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DOMINANT RATIONAL MAPS IN THE CATEGORY OF LOG SCHEMES

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Abstract. Kobayashi-Ochiai's theorem states that the set of dominant rational maps from a complex variety to a complex variety of general type is finite. Kazuya Kato conjectured a similar result in the category of log schemes. Our main theorem of this paper is a solution to his conjecture.

Introduction. In the paper [6], Kobayashi and Ochiai proved that the set of dominant rational maps from a complex variety to a complex variety of general type is finite. This result was generalized to the case over a field of positive characteristic by Dechamps and Menegaux [2]. Furthermore, Tsushima [13] established finiteness for open varieties over a field of characteristic zero. With these foregoing results, Kazuya Kato conjectured a similar result in the category of log schemes. As we know, logarithmic geometry is a general framework to cover compactification and singularities in degeneration. The most typical example of these mixed phenomena is a logarithmic structure on a semistable variety (cf. Conventions and terminology 9 below). Actually, we deal with a log rational map on a semistable variety with a logarithmic structure. The following finiteness theorem is our solution to Kato's conjecture:

THEOREM A (Finiteness theorem). *Let k be an algebraically closed field and M_k a fine log structure on* $\mathrm{Spec}(k)$. *Let X and Y be proper semistable varieties over k, endowed with fine log structures M_X and M_Y over M_k, respectively, such that*

$$(X, M_X) \to (\mathrm{Spec}(k), M_k) \quad \textit{and} \quad (Y, M_Y) \to (\mathrm{Spec}(k), M_k)$$

are log smooth and integral. We assume that (Y, M_Y) is of log general type over $(\mathrm{Spec}(k), M_k)$, *that is,* $\det(\Omega^1_{Y/k}(\log(M_Y/M_k)))$ *is a big line bundle on Y (see Conventions and terminology* 10 *below). Then the set of all log rational maps*

$$(\phi, h) : (X, M_X) \dashrightarrow (Y, M_Y)$$

over $(\mathrm{Spec}(k), M_k)$ *with the following properties* (1) *and* (2) *is finite*:

(1) $\phi : X \dashrightarrow Y$ *is a rational map defined over a dense open set U with* $\mathrm{codim}(X \setminus U) \geq 2$, *and* $(\phi, h) : (U, M_X|_U) \to (Y, M_Y)$ *is a log morphism over* $(\mathrm{Spec}(k), M_k)$.

(2) *For any irreducible component X' of X, there is an irreducible component Y' of Y such that $\phi(X') \subseteq Y'$ and the induced rational map $\phi' : X' \dashrightarrow Y'$ is dominant and separable.*

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As an immediate consequence of the above theorem, we have the following corollary.

COROLLARY B. *Let X be a proper semistable variety over k, endowed with a fine log structure M_X over M_k, such that $(X, M_X) \to (\mathrm{Spec}(k), M_k)$ is log smooth and integral. If (X, M_X) is of log general type over $(\mathrm{Spec}(k), M_k)$, then the set of automorphisms of (X, M_X) over $(\mathrm{Spec}(k), M_k)$ is finite.*

Let us explain how we can obtain the main results of [2] and [13] from Theorem A. Let $(X, M_X) \to (\mathrm{Spec}(k), M_k)$ and $(Y, M_Y) \to (\mathrm{Spec}(k), M_k)$ be log smooth and integral morphisms to a fine log scheme $(\mathrm{Spec}\, k, M_k)$. Assume that X and Y are proper semistable varieties and (Y, M_Y) is of log general type over $(\mathrm{Spec}(k), M_k)$. If we suppose further that M_X, M_Y and M_k are trivial log structures, then Theorem A is nothing but [2, Thèorém 2]. In virtue of Hironaka's resolution of singularities ([3]) and Nagata's compactification theorem ([9]), the result [13, Theorem] follows from Theorem A in the case when X and Y are smooth over the field k of characteristic zero, M_X (resp. M_Y) is a fine log structure arising from a normal crossing divisor $D_X \subset X$ (resp. $D_Y \subset Y$) and M_k is a trivial log structure. The main advance of Theorem A relative to [2] and [13] is that we can allow X and Y to have a certain kind of singularities. Since our work is partly motivated by logarithmic compactification problems for moduli spaces, a semistable variety X endowed with a smooth log structure M_X is a quite natural object to study. Roughly speaking, our proof of Theorem A except analyses of log structures is technically a modification of the algebraic proof in [2] (see also [13, page 96–98]). Since (X, M_X) and (Y, M_Y) behave as if they are smooth objects in the category of log schemes (for example, their log differential sheave are locally free), the argument in [2] works in our situation. Let us give a sketch of the proof of Theorem A. For this purpose, we need to consider the following two problems:

(i) The finiteness of the underlying rational maps.

(ii) How many do log morphisms exist for a fixed underlying rational map?

As mentioned above, the first problem is closely related to the classical case, that is, the case where $M_k = k^\times$, and X and Y are smooth over k. In this case, we can use similar arguments as in [2]. Actually, we prove it under weaker conditions (cf. Theorem 7.1). In this sense, from the viewpoint of logarithmic geometry, the second problem is crucial for our consideration. The following rigidity theorem of log morphisms over a fixed scheme morphism, which is one of the main results of this paper, is our answer to the second problem.

THEOREM C (Rigidity theorem). *Let X and Y be semistable varieties over k, endowed with fine log structures M_X and M_Y over M_k, respectively, such that (X, M_X) and (Y, M_Y) are log smooth and integral over $(\mathrm{Spec}(k), M_k)$. Let $\mathrm{Supp}(M_Y/M_k)$ be the union of $\mathrm{Sing}(Y)$ and the boundaries of the log structure of M_Y over M_k, that is,*

$$\mathrm{Supp}(M_Y/M_k) = \{y \in Y \mid M_k \times \mathcal{O}^\times_{Y,\bar{y}} \to M_{Y,\bar{y}} \text{ is not surjective}\}.$$

Let $\phi : X \to Y$ be a morphism over k such that $\phi(X') \not\subseteq \mathrm{Supp}(M_Y/M_k)$ for any irreducible component X' of X. If $(\phi, h) : (X, M_X) \to (Y, M_Y)$ and $(\phi, h') : (X, M_X) \to (Y, M_Y)$ are morphisms of log schemes over $(\mathrm{Spec}(k), M_k)$, then $h = h'$.

For the proof of the rigidity theorem, we have to determine a local description of a log structure. Indeed, we have the following theorem, which is a generalization of results in [4, Theorem 1.3 and (1.8)] and [12, Theorem 2.7].

