambda = \psi^{-1}(\lambda)/\mathrm{T}^k$$

is a symplectic manifold on which K acts, since it commutes with T^k, in a Hamiltonian fashion with moment map

$$\bar{\mu} : M_\lambda \to \mathfrak{k}, \quad \bar{\mu}([x]) = \mu(x) .$$

This proves that $\bar{\mu}([x]) = 0$ if and only if $\mu(x) = 0$ so that if and only if $\phi(x) \in \mathfrak{t}^k$.

Let $[p] \in M_\lambda$. It is easy to see that $k[p] = [p]$ if and only if there exists $r(k) \in \mathrm{T}^k$ such that $kp = r(k)p$, which is unique, since T^k acts freely on $\psi^{-1}(\lambda)$. This means that the following homomorphism of Lie groups,

$$K_{[p]} \xrightarrow{F} (\mathrm{T}^k \cdot K)_p, \quad F(k) = kr(k)^{-1} ,$$

is a covering map, due to the fact that K is semisimple. Hence

$$\dim K[p] = \dim(\mathrm{T}^k \cdot K)p - \dim \mathrm{T}^k .$$

Since $\dim M = \dim M_\lambda - 2 \dim \mathrm{T}^k$, we have proved the following result.

PROPOSITION 3.1. *Let M be a connected symplectic manifold on which a compact connected Lie group G acts in a Hamiltonian fashion on M. Let K be a compact connected semisimple Lie subgroup of G and* T^k *a k-dimensional torus which centralizes K in G. Let $\lambda \in \mathfrak{t}_k$ be such that* T^k *acts freely on* $\psi^{-1}(\lambda)$, *where ψ is a moment map of the* T^k*-action on M. Then* $(\mathrm{T}^k \cdot K)p$, $p \in \psi^{-1}(\lambda)$, *is Lagrangian if and only if $K[p]$ in M_λ is.*

REMARK 3.2. It should be remarked that Proposition 3.1 holds even if we only assume that $\lambda \in \mathfrak{t}^k$ is a regular value. In this case the symplectic reduction could be an orbifold (see [11]), since T^k could act almost freely on the level set $\psi^{-1}(\lambda)$. Hence the following map,

$$F : K_{[p]} \to (\mathrm{T}^k \cdot K)_p/\mathrm{T}^k_p ,$$

is a covering map, which implies $\dim K_{[p]} = \dim(\mathrm{T}^k \cdot K)_p$, since T^k_p is finite.

It is well-known that complex projective n-space is the Kähler reduction for the standard S^1-action on $\boldsymbol{C}^{n+1}$, (see [11]). Therefore, from Proposition 3.1, we deduce the following

COROLLARY 3.3. *Let K be a compact semisimple connected Lie group acting in a Hamiltonian fashion on $\boldsymbol{CP}^n$. Then K admits a Lagrangian orbit if and only if the* $\mathrm{S}^1 \times K$*-action on $\boldsymbol{C}^{n+1}$ admits a Lagrangian orbit.*

REMARK 3.4. From the classification given in [4], one can deduce which compact simple Lie groups K have the $K \times \mathrm{S}^1$-actions with Lagrangian orbits on $\boldsymbol{C}^n$.

Now, we consider a one-dimensional torus T^1 which centralizes a compact connected Lie subgroup K in G. We may consider the symplectic cut (see [10, 24]), which is briefly described as follows. In the sequel we regard the moment maps for the T^1-actions as $\boldsymbol{R}$-valued maps.

We consider the symplectic manifold $M \times \boldsymbol{C}$ with symplectic form $\omega - i dz \wedge d\bar{z}$. T^1 acts diagonally on $M \times \boldsymbol{C}$ as $t(m, z) = (tm, tz)$ with moment map $\bar{\psi} = \psi + \|z\|^2$.

Let $\lambda \in \boldsymbol{R}$ =Lie(T^1) be such that T^1 acts freely on $\psi^{-1}(\lambda)$. Then T^1 acts freely on $\bar{\psi}^{-1}(\lambda)$, so that we may consider the symplectic reduction,

$$M^\lambda = \bar{\psi}^{-1}(\lambda)/\mathrm{T}^1\,,$$

which is called the symplectic cut. Note that $\dim M = \dim M^\lambda$. K acts on M^λ as $k[m, z] = [km, z]$ with moment map

$$\bar{\mu}([x, z]) = \mu(x)\,,$$

where μ is the moment map of the K-action on M. Since if $z \neq 0$, $K_{[m,z]} = K_m$, we deduce that Km is Lagrangian if and only if $K[m, z]$ is.

PROPOSITION 3.5. *Let (M, ω) be a symplectic manifold on which a compact connected Lie group G acts in a Hamiltonian fashion. Assume that K is a compact connected subgroup of G whose centralizer in G contains a one-dimensional torus* T^1. *Then K admits a Lagrangian orbit in the open subset $\{x \in M; \psi(x) < \lambda\}$, ψ being the corresponding moment map of the* T^1*-action on M, if and only if the K-action on the symplectic cut, obtained from the* T^1*-action on M, admits a Lagrangian orbit.*

EXAMPLE 3.6. Let T^1 act on $\boldsymbol{CP}^n$ by

$$(t, [z_o, \ldots, z_n]) \to [t^{-1}z_o, z_1, \ldots, z_n]\,.$$

This is a Hamiltonian action with moment map

$$\phi([z_o, \ldots, z_n]) = \frac{1}{2}\frac{\|z_o\|^2}{\|z_o\|^2 + \cdots + \|z_n\|^2}\,.$$

One can prove that $\phi([1, \ldots, 0])$ is the global maximum of the moment map ϕ and $\phi^{-1}(1/2) = [1, \ldots, 0]$. Hence, as in [10, p. 5], if $\lambda = 1/2 - \varepsilon$, $\varepsilon \approx 0$, then the Kähler cut $(\boldsymbol{CP}^n)^\lambda$ is the blow-up of $\boldsymbol{CP}^n$ at $[1, \ldots, 0]$, which we shall indicate by $\boldsymbol{CP}^n_{[1,\ldots,0]}$. The torus action T^n on $\boldsymbol{CP}^n$ given by

$$(t_1, \ldots, t_n)([z_o, \ldots, z_n] = [z_o, t_1^{-1}z_1, \ldots, t_n^{-1}z_n]\,,$$

is Hamiltonian and the principal orbits are Lagrangian. Note that T^n acts in a Hamiltonian fashion on $\boldsymbol{CP}^n_{[1,\dots,0]}$ and for the above discussion we conclude that the T^n-action on $\boldsymbol{CP}_{[1,\dots,0]}$ admits a Lagrangian orbit. Moreover, it acts coisotropically and its principal orbits are Lagrangian.

Let $T^1 = Z(S(U(1) \times U(4))$ and $K = T^1 \times U(2)$ act on $\boldsymbol{CP}^4$ by

$$(t, A)(Z) = (t, A)[z_o, z_1, z_2, w_1, w_2] = (t, A)[z_o, z, w] = [tz_o, At^{-4}z, At^{-4}w] .$$

This action is Hamiltonian with moment map

$$\mu(Z) = \frac{i}{2\,\|Z\|^2}\begin{pmatrix} \|z\|^2 + \|w\|^2 & z_1\overline{z_2} + w_1\overline{w_2} \\ z_2\overline{z_1} + w_2\overline{w_1} & \|z\|^2 + \|w\|^2 \end{pmatrix}$$
$$+\frac{1}{2\,\|Z\|^2}\Big(-\,\|z_o\|^2 + \frac{1}{2}\,\|(z, w)\|^2\Big) .$$

Let $p = [z_o, 1, 1, 1, -1] \in \boldsymbol{CP}^4$. Note that $\mu(p) \in \mathfrak{z}(\mathfrak{u}(2)) \oplus \mathfrak{t}^1$, and if $z_o \neq 0$, then $\dim Kp = 4$, which implies that Kp is Lagrangian.