THEOREM D (Local structure theorem). *Let X be a semistable variety over k, which is endowed with a fine log structure M_X of X over M_k such that $(X, M_X) \to (\mathrm{Spec}(k), M_k)$ is log smooth and integral. Let us take a fine sharp monoid Q with $M_k = Q \times k^{\times}$. For a closed point $x \in X$, there is a good chart $(Q \to M_k,\ P \to M_{X,\bar{x}},\ Q \to P)$ of $(X, M_X) \to (\mathrm{Spec}(k), M_k)$ at x, namely,*

(a) *$Q \to M_k/k^{\times}$ and $P \to M_{X,\bar{x}}/\mathcal{O}^{\times}_{X,\bar{x}}$ are bijective,*

(b) *the diagram*

$$\begin{array}{ccc} Q & \longrightarrow & P \\ \big\downarrow & & \big\downarrow \\ M_k & \longrightarrow & M_{X,\bar{x}} \end{array}$$

is commutative,

(c) *$k \otimes_{k[Q]} k[P] \to \mathcal{O}_{X,\bar{x}}$ is smooth.*

Moreover, using the good chart $(Q \to M_k, P \to M_{X,\bar{x}}, Q \to P)$, we can determine the local structure in the following manner:

(1) *If the multiplicity of X at x is equal to* 1, *then $Q \to P$ splits and $P \simeq Q \times \boldsymbol{N}^r$ for some r.*

(2) *If the multiplicity of X at x is equal to* 2, *then we have one of the following:*

(2.1) *If $Q \to P$ does not split, then P is of semistable type over Q.*

(2.2) *If $Q \to P$ splits, then* $\mathrm{char}(k) \neq 2$ *and there is a submonoid N of P such that $P \simeq Q \times N$ and N is isomorphic to the momoid arising from the monomials of $k[T_1, T_2, \dots, T_a]/(T_1^2 - T_2^2)$ for some $a \geq 2$.*

(3) *If the multiplicity of X at x is greater than or equal to* 3, *then $Q \to P$ does not split and P is of semistable type over Q.*

For the definition of a monoid of semistable type, see §2.

To verify the existence of the good charts above, we use a general result of good charts due to Ogus (cf. [10, Theorem 2.13]). It is however currently difficult to obtain the preprint [10]. By this reason, we show the well-known result of good charts for the benefit of readers in Appendix.

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CONVENTION AND TERMINOLOGY. Here we fix some of our convention and terminology in this paper.

1. Throughout this paper, we work within the logarithmic structures in the sense of J.-M Fontaine, L. Illusie and K. Kato. For the details, we refer to [5]. All log structures on

schemes are considered with respect to the étale topology. We often denote the log structure on a scheme X by M_X and the quotient $M_X/\mathcal{O}_X^\times$ by $\bar{M}_X$.

2. We denote by $\boldsymbol{N}$ the set of natural integers. Note that $0 \in \boldsymbol{N}$. For $I = (a_1, \dots, a_n) \in \boldsymbol{N}^n$, we define $\mathrm{Supp}(I)$ and $\deg(I)$ to be

$$\mathrm{Supp}(I) = \{i \mid a_i > 0\} \quad \text{and} \quad \deg(I) = \sum_{i=1}^{n} a_i\,.$$

The i-th entry of I is denoted by $I(i)$, i.e., $I(i) = a_i$. For $I, J \in \boldsymbol{N}^n$, a partial order $I \geq J$ is defined by $I(i) \geq J(i)$ for all $i = 1, \dots, n$. The non-negative integer g with $g\boldsymbol{Z} = \boldsymbol{Z}I(1) + \cdots + \boldsymbol{Z}I(n)$ is denoted by $\gcd(I)$.

3. Here let us briefly recall some generalities on monoids. All monoids in this paper are commutative with the unit element. The binary operation of a monoid is often written additively. We say a monoid P is *finitely generated* if there are $p_1, \dots, p_n \in P$ such that $P = \boldsymbol{N}p_1 + \cdots + \boldsymbol{N}p_r$. Moreover, P is said to be *integral* if whenever $x + z = y + z$ for elements $x, y, z \in P$, we have $x = y$. An integral and finitely generated monoid is said to be *fine*. We say P is *sharp* if whenever $x + y = 0$ for $x, y \in P$, then $x = y = 0$. For a sharp monoid P, an element x of P is said to be *irreducible* if whenever $x = y + z$ for $y, z \in P$, then either $y = 0$ or $z = 0$. It is well known that if P is fine and sharp, then there are only finitely many irreducible elements and P is generated by irreducible elements (cf. [11, Lemma 3.9]). If k is a field and P is a sharp monoid, then $M_P = \bigoplus_{x \in P \setminus \{0\}} k \cdot x$ forms the maximal ideal of $k[P]$. This M_P is called *the origin of* $k[P]$. An integral monoid P is said to be *saturated* if $nx \in P$ for $x \in P^{\mathrm{gp}}$ and $n > 0$, then $x \in P$, where P^{gp} is the Grothendieck group associated with P. A homomorphism $f : Q \to P$ of monoids is said to be *integral* if f is injective and an equation $f(q) + p = f(q') + p'$ $(p, p' \in P, q, q' \in Q)$ implies that $p = f(q_1) + p''$ and $p' = f(q_2) + p''$ for some $p'' \in P$ and some $q_1, q_2 \in Q$ with $q + q_1 = q' + q_2$. Moreover, we say an injective homomorphism $f : Q \to P$ *splits* if there is a submonoid N of P with $P = f(Q) \times N$. Finally, let us recall a *congruence relation*. A congruence relation on a monoid P is a subset $S \subset P \times P$ which is both a submonoid and a set-theoretic equivalence relation. We say that a subset $T \subset S$ *generates the congruence relation* S if S is the smallest congruence relation on P containing T. Let S be an equivalence relation on P. It is easy to see that $P \to P/S$ gives rise a structure of a monoid on P/S if and only if S is a congruence relation.

4. Let P and Q be integral monoids and let $f : \boldsymbol{N} \to P$ and $g : \boldsymbol{N} \to Q$ be homomorphisms with $p = f(1)$ and $q = g(1)$. Let $P \times_{\boldsymbol{N}} Q$ be the pushout of $f : \boldsymbol{N} \to P$ and $g : \boldsymbol{N} \to Q$ in the category of integral monoids:

$$\begin{array}{ccc} \boldsymbol{N} & \longrightarrow & Q \\ \downarrow & & \downarrow \\ P & \longrightarrow & P \times_{\boldsymbol{N}} Q\,. \end{array}$$

Namely, $P \times_{N} Q = P \times Q/\sim$, where

$$(p, q) \sim (p', q') \Longleftrightarrow (p, q) + (f(x), g(y)) = (p', q') + (f(y), g(x)) \text{ for some } x, y \in \boldsymbol{N}.$$

We denote this pushout $P \times_{N} Q$ by $P \times_{(p,q)} Q$.

5. Let k be a field and let R be either the ring of polynomials of n-variables over k, or the ring of formal power series of n-variables over k, that is, $R = k[X_1, \dots, X_n]$ or $k[\![X_1, \dots, X_n]\!]$. For $I \in \boldsymbol{N}^n$, we denote the monomial $X_1^{I(1)} \cdots X_n^{I(n)}$ by X^I.

6. Let P be a monoid, $p_1, \dots, p_n \in P$ and $I \in \boldsymbol{N}^n$. For simplicity, $\sum_{i=1}^{n} I(i)p_i$ is often denoted by $I \cdot p$.

7. Let (X, M_X) be a log scheme and $\alpha : M_X \to \mathcal{O}_X$ the structure homomorphism. Then, $\alpha(M_X) \setminus \{\text{zero divisors of } \mathcal{O}_X\}$ gives rise to a log structure because

$$\mathcal{O}_X^{\times} \subseteq \alpha(M_X) \setminus \{\text{zero divisors of } \mathcal{O}_X\}.$$

$\alpha(M_X) \setminus \{\text{zero divisors of } \mathcal{O}_X\}$ is called *the underlying log structure* of M_X and is denoted by M_X^u. Let $f : (X, M_X) \to (Y, M_Y)$ be a morphism of log schemes such that one of the following conditions is satisfied:

(1) $X \to Y$ is flat.