Since K commutes with the above T^1-action, we obtain that the K-action on $\boldsymbol{CP}^4_{[1,\dots,0]}$ admits a Lagrangian orbit, which is $K[p]$.

Next, we prove the Lagrangian slice theorem for G-action with a fixed point.

PROPOSITION 3.7. *Let G be a compact connected Lie group acting in a Hamiltonian fashion with a fixed point on a symplectic manifold M. If the slice representation at the fixed point has a Lagrangian orbit, then the G-action on M admits a Lagrangian orbit.*

PROOF. It follows immediately from symplectic slice. Indeed, if $Gp = p$, then $\mu(p) = \beta \in \mathfrak{z}(\mathfrak{g})$ and from symplectic slice follows that the moment map is locally given by

$$\mu(gp, m) = \beta + Ad(g)(\mu_{T_pM}(m)) .$$

If Gm is a Lagrangian orbit of the G-action on the slice, then $\dim Gm = \dim M/2$ and $\mu_{T_pM}(m) \in \mathfrak{z}(\mathfrak{g})$. In particular, $\dim G[p, m] = \dim M/2$ and $\mu([p, m]) \in \mathfrak{z}(\mathfrak{g})$, which implies that $G[p, m]$ is Lagrangian. □

PROOF OF THE PROPOSITION 1.5. Except for the first and the last cases, the proof follows from the classification given in [4], Corollary 3.3, and finally from Proposition 3.7. In the sequel we always refer to Table 1 in [4, p. 16], which is included at the end of Section 5, for compact simple Lie groups acting with a Lagrangian orbit in $\boldsymbol{CP}^n$. We briefly explain our method.

The semisimple part of G admits a Lagrangian orbit on the complex projective space of the slice, by Table 1. Hence, by Corollary 3.1, G has a Lagrangian orbit on the slice, which implies, from Proposition 3.7, that G admits a Lagrangian orbit on M. We consider only Hermitian symmetric spaces of compact type, since the noncompact case is exactly the same.

(1) $G = G_2$ acting on $M = SO(8)/U(4)$. We use the same argument as in [8]. Since $G_2 \cap U(4) = SU(3)$, the orbit through $[U(4)]$ is Lagrangian. Indeed, let

$$\phi : M \to \mathfrak{g}_2^*$$

be the moment map of the G_2-action on M. One may check that

$$\dim G_2[U(4)] = \frac{1}{2} \dim SO(8)/U(4) ,$$

$G_2\phi([U(4)]) = G_2/P$ is a generalized flag manifold and $SU(3) \subseteq P$. Since SU(3) is a maximal subgroup of G_2 which does not centralize a torus, we deduce that $P = G_2$, proving that $\phi([U(4)]) = 0$. Hence $G_2[U(4)]$ is Lagrangian.

(2) $G = SO(2) \times SO(n)$ acting on $M = SO(n+2)/SO(2) \times SO(n)$. The slice is $\boldsymbol{C}^n$ on which SO(n) acts with Λ_1. Since SO(n) admits a Lagrangian orbit on $\boldsymbol{CP}^{n-1}$, from Corollary 3.3, G admits a Lagrangian orbit on the slice, which implies, from Proposition 3.7, that G admits a Lagrangian orbit on M.

(3) $G = Z(S(U(1) \times U(n))) \times SO(n)$ acting on $M = \boldsymbol{CP}^n$. Since SO(n) has a Lagrangian orbit on $\boldsymbol{CP}^{n-1}$, G admits a Lagrangian orbit on the slice, which implies that G has a Lagrangian orbit on M.

(4) $G = Z(S(U(2) \times U(2n))) \times Sp(n)$ acting on $M = SU(2n+2)/S(U(2) \times U(2n))$. The slice is given by $\boldsymbol{C}^{2n} \oplus \boldsymbol{C}^{2n}$ on which Sp(n) acts diagonally, while the one dimensional torus acts as

$$t(v, w) = (t^{(1-n)/n} v, t^{(1-n)/n} w) .$$

Since Sp(n) admits a Lagrangian orbit on $\boldsymbol{CP}^{4n-1}$, we see that so does G on M.

(5) $G = U(2n)$ acting on $M = SO(4n)/U(2n)$. The slice is given by $\Lambda^2(\boldsymbol{C}^{2n})$ and SU($2n$) acts with a Lagrangian orbit on its complex projective space. Hence G admits a Lagrangian orbit on M.

(6) $G = U(2n+1)$ acting on $M = SO(4n+4)/U(2n+2)$. The slice is given by

$$\Lambda^2(\boldsymbol{C}^{2n+1}) \oplus \boldsymbol{C} \otimes \boldsymbol{C}^{2n+1} .$$

Note that the center acts as $t(X, v) = (t^2 X, tv)$. As in [4], one may prove that Gp is Lagrangian, where

$$p = (\begin{pmatrix} 0 & 0 \\ 0 & J_n \end{pmatrix}, e_1) \quad \text{and} \quad J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} ,$$

which implies that G admits a Lagrangian orbit on M;

(7) $G = U(n)$ acting on $Sp(n)/U(n)$. The slice is given by $S^2(\boldsymbol{C}^n)$ and SU(n) has a Lagrangian orbit on the complex projective space of $S^2(\boldsymbol{C}^n)$. Then G admits a Lagrangian orbit on M.

(8) $G = T^1 \cdot E_6$ acting on $M = E_7/T^1 \cdot E_6$. The slice is given by $\boldsymbol{C}^{27}$ on which G acts with the Λ_1 representation. Since E_6 admits a Lagrangian orbit on $\boldsymbol{CP}^{26}$, so does G on M.

(9) $G = T^1 \cdot Spin(9) \subseteq T^1 \cdot Spin(10)$ acting on $E_6/T^1 \cdot Spin(10)$. The slice is given by $\boldsymbol{C}^{16}$ on which G acts with the *spin-representation*. Since Spin(9) has a Lagrangian orbit on $\boldsymbol{CP}^{15}$, G admits a Lagrangian orbit on M from Corollary 3.3 together with Proposition 3.7.

(10) This case is proved in [7, p. 1736]. □

4. Homogeneous Lagrangian submanifolds on weighted projective spaces. In [4] it was proved that the following actions

G	ρ	$\dim_{\mathbf{C}} \boldsymbol{P}(V)$	G_p^o
$\mathrm{SU}(n)$	$\Lambda_1 \oplus \Lambda_1^*$	$2n-1$	$\mathrm{SO}(n)$
$\mathrm{SU}(2n+1)$	$\Lambda_2 \oplus \Lambda_1$	$2n^2+3n+1$	$\mathrm{Sp}(n)$
$\mathrm{Sp}(n)$	$\Lambda_1 \oplus \Lambda_1$	$4n-1$	$\mathrm{Sp}(n-1)$
$\mathrm{Spin}(10)$	$\Lambda_e \oplus \Lambda_e$	31	$\mathrm{SU}(5)$

admit a Lagrangian orbit. We briefly explain the notation, which follows that given in [4]. ρ denotes the representation, while G_p^o denotes the connected component of the identity of the isotropy of the Lagrangian orbit. Moreover, we identify the fundamental weights Λ_i with the corresponding irreducible representations.

We shall prove that if G appears in the above table, then it induces an action on some weighted projective space with a Lagrangian orbit. Let T^1 be a one-dimensional torus acting on $V = V_1 \oplus V_2$ as

$$t(v, w) = (t^{-k}v, t^{-s}),$$

where k, s are distinct natural numbers. This actions is Hamiltonian with moment map

$$\psi((v, w)) = (1/2)(k\,\|v\|^2 + s\|w\|^2).$$

Let $\lambda = 1/2$. The reduced space M_λ is the weighted projective space, denoted by $\boldsymbol{P}(V)_{[k,s]}$, on which G acts in a Hamiltonian fashion, since it commutes with the T^1-action. The map $\mu : V \to \mathfrak{g}^*$, defined for every $(v, w) \in V$ and for every $X \in \mathfrak{g}$ by

$$\mu((v, w)) = -i\langle X(v, w), (v, w)\rangle,$$

is the moment map for the G-action on V, where $\langle\cdot,\cdot\rangle$ denotes the natural hermitian scalar product on V.