(2) X and Y are integral schemes and $X \to Y$ is a dominant morphism.

Then we have the induced morphism $f^u : (X, M_X^u) \to (Y, M_Y^u)$.

8. Let X and Y be reduced noetherian schemes. Let $\phi : X \dashrightarrow Y$ be a rational map. We say ϕ is *dominant* (resp. *separably dominant*) if for any irreducible component X' of X, there is an irreducible component Y' of Y such that $\phi(X') \subseteq Y'$ and the induced rational map $\phi' : X' \dashrightarrow Y'$ is dominant (resp. dominant and separable). Moreover, we say ϕ is *defined in codimension one* if there is a dense open set U of X such that ϕ is defined over U and $\operatorname{codim}(X \setminus U) \geq 2$.

Let $f : X \to T$ and $g : Y \to T$ be morphisms of reduced noetherian schemes. A rational map $\phi : X \dashrightarrow Y$ is called a *relative rational map* if there is a dense open set U of X such that ϕ is defined on U, $\phi : U \to Y$ is a morphism over T (i.e., $f = g \cdot \phi$) and $X_t \cap U \neq \emptyset$ for all $t \in T$.

9. Let k be an algebraically closed field and X a reduced algebraic scheme over k (i.e., reduced algebraic scheme of finite type over k). We say X is a *semistable variety* if for any closed point $x \in X$, the completion $\hat{\mathcal{O}}_{X,x}$ at x is isomorphic to a ring of the type $k[\![X_1, \dots, X_n]\!]/(X_1 \cdots X_l)$.

10. Let k be an algebraically closed field. Let X be a proper reduced algebraic scheme over k and H a line bundle on X. We say H is *very big* if there is a dense open set U of X such that $H^0(X, H) \otimes \mathcal{O}_X \to H$ is surjective on U and the induced rational map $X \dashrightarrow \boldsymbol{P}(H^0(X, H))$ is birational to the image. Moreover, H is said to be *big* if $H^{\otimes m}$ is very big for some positive integer m.

1. Existence of a good chart on a polysemistable variety. Let k be an algebraically closed field and X an algebraic scheme over k. We say X is a *polysemistable variety* if, for any closed point x of X, the completion $\hat{\mathcal{O}}_{X,x}$ of $\mathcal{O}_{X,x}$ is isomorphic to a ring of the following type:

$$k[\![T_1, \dots, T_e]\!]/(T^{A_1}, \dots, T^{A_l}),$$

where $A_1, \dots, A_l$ are elements of $\boldsymbol{N}^e \setminus \{0\}$ such that $A_i(j)$ is either 0 or 1 for all i, j (cf. Convention and terminology 2 and 5). Note that a polysemistable variety is a reduced scheme (cf. Lemma 1.5).

Let M_k and M_X be fine log structures on $\mathrm{Spec}(k)$ and X, respectively. We assume that (X, M_X) is log smooth and integral over $(\mathrm{Spec}(k), M_k)$. Since the map $x \mapsto x^n$ on k is surjective for any positive integer n, the projection $M_k \to \bar{M}_k$ splits (for the definition of $\bar{M}$ of a log structure M, see Convention and terminology 1). Thus, there is a fine sharp monoid Q together with a chart $\pi_Q : Q \to M_k$ such that $Q \to M_k \to \bar{M}_k$ is bijective.

Next, let us choose a closed point x of X. In the case where X is a polysemistable variety, we would like to construct a chart $\pi_P : P \to M_{X,\bar{x}}$ together with a homomorphism $f : Q \to P$ such that $P \to M_{X,\bar{x}} \to \bar{M}_{X,\bar{x}}$ is bijective, the natural morphism $X \to \mathrm{Spec}(k) \times_{k[Q]} \mathrm{Spec}(k[P])$ is smooth and the following diagram is commutative:

$$\begin{array}{ccc} Q & \xrightarrow{f} & P \\ {\scriptstyle \pi_Q}\big\downarrow & & \big\downarrow{\scriptstyle \pi_P} \\ M_k & \longrightarrow & M_{X,\bar{x}} . \end{array}$$

Then, the triple $(Q \to M_k, P \to M_{X,\bar{x}}, Q \to P)$ is called *a good chart of* $(X, M_X) \to (\mathrm{Spec}(k), M_k)$ *at* x. For this purpose, we need to see the following theorem.

THEOREM 1.1. *Let $\mu : (X, M_X) \to (Y, M_Y)$ be a log smooth and integral morphism of fine log schemes. Let $x \in X$ and $y = \mu(x)$. Let k be the algebraic closure of the residue field at x and $\eta : \mathrm{Spec}(k) \to X \xrightarrow{\mu} Y$ the induced morphism. If $X \times_Y \mathrm{Spec}(k)$ is a polysemistable variety over k, then the torsion part of $\mathrm{Coker}(\bar{M}^{\mathrm{gp}}_{Y,\bar{y}} \to \bar{M}^{\mathrm{gp}}_{X,\bar{x}})$ is a finite group of order invertible in $\mathcal{O}_{X,\bar{x}}$.*

PROOF. We denote $X \times_Y \mathrm{Spec}(k)$ by X'. Then we have the following commutative diagram:

$$\begin{array}{ccc} X & \xleftarrow{\tilde{\eta}} & X' \\ {\scriptstyle \mu}\big\downarrow & & \big\downarrow{\scriptstyle \mu'} \\ Y & \xleftarrow{\eta} & \mathrm{Spec}(k) . \end{array}$$

Note that the natural morphism $\eta' : \mathrm{Spec}(k) \to X'$ gives rise to a section of $\mu' : X' \to \mathrm{Spec}(k)$. Let x' be the image of η'. We consider the natural commutative diagram:

$$\begin{array}{ccccc} \bar{M}_{X,\bar{x}} & \longrightarrow & \overline{\tilde{\eta}^*(M_X)}_{X',\bar{x}'} & \longrightarrow & \overline{\eta'^*(\tilde{\eta}^*(M_X))} \\ \big\uparrow & & \big\uparrow & & \big\uparrow \\ \bar{M}_{Y,\bar{y}} & \longrightarrow & \overline{\eta^*(M_Y)} & = & \overline{\eta^*(M_Y)} . \end{array}$$

Notice that

$$\bar{M}_{Y,\bar{y}} \to \overline{\eta^*(M_Y)} \quad \text{and} \quad \overline{\tilde{\eta}^*(M_X)}_{X',\bar{x}'} \to \overline{\eta'^*(\tilde{\eta}^*(M_X))}$$

are bijective. Moreover, since $\eta'^*(\tilde{\eta}^*(M_X)) = (\tilde{\eta} \cdot \eta')^*(M_X)$, the composition

$$\bar{M}_{X,\bar{x}} \to \overline{\tilde{\eta}^*(M_X)}_{X',\bar{x}'} \to \overline{\eta'^*(\tilde{\eta}^*(M_X))}$$

is also bijective. Thus, we can see that

$$\bar{M}_{X,\bar{x}} \to \overline{\tilde{\eta}^*(M_X)}_{X',\bar{x}'}$$

is an isomorphism. Moreover, $(X', \tilde{\eta}^*(M_X)) \to (\mathrm{Spec}(k), \eta^*(M_Y))$ is smooth and integral. Thus, we may assume that $Y = \mathrm{Spec}(k)$, X is a polysemistable variety over k and x is a closed point of X.

Clearly, we may assume that $p = \mathrm{char}(k) > 0$. We can take a fine sharp monoid Q with $M_k = Q \times k^\times$. Let $f : Q \to M_{X,\bar{x}}$ and $\bar{f} : Q \to \bar{M}_{X,\bar{x}}$ be the canonical homomorphisms.

Let us choose $t_1, \dots, t_r \in M_{X,\bar{x}}$ such that $d\log(t_1), \dots, d\log(t_r)$ form a free basis of $\Omega^1_{X/k,\bar{x}}(\log(M_X/M_k))$. Then, in the same way as in [5, (3.13)], we have the following:

(i) If we set $P_1 = \mathbf{N}^r \times Q$ and a homomorphism $\pi_1 : P_1 \to M_{X,\bar{x}}$ by

$$\pi_1(a_1, \dots, a_r, q) = a_1 t_1 + \cdots + a_r t_r + f(q)\,,$$

then there is a fine monoid P such that $P \supseteq P_1$, $P^{\mathrm{gp}}/P_1^{\mathrm{gp}}$ is a finite group of order invertible in $\mathcal{O}_{X,\bar{x}}$ and that $\pi_1 : P_1 \to M_{X,\bar{x}}$ extends to the surjective homomorphism $\pi : P \to \bar{M}_{X,\bar{x}}$. Moreover, P gives a local chart around x. Here we have the natural homomorphism $h : Q \to P_1 \hookrightarrow P$. Then the following diagram is commutative:

$$\begin{array}{ccc} Q & \xrightarrow{h} & P \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ M_k & \longrightarrow & M_{X,\bar{x}}\,. \end{array}$$

(ii) The natural morphism $g : X \to \mathrm{Spec}(k) \times_{\mathrm{Spec}(k[Q])} \mathrm{Spec}(k[P])$ is étale around x.

Let $\bar{p}_1, \dots, \bar{p}_l$ be all irreducible elements of $\bar{M}_{X,\bar{x}}$ not lying in the image $Q \to \bar{M}_{X,\bar{x}}$. Let us choose $p_1, \dots, p_l \in M_{X,\bar{x}}$ such that the image of p_i in $\bar{M}_{X,\bar{x}}$ is $\bar{p}_i$. Let $\alpha : M_X \to \mathcal{O}_X$ be the canonical homomorphism. We set $z_i = \alpha(p_i)$ for $i = 1, \dots, l$. Since $\hat{\mathcal{O}}_{X,x} \simeq (k \otimes_{k[Q]} k[P])^\wedge_{g(y)}$ and $\bar{p}_i \notin h(Q \setminus \{0\}) + P$, we can see that $z_i \neq 0$ in $\mathcal{O}_{X,\bar{x}}$ for all i.

Note that $M_{X,\bar{x}}$ is generated by $p_1, \dots, p_l$, $\mathcal{O}^\times_{X,\bar{x}}$ and the image of Q in $M_{X,\bar{x}}$, so that, from now on, we always choose $t_1, \dots, t_r$ from elements of the following types:

$$p_i u \ \ (u \in \mathcal{O}^\times_{X,\bar{x}},\ i = 1, \dots, l) \quad \text{and} \quad v \ \ (v \in \mathcal{O}^\times_{X,\bar{x}})\,.$$

We set $x_i = \alpha(t_i)$ for $i = 1, \dots, r$.

CLAIM 1.1.1. (a) $x_1^{a_1} \cdots x_r^{a_r} \neq 0$ *for any non-negative integers* $a_1, \dots, a_r$.

(b) *If* $x_1^{a_1} \cdots x_r^{a_r} = x_1^{a'_1} \cdots x_r^{a'_r}$ *for non-negative integers* $a_1, \dots, a_r, a'_1, \dots, a'_r$, *then*

$$(a_1, \dots, a_r) = (a'_1, \dots, a'_r)\,.$$

PROOF. Let T_i be an element of $k \otimes_{k[Q]} k[P]$ arising from $e_i = (0, \dots, 1, \dots, 0) \in \boldsymbol{N}^r$ (i-th standard basis of $\boldsymbol{N}^r$), namely, $T_i = 1 \otimes e_i$. Let us choose $u_1, \dots, u_a \in P$ such that the kernel of $P^{\mathrm{gp}} \to \bar{M}^{\mathrm{gp}}_{X,\bar{x}}$ is generated by $u_1, \dots, u_a$. Let P' be the submonoid of P^{gp} generated by $\pm e_1, \dots, \pm e_r, \pm u_1, \dots, \pm u_a$ and P. Since the map $k[Q] \to k[\bar{\pi}(P')]$ is flat, thus $\bar{f} : Q \to \bar{\pi}(P')$ is integral by [5, Proposition (4.1)]. By using this fact, we can easily observe that the natural injective homomorphism $\nu : Q \times \boldsymbol{Z}^r \to P'$ given by $\nu(q, I) = f(q) + I \cdot e$ is also integral. Therefore, by [5, Proposition (4.1)], $k[P']$ is flat over $k[Q \times \boldsymbol{Z}^r]$. Moreover, since

$$k \otimes_{k[Q]} k[P'] \simeq (k \otimes_{k[Q]} k[Q \times \boldsymbol{Z}^r]) \otimes_{k[Q \times \boldsymbol{Z}^r]} k[P'],$$

the following diagram

$$\begin{array}{ccc} \mathrm{Spec}(k \otimes_{k[Q]} k[P']) & \longrightarrow & \mathrm{Spec}(k[P']) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k \otimes_{k[Q]} k[Q \times \boldsymbol{Z}^r]) & \longrightarrow & \mathrm{Spec}(k[Q \times \boldsymbol{Z}^r]) \end{array}$$

is Cartesian. Therefore,

$$\mathrm{Spec}(k \otimes_{k[Q]} k[P']) \to \mathrm{Spec}(k \otimes_{k[Q]} k[Q \times \boldsymbol{Z}^r]) = \mathrm{Spec}(k[\boldsymbol{Z}^r])$$

is flat. In particular,

$$\beta : k[\boldsymbol{Z}^r] = k \otimes_{k[Q]} k[Q \times \boldsymbol{Z}^r] \to k \otimes_{k[Q]} k[P']$$

is injective because $k[\boldsymbol{Z}^r]$ is a integral domain. Further, $\beta(Y_i) = T_i$ for $i = 1, \dots, r$, where $k[\boldsymbol{Z}^r] = k[Y_1^{\pm}, \dots, Y_r^{\pm}]$.