We claim that $L = G \times \mathrm{T}^1$ admits a Lagrangian orbit which lies in $\psi^{-1}(1/2)$. Note that the corresponding moment map for the L-action is $\xi = \mu + \psi$. Our approach can be described as follows.

Let $p = (v, w) \in V$ be such that $G[p]$ is Lagrangian in $\boldsymbol{P}(V)$, which has been calculated in [4]. Then we may prove that $L\bar{p}$, where $\bar{p} = p/\sqrt{k\,\|v\|^2 + s\|w\|^2}$, is Lagrangian. This implies that $G[\bar{p}]$ is Lagrangian in $\boldsymbol{P}(V)_{[k,s]}$.

(1) $G = \mathrm{SU}(n)$ and $\rho = \Lambda_1 \oplus \Lambda_1^*$. $p = (e_1, e_1^*)$ and one may prove that the L-orbit through $\bar{p}$ has dimension $2n$. Since $\xi(\bar{p}) = \psi(\bar{p})$, $L\bar{p}$ is Lagrangian.

(2) $G = \mathrm{SU}(2n+1)$ and $\rho = \Lambda_2 \oplus \Lambda_1$. $p = (J_n, e_1)$, where J_n is the same as in case (6) of the proof of Proposition 3.7. Since $\dim L\bar{p} = 2n^2 + 3n + 2$, $L\bar{p}$ is Lagrangian.

(3) $\mathrm{Sp}(n)$ and $\rho = \Lambda_1 \oplus \Lambda_1$. $p = (e_1, e_2)$ and it is easy to check that $\dim L\bar{p} = 4n$, which implies that $L\bar{p}$ is Lagrangian.

(4) $G = \mathrm{Spin}(10)$ and $\rho = \Lambda_e \oplus \Lambda_e$. $p = (1 + e_{1234}, e_{15} + e_{2345})$ (see [31] for the notation). One may prove that $\dim L\bar{p} = 32$, which means that L admits a Lagrangian orbit.

5. Homogeneous Lagrangian submanifolds on the blow-up at one point of the complex projective spaces. Let K be compact Lie group acting linearly on V with two sub-modules, i.e., $V = V_1 \oplus V_2$ and K preserves V_i for $i = 1, 2$. Suppose that K admits a Lagrangian orbit on the complex projective space $P(V)$. If we consider a one-dimensional torus T^1 acting on V_1 or on V_2 which commutes with K, we may induce a Hamiltonian action of K on the Kähler cut $\boldsymbol{P}(V)^\lambda$ and this action admits a Lagrangian orbit. In [4] it was proved that the following actions

G	ρ	$\dim_{\boldsymbol{C}} \boldsymbol{P}(V)$	G_p^o
$\mathrm{SU}(n)$	$\Lambda_1 \oplus \Lambda_1^*$	$2n - 1$	$\mathrm{SO}(n)$
$\mathrm{SU}(2n+1)$	$\Lambda_2 \oplus \Lambda_1$	$2n^2 + 3n + 1$	$\mathrm{Sp}(n)$
$\mathrm{Sp}(n)$	$\Lambda_1 \oplus \Lambda_1$	$4n - 1$	$\mathrm{Sp}(n-1)$
$\mathrm{Spin}(10)$	$\Lambda_e \oplus \Lambda_e$	31	$\mathrm{SU}(5)$

admit a Lagrangian orbit.

We shall prove that these groups admit a Lagrangian orbit on the blow-up at one point of $\boldsymbol{P}(V)$. We analyze in detail only the first case; the other cases are similar.