Let U be an étale neighborhood at x and V a non-empty open set of $\mathrm{Spec}(k \otimes_{k[Q]} k[P])$ such that $V = g(U)$ and $g : U \to V$ is étale. Moreover, we set $W = \mathrm{Spec}(k \otimes_{k[Q]} k[P'])$. Then, W is an open set of $\mathrm{Spec}(k \otimes_{k[Q]} k[P])$, i.e.,

$$W = \{t \in \mathrm{Spec}(k \otimes_{k[Q]} k[P]) \mid T_i(t) \neq 0 \text{ for all } i \text{ and } (1 \otimes u_j)(t) \neq 0 \text{ for all } j\}.$$

Let $\bar{W}$ be the closure of W. Note that

$$\begin{aligned} &\mathrm{Spec}(k \otimes_{k[Q]} k[P]) \\ &\qquad = \bar{W} \cup \{T_1 = 0\} \cup \cdots \cup \{T_r = 0\} \cup \{1 \otimes u_1 = 0\} \cup \cdots \cup \{1 \otimes u_a = 0\}. \end{aligned}$$

Moreover, if we set $y = g(\bar{x})$, then $(1 \otimes u_j)(y) \neq 0$ for all j because $\pi(u_j) \in \mathcal{O}^{\times}_{X,\bar{x}}$. Note that the local ring $(k \otimes_{k[Q]} k[P])_y$ is reduced, since $g^* : (k \otimes_{k[Q]} k[P])_y \to \mathcal{O}_{X,\bar{x}}$ is étale. Therefore, if $y \notin \bar{W}$, then $T_i = 0$ in $(k \otimes_{k[Q]} k[P])_y$. This contradicts the fact that $z_i \neq 0$ in $\mathcal{O}_{X,\bar{x}}$ for all i because $g^*(T_i) = x_i$. Thus, $y \in \bar{W}$. Let us consider

$$\gamma : k[\boldsymbol{Z}^r] \xrightarrow{\beta} \mathcal{O}_W \longrightarrow \mathcal{O}_{W \cap V} \xrightarrow{g^*} \mathcal{O}_{g^{-1}(W \cap V)}.$$

Then, $\gamma(Y_i) = x_i$. Further, γ is injective, since β and g^* are injective and $k[\boldsymbol{Z}^r]$ is an integral domain. Thus, we get the claim. □

Fix $t_1, \dots, t_r \in M_{X,\bar{x}}$ with the following properties:

(1) t_i is equal to either $p_j u$ ($u \in \mathcal{O}_{X,\bar{x}}$) or a unit v for all i.
(2) $d\log(t_1), \dots, d\log(t_r)$ form a free basis of $\Omega^1_{X/k,\bar{x}}(\log(M_X/M_k))$.
(3) If we replace the non-unit $t_i \notin \mathcal{O}^\times_{X,\bar{x}}$ by a unit $t'_i \in \mathcal{O}^\times_{X,\bar{x}}$, then

$$d\log(t_1), \dots, d\log(t'_i), \dots, d\log(t_r)$$

do not form a free basis of $\Omega^1_{X/k,\bar{x}}(\log(M_X/M_k))$.

CLAIM 1.1.2. *For a non-unit t_i and $u \in \mathcal{O}^\times_{X,\bar{x}}$,*

$$d\log(t_1), \dots, d\log(t_i u), \dots, d\log(t_r)$$

form a free basis of $\Omega^1_{X/k,\bar{x}}(\log(M_X/M_k))$.

PROOF. We set $d\log(u) = f_1 d\log(t_1) + \cdots + f_r d\log(t_r)$. If $f_i \in \mathcal{O}^\times_{X,\bar{x}}$, then $d\log(t_i)$ belongs to a submodule generated by

$$d\log(u), d\log(t_1), \dots, d\log(t_{i-1}), d\log(t_{i+1}), \dots, d\log(t_r)\,.$$

Thus, $d\log(u), d\log(t_1), \cdots, d\log(t_{i-1}), d\log(t_{i+1}), \cdots, d\log(t_r)$ form a basis, so that f_i belongs to the maximal ideal of $\mathcal{O}_{X,\bar{x}}$. Therefore,

$$d\log(t_i u) = (1 + f_i) d\log(t_i) + \sum_{j \neq i} f_j d\log(t_j)\,,$$

and $1 + f_i \in \mathcal{O}^\times_{X,\bar{x}}$. Thus, we get the claim. □

Renumbering $t_1, \dots, t_r$, we may assume that

$$\{t_1, \dots, t_s\} = \{t_i \mid t_i \text{ is not a unit}\}\,.$$

CLAIM 1.1.3. *Let $a_1, \dots, a_s, a'_1, \dots, a'_s$ be non-negative integers such that either a_i or a'_i is zero for all i. For $u \in \mathcal{O}^\times_{X,\bar{x}}$, if*

$$x_1^{a_1} \cdots x_s^{a_s} = u x_1^{a'_1} \cdots x_s^{a'_s}\,,$$

then $a_1 = \cdots = a_s = a'_1 = \cdots = a'_s = 0$ and $u = 1$.

PROOF. Assume the contrary. Let us choose a non-negative integer k such that $a_i = p^k b_i$ and $a'_i = p^k b'_i$ for all i and that

$$\gcd(b_1, \dots, b_s, b'_1, \dots, b'_s)$$

is prime to p. Then, by Lemma 1.3, there is $v \in \mathcal{O}^\times_{X,\bar{x}}$ with

$$x_1^{a_1} \cdots x_s^{a_s} = v^{p^k} x_1^{a'_1} \cdots x_s^{a'_s}\,.$$

Moreover by our construction, replacing v by v^{-1} if necessarily, we can find b'_i prime to p. Thus, there is $v' \in \mathcal{O}^\times_{X,\bar{x}}$ with $v'^{b'_i} = v$. Hence, if we replace t_i by $v' t_i$, then we have $x_1^{a_1} \cdots x_s^{a_s} = x_1^{a'_1} \cdots x_s^{a'_s}$. Therefore, by Claim 1.1.1 and Claim 1.1.2, $a_1 = a'_1, \dots, a_s = a'_s$, which implies that $a_1 = \cdots = a_s = a'_1 = \cdots = a'_s = 0$. This is a contradiction. □

CLAIM 1.1.4. *$t_1, \dots, t_s$ are linearly independent over $\mathbf{Z}$ in* $\mathrm{Coker}(Q^{\mathrm{gp}} \to \bar{M}^{\mathrm{gp}}_{X,\bar{x}})$.

PROOF. We assume that a non-trivial relation $a_1t_1 + \cdots + a_st_s = 0$ $(a_1, \ldots, a_s \in \mathbf{Z})$ holds in $\mathrm{Coker}(Q^{\mathrm{gp}} \to \bar{M}^{\mathrm{gp}}_{X,\bar{x}})$. Let $\bar{t}_i$ be the class of t_i in $\bar{M}_{X,\bar{x}}$. Then, $a_1\bar{t}_1 + \cdots + a_s\bar{t}_s = \bar{f}(q)$ for some $q \in Q^{\mathrm{gp}}$. Renumbering $t_1, \ldots, t_s$, we may assume that $a_1, \ldots, a_l > 0$ and $a_{l+1}, \ldots, a_s \leq 0$. Thus, we have

$$b_1\bar{t}_1 + \cdots + b_l\bar{t}_l + \bar{f}(q_1) = b_{l+1}\bar{t}_{l+1} + \cdots + b_s\bar{t}_s + \bar{f}(q_2)$$

for some $q_1, q_2 \in Q$, where $b_1 = a_1, \ldots, b_l = a_l$ and $b_{l+1} = -a_{l+1}, \ldots, b_s = -a_s$. Since $\bar{f}$ is integral, there are $q_3, q_4 \in Q$, $x \in M_{X,\bar{x}}$ and $u, u' \in \mathcal{O}^{\times}_{X,\bar{x}}$ with

$$\begin{cases} q_1 + q_3 = q_2 + q_4\,, \\ b_1t_1 + \cdots + b_lt_l = f(q_3) + x + u\,, \\ b_{l+1}t_{l+1} + \cdots + b_st_s = f(q_4) + x + u'\,. \end{cases}$$

Thus, if $q_3 \neq 0$, then $x_1^{b_1} \cdots x_s^{b_s} = 0$, which contradicts to Claim 1.1.1. Therefore, $q_3 = 0$. In the same way, $q_4 = 0$. Thus, we get

$$b_1t_1 + \cdots + b_lt_l = b_{l+1}t_{l+1} + \cdots + b_st_s + v_0$$

for some $v_0 \in \mathcal{O}^{\times}_{X,\bar{x}}$. Hence $x_1^{b_1} \cdots x_l^{b_l} = v_0x_{l+1}^{b_{l+1}} \cdots x_s^{b_s}$. Therefore, by Claim 1.1.3, $b_1 = \cdots = b_l = b_{l+1} = \cdots = b_s = 0$. This is a contradiction. □

Let $\lambda : P^{\mathrm{gp}} \to \bar{M}^{\mathrm{gp}}_{X,\bar{x}}$ be the natural surjective homomorphism and

$$\lambda' : \mathrm{Coker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}}) \to \mathrm{Coker}(Q^{\mathrm{gp}} \to \bar{M}^{\mathrm{gp}}_{X,\bar{x}})$$

the induced homomorphism. Then, by using Claim 1.1.4, if we set

$$T = \mathrm{Coker}(\mathbf{Z}t_1 \oplus \cdots \oplus \mathbf{Z}t_r \to \mathrm{Coker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}}))$$

and

$$T' = \mathrm{Coker}(\mathbf{Z}t_1 \oplus \cdots \oplus \mathbf{Z}t_s \to \mathrm{Coker}(Q^{\mathrm{gp}} \to \bar{M}^{\mathrm{gp}}_{X,\bar{x}}))\,,$$

then we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}t_1 \oplus \cdots \oplus \mathbf{Z}t_r \arrow[r] \arrow[d, "\text{projection}"] & \mathrm{Coker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}}) \arrow[r] \arrow[d, "\lambda'"] & T \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{Z}t_1 \oplus \cdots \oplus \mathbf{Z}t_s \arrow[r] \arrow[d] & \mathrm{Coker}(Q^{\mathrm{gp}} \to \bar{M}^{\mathrm{gp}}_{X,\bar{x}}) \arrow[r] \arrow[d] & T' \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0\,. &
\end{tikzcd}$$

Here T is a torsion group of order prime to p. Therefore, we get our assertion. □

LEMMA 1.2. *Let R be a ring and $f : Q \to P$ a homomorphism of commutative monoids with the unity. Then, the kernel of the induced ring homomorphism $R[Q] \to R[P]$ by f is generated by elements of type $[q] - [q']$ with $f(q) = f(q')$.*

PROOF. The proof of this is left to the reader. □

LEMMA 1.3. *Let X be a polysemistable variety over an algebraically closed field k of characteristic $p > 0$ and x a closed point of X. Let $\mathcal{O}_{X,\bar{x}}$ be the local ring at x in the étale topology. Let H and G be elements of $\mathcal{O}_{X,\bar{x}}$ and $u \in \mathcal{O}_{X,\bar{x}}^{\times}$. If $H^{p^k}u = G^{p^k}$, then there is $v \in \mathcal{O}_{X,\bar{x}}^{\times}$ with $(Hv)^{p^k} = G^{p^k}$.*

PROOF. By Artin's approximation theorem, it is sufficient to find v in $\hat{\mathcal{O}}_{X,\bar{x}}$. Since X is a polysemistable variety, we can set

$$\hat{\mathcal{O}}_{X,\bar{x}} = k[\![T_1, \ldots, T_e]\!]/(T^{A_1}, \ldots, T^{A_l}),$$

where $A_1, \ldots, A_l \in \boldsymbol{N}^e \setminus \{0\}$. We set

$$\Omega = \bigcup_{i=1}^{l}(A_i + \boldsymbol{N}^e), \quad \Sigma = \boldsymbol{N}^e \setminus \bigcup_{i=1}^{l}(A_i + \boldsymbol{N}^e) \quad \text{and} \quad \Sigma_k = \{I \in \Sigma \mid p^k | A(i) \text{ for all } i\}.$$

Then, any elements of $\hat{\mathcal{O}}_{X,\bar{x}}$ can be uniquely written in a form

$$\sum_{I \in \Sigma} \alpha_I T^I.$$

We set $u = \sum_{I \in \Sigma} a_I T^I$ and $H = \sum_{I \in \Sigma} b_I T^I$. Moreover, we set

$$u' = \sum_{I \in \Sigma_k} a_I T^I \quad \text{and} \quad u'' = \sum_{I \notin \Sigma_k} a_I T^I.$$

Then, $u = u' + u''$ and there is a unit v with $v^{p^k} = u'$. Thus, $H^{p^k}u'' = (G - Hv)^{p^k}$. Therefore,

$$(G - Hv)^{p^k} = \Big(\sum_{I \in \Sigma} b_I^{p^k} T^{p^k I}\Big)\Big(\sum_{I \notin \Sigma_k} a_I T^I\Big).$$

Even if we delete the terms T^J with $J \in \Omega$, the left hand side of the above equation consists of the terms T^J with $J \in \Sigma_k$ and the right hand side does not contain the terms T^J with $J \in \Sigma_k$. Thus, $(G - Hv)^{p^k} = 0$. □

As a corollary of Theorem 1.1, we have the following existence of a good chart of a log morphism.

COROLLARY 1.4. *Let X be a polysemistable variety over an algebraically closed field k. Let M_k and M_X be fine log structures on* Spec(k) *and X, respectively. We assume that (X, M_X) is log smooth and integral over* (Spec(k), M_k). *Let Q be a fine sharp monoid with $M_k \simeq Q \times k^{\times}$ and $\pi_Q : Q \to M_k$ the composition of $Q \to Q \times k^{\times}$ ($q \mapsto (q, 1)$) and $Q \times k^{\times} \xrightarrow{\sim} M_k$. Moreover, let x be a closed point of X. Then, there is a fine sharp monoid P together with homomorphisms $\pi_P : P \to M_{X,\bar{x}}$ and $f : Q \to P$ such that a triple ($\pi_Q : Q \to M_k$, $\pi_P : P \to M_{X,\bar{x}}$, $f : Q \to P$) is a good chart of $(X, M_X) \to$ (Spec(k), M_k) at x, namely, the following properties are satisfied:*

(1) *The diagram*

$$\begin{array}{ccc} Q & \xrightarrow{f} & P \\ {\scriptstyle\pi_Q}\downarrow & & \downarrow{\scriptstyle\pi_P} \\ M_k & \longrightarrow & M_{X,\bar{x}} \end{array}$$

is commutative.