Let T^1 be a torus acting on $\boldsymbol{P}(\boldsymbol{C}^n \oplus (\boldsymbol{C}^n)^*)$ as $t([(v, w)]) = [(t^{-1}v, w)]$. This action is Hamiltonian with moment map

$$\mu([(v, w)]) = \frac{1}{2} \frac{\|v\|^2}{\|[v, w]\|^2}.$$

Hence $p = [(1, \dots, 1), (0 \dots, 0)]$ is the global maximum and, as usual (see [10]), given $\lambda = 1/2 - \varepsilon$, $\varepsilon \approx 0$, the Kähler cut is the blow-up of the complex projective space $\boldsymbol{P}(V)$ at p, which we denote by $\boldsymbol{P}(\boldsymbol{C}^n \oplus (\boldsymbol{C}^n)^*)_{[p]}$. Since the $\mathrm{SU}(n)$-action on $\Lambda_1 \oplus \Lambda_1^*$ commutes with the T^1-action, it follows from Proposition 3.5 that $\mathrm{SU}(n)$ admits a Lagrangian orbit on $\boldsymbol{P}(\boldsymbol{C}^n \oplus (\boldsymbol{C}^n)^*)_{[p]}$. Summing up, we have the following homogeneous Lagrangian submanifolds.

G	$\boldsymbol{P}(V)_{[p]}$	G_p^o
$\mathrm{SU}(n)$	$\boldsymbol{P}(\boldsymbol{C}^n \oplus (\boldsymbol{C}^n)^*)_{[(1,\dots,1),(0,\dots,0)]}$	$\mathrm{SO}(n)$
$\mathrm{SU}(2n+1)$	$\boldsymbol{P}(\Lambda^2(\boldsymbol{C}^{2n+1}) \oplus \boldsymbol{C}^{2n+1})_{[(0,\dots,0),(1,\dots,1)]}$	$\mathrm{Sp}(n)$
$\mathrm{Sp}(n)$	$\boldsymbol{P}(\boldsymbol{C}^{2n} \oplus \boldsymbol{C}^{2n})_{[(1,\dots,1),(0,\dots,0)]}$	$\mathrm{Sp}(n-1)$
$\mathrm{Spin}(10)$	$\boldsymbol{P}(\boldsymbol{C}^{16} \oplus \boldsymbol{C}^{16})_{[(1,\dots,1),(0,\dots,0)]}$	$\mathrm{SU}(5)$

TABLE. Lagrangian orbits of simple Lie groups in complex projective spaces.

G	ρ	$\dim_{\boldsymbol{C}} \boldsymbol{P}(V)$	cond.
SU(n)	$2\Lambda_1$	$n(n+1)/2-1$	
SU(n)	$\Lambda_1 \oplus \Lambda_1^*$	$2n-1$	
SU(n)	$\underbrace{\Lambda_1 \oplus \cdots \oplus \Lambda_1}_{n}$	n^2-1	
SU($2n$)	Λ_2	$n(2n-1)-1$	$n \geq 3$
SU($2n+1$)	$\Lambda_2 \oplus \Lambda_1$	$2n^2+3n+1$	$n \geq 2$
SU(2)	$3\Lambda_1$	3	
SU(6)	Λ_3	19	
SU(7)	Λ_3	34	
SU(8)	Λ_3	55	
Sp(n)	$\Lambda_1 \oplus \Lambda_1$	$4n-1$	
Sp(3)	Λ_3	13	
SO(n)	Λ_1	$n-1$	$n \geq 3$
Spin(7)	spin.rep.	7	
Spin(9)	spin.rep.	15	
Spin(10)	$\Lambda_e \oplus \Lambda_e$	31	
Spin(11)	spin.rep.	31	
Spin(12)	Λ_e	31	
Spin(14)	Λ_e	63	
E_6	Λ_1	26	
E_7	Λ_1	55	
G_2	Λ_2	6	

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DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI PARMA
VIALE G. USBERTI 53/A
43100, PARMA
ITALY

E-mail address: leonardo.biliotti@unipr.it