(2) *The homomorphism $P \to M_{X,\bar{x}} \to \bar{M}_{X,\bar{x}}$ is an isomorphism.*

(3) *The natural morphism $g : X \to \mathrm{Spec}(k) \times_{\mathrm{Spec}(k[Q])} \mathrm{Spec}(k[P])$ is smooth in the usual sense.*

PROOF. This is a corollary of Theorem 1.1 together with Proposition A.1 and Proposition A.2. □

Finally, let us consider the following lemma, which is in use to show that a poly-semistable variety is a reduced scheme.

LEMMA 1.5. *Let $k[[T_1, \ldots, T_e]]$ be the ring of formal power series over k. Let $A_1, \ldots, A_l$ be elements of $\boldsymbol{N}^e \setminus \{0\}$ such that $A_i(j)$ is either 0 or 1 for all i, j. Let I be an ideal of $k[[T_1, \ldots, T_e]]$ generated by $T^{A_1}, \ldots, T^{A_l}$. Then, I is reduced, i.e., $\sqrt{I} = I$.*

PROOF. We prove this by induction on e. If $e = 1$, our assertion is obvious, so that we assume that $e > 1$. Let $f \in \sqrt{I}$. Then, there is $n > 0$ with $f^n \in I$. It is easy to see that there are $a_1, \ldots, a_e \in k[[T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_e]]$ and $b \in k[[T_1, \ldots, T_e]]$ with

$$f = a_1 + T_1 a_2 + \cdots + T_1 \cdots T_{i-1} a_i + \cdots + T_1 \cdots T_{e-1} a_e + T_1 \cdots T_e b\,.$$

Then, $f(0, T_2, \ldots, T_e) = a_1 \in k[[T_2, \ldots, T_e]]$. If $1 \in \mathrm{Supp}(A_i)$ for all i, then

$$f(0, T_2, \ldots, T_e)^n = 0\,.$$

Thus, $a_1 = 0$. In particular, $a_1 \in I$. Otherwise,

$$a_1^n = f(0, T_2, \ldots, T_e)^n \in \sum_{1 \notin \mathrm{Supp}(A_i)} T^{A_i} k[[T_2, \ldots, T_e]]\,.$$

Thus, by induction hypothesis , $a_1 \in I$. Therefore, $(f - a_1)^n \in I$. Note that $(f - a_1)(T_1, 0, T_3, \ldots, T_e) = T_1 a_2$. Thus, in the same way as before, we can see that $T_1 a_2 \in I$. Hence, $(f - a_1 - T_1 a_2)^n \in I$. Proceeding with the same argument, $T_1 \cdots T_{i-1} a_i \in I$ for all i. On the other hand, $T_1 \cdots T_e \in I$. Therefore, $f \in I$. □

REMARK 1.6. It is very natural to ask a generalization of Theorem 1.1 to the case of idealized log schemes. However we do not use idealized log schemes in this paper. This problem is left to the reader.

2. Monoids of semistable type. In this section, we consider a monoid of semistable type. First of all, let us give its definition. Let $f : Q \to P$ be an integral homomorphism of fine sharp monoids with $Q \neq \{0\}$. We say P is *of semi-stable type*

$$(r, l, p_1, \dots, p_r, q_0, b_{l+1}, \dots, b_r)$$

over Q if the following conditions are satisfied:

(1) r and l are positive integers with $r \geq l$, $p_1, \dots, p_r \in P$, $q_0 \in Q \setminus \{0\}$, and $b_{l+1}, \dots, b_r$ are non-negative integers.

(2) P is generated by $f(Q)$ and $p_1, \dots, p_r$. The submonoid of P generated by $p_1, \dots, p_r$ in P, which is denoted by N, is canonically isomorphic to $\boldsymbol{N}^r$, namely, the homomorphism $\boldsymbol{N}^r \to N$ given by $(t_1, \dots, t_r) \mapsto \sum_i t_i p_i$ is an isomorphism.

(3) We set $\Delta_l, B \in \boldsymbol{N}^r$ as follows:

$$\Delta_l = (\underbrace{1, \dots, 1}_{l}, \underbrace{0, \dots, 0}_{r-l}) \quad \text{and} \quad B = (\underbrace{0, \dots, 0}_{l}, b_{l+1}, \dots, b_r)\,.$$

Then, $\Delta_l \cdot p = f(q_0) + B \cdot p$, i.e., $p_1 + \cdots + p_l = f(q_0) + \sum_{i>l} b_i p_i$ (cf. Convention and terminology 6).

(4) If we have a relation

$$I \cdot p = f(q) + J \cdot p \quad (I, J \in \boldsymbol{N}^r)$$

with $q \neq 0$, then $I(i) > 0$ for all $i = 1, \dots, l$ (cf. Convention and terminology 2).

REMARK 2.1. Under the assumption as above, let $U \subset P$ (resp. $V \subset P$) be the submonoid of P generated by $p_1, \dots, p_l$ (resp. $f(Q)$ and $p_{l+1}, \dots, p_r$). According to (3), there is a natural map

$$U \times_{(\Delta_l \cdot p,\ f(q_0)+B\cdot p)} V \to P\,.$$

See Convention and terminology 4 for the definition of $U \times_{(\Delta_l \cdot p,\ f(q_0)+B\cdot p)} V$.

REMARK 2.2. In the case where $l = 1$, by using (2) of the following proposition, we can see $P = f(Q) \times \boldsymbol{N} p_2 \times \cdots \times \boldsymbol{N} p_r$. Conversely, if P has a form $f(Q) \times \boldsymbol{N}^{r-1}$ and $Q \neq \{0\}$, then P is of semistable type in the following way: Let q_0 be an irreducible element of Q and $p_1 = f(q_0)$. Let e_i be the standard basis of $\boldsymbol{N}^{r-1}$. We set $p_i = (0, e_{i-1})$ for $i = 2, \dots, r$. Then, since Q is sharp, $\boldsymbol{N} p_1 \simeq \boldsymbol{N}$. Thus, the submonoid generated by $p_1, \dots, p_r$ in P is isomorphic to $\boldsymbol{N}^r$. Finally, let us consider a relation $\sum_i a_i p_i = f(q) + \sum_i c_i p_i$ with $q \neq 0$. Then,

$$f(a_1 q_0) + \sum_{i \geq 2} a_i p_i = f(q + c_1 q_0) + \sum_{i \geq 2} c_i p_i\,.$$

Thus, $a_1 q_0 = q + c_1 q_0$. Hence, if $a_1 = 0$, then $q = 0$. Therefore, $a_1 > 0$.

First, let us see elementary properties of a monoid of semistable type.

PROPOSITION 2.3. *Let $f : Q \to P$ be an integral homomorphism of fine sharp monoids. We assume that P is of semi-stable type*

$$(r, l, p_1, \dots, p_r, q_0, b_{l+1}, \dots, b_r)$$

over Q. Then we have the following:

(1) *Let $I \cdot p = f(q) + J \cdot p$ $(I, J \in \boldsymbol{N}^r)$ be a relation with $q \neq 0$. Then, $q = nq_0$ for some $n \in \boldsymbol{N}$. Moreover, if* $\mathrm{Supp}(I) \cap \mathrm{Supp}(J) = \emptyset$, *then $I = n\Delta_l$ and $J = nB$.*

(2) *Let us consider two elements*

$$f(q) + T \cdot p \quad \textit{and} \quad f(q') + T' \cdot p$$

of P such that there are i and j with $1 \leq i, j \leq l$ and $T(i) = T'(j) = 0$. If $f(q) + T \cdot p = f(q') + T' \cdot p$, then $q = q'$ and $T = T'$.

(3) *Let U (resp. V) be the submonoid of P generated by $p_1, \dots, p_l$ (resp. $f(Q)$ and $p_{l+1}, \dots, p_r$) (cf. Remark* 2.1). *Then, $U \simeq \boldsymbol{N}^l$, $V \simeq Q \times \boldsymbol{N}^{r-l}$ and the natural homomorphism*

$$U \times_{(\Delta_l \cdot p,\ f(q_0)+B \cdot p)} V \to P$$

is bijective.

PROOF. (1) First we assume that $\mathrm{Supp}(I) \cap \mathrm{Supp}(J) = \emptyset$. We set

$$n = \min\{I(1), \dots, I(l)\} \quad \text{and} \quad I' = I - n\Delta_l\,.$$

Then, $I'(i) = 0$ for some i with $1 \leq i \leq l$ and $I \cdot p = n\Delta_l \cdot p + I' \cdot p$. Thus,

$$f(nq_0) + (nB + I') \cdot p = f(q) + J \cdot p\,.$$

Therefore, since $f : Q \to P$ is integral, there are $q_1, q_2 \in Q$ and $T \in \boldsymbol{N}^r$ such that $nq_0 + q_1 = q + q_2$,

$$(nB + I') \cdot p = f(q_1) + T \cdot p \quad \text{and} \quad J \cdot p = f(q_2) + T \cdot p\,.$$

Note that $(nB + I')(i) = 0$ for some i $(1 \leq i \leq l)$. Thus, $q_1 = 0$ by Property (4). Moreover, since $\{1, \dots, l\} \subseteq \mathrm{Supp}(I)$, we have $\mathrm{Supp}(J) \subseteq \{l+1, \dots, r\}$, so that $q_2 = 0$ by Property (4). Therefore, $q = nq_0$ and $(nB + I') \cdot p = J \cdot p$. In particular, $nB + I' = J$. Note that $(nB + I')(i) = I'(i)$ and $J(i) = 0$ for $i = 1, \dots, l$. Thus, $I'(1) = \cdots = I'(l) = 0$. We assume that $\mathrm{Supp}(I') \neq \emptyset$. Let us choose $i \in \mathrm{Supp}(I')$. Then, $i > l$ and $J(i) = 0$. Thus, $nB(i) + I'(i) = 0$, which implies $I'(i) = 0$. This is a contradiction. Hence, $I' = 0$. Therefore, $q = nq_0$, $I = n\Delta_l$ and $J = nB$.

Next let us consider the general case. We define $T \in \boldsymbol{N}^r$ by $T(i) = \min\{I(i), J(i)\}$, and we set $I' = I - T$ and $J' = J - T$. Then, $I' \cdot p = f(q) + J' \cdot p$ and $\mathrm{Supp}(I') \cap \mathrm{Supp}(J') = \emptyset$. Thus, $q = nq_0$ for some $n \in \boldsymbol{N}$.

(2) Since $f : Q \to P$ is integral, there are $q_1, q_2 \in Q$ and $h \in \boldsymbol{N}p_1 + \cdots + \boldsymbol{N}p_r$ such that $T \cdot p = f(q_1) + h$, $T' \cdot p = f(q_2) + h$ and $q + q_1 = q' + q_2$. Here $T(i) = 0$ for some $i = 1, \dots, l$. Thus, $q_1 = 0$. In the same way, $q_2 = 0$. Therefore, $q = q'$. Hence $T \cdot p = T' \cdot p$.

(3) By (2), it is easy to see that $U \simeq \boldsymbol{N}^l$ and $V \simeq Q \times \boldsymbol{N}^{r-l}$. Let us choose $I, I', J, J' \in \boldsymbol{N}^r$ such that $\mathrm{Supp}(I), \mathrm{Supp}(I') \subseteq \{1, \dots, l\}$ and $\mathrm{Supp}(J), \mathrm{Supp}(J') \subseteq \{l+1, \dots, r\}$. It is sufficient to see that if

$$I \cdot p + f(q) + J \cdot p = I' \cdot p + f(q') + J' \cdot p$$

for some $q, q' \in Q$, then

$$(I \cdot p,\ f(q) + J \cdot p) \sim (I' \cdot p,\ f(q') + J' \cdot p)$$

in $U \times_{(\Delta_l \cdot p,\ f(q_0)+B\cdot p)} V$. We set

$$n = \min\{I(1), \dots, I(l)\} \quad \text{and} \quad n' = \min\{I'(1), \dots, I'(l)\}\,.$$

Moreover, we set $T = I - n\Delta_l$ and $T' = I' - n'\Delta_l$. Then

$$(T + J + nB) \cdot p + f(q + nq_0) = (T' + J' + n'B) \cdot p + f(q' + n'q_0)\,.$$

Thus, by (2), $T + J + nB = T' + J' + n'B$ and $q + nq_0 = q' + n'q_0$. In particular, $T = T'$ and $J + nB = J' + n'B$. Therefore, since $(\Delta_l \cdot p,\ 0) \sim (0, f(q_0) + B \cdot p)$,

$$\begin{aligned}(I \cdot p,\ f(q) + J \cdot p) &= ((T + n\Delta_l) \cdot p,\ f(q) + J \cdot p)\\ &\sim (T \cdot p,\ f(q + nq_0) + (J + nB) \cdot p)\\ &= (T' \cdot p,\ f(q' + n'q_0) + (J' + n'B) \cdot p)\\ &\sim ((T' + n'\Delta_l) \cdot p,\ f(q') + J' \cdot p)\\ &= (I' \cdot p,\ f(q') + J' \cdot p)\,.\end{aligned}$$

□

REMARK 2.4. By using a result of congruence relations [12, Lemma 2.8 (3)], we can prove Proposition 2.3 in more direct way.

REMARK 2.5. By the properties above, $k \otimes_{k[Q]} k[P]$ is canonically isomorphic to

$$k[X_1, \dots, X_r]/(X_1 \cdots X_l)\,.$$

The converse of the above remark holds under a kind of assumptions of P.

PROPOSITION 2.6. *Let k be a field and $f : Q \to P$ an integral homomorphism of fine sharp monoids with $Q \neq \{0\}$. Let R be the completion of $k \otimes_{k[Q]} k[P]$ at the origin and m the maximal ideal of R, where the homomorphism $k[Q] \to k = k[Q]/M_Q$ is given by the origin M_Q of $k[Q]$. We assume the following*:

(1) *$f : Q \to P$ does not split, i.e., there is no submonoid N of P with $P = f(Q) \times N$.*

(2) *Let $R' = R[\![T_1, \dots, T_e]\!]$ be the ring of formal power series of e-variables over R and m' the maximal ideal of R'. Then, R' is reduced,* $\dim_k m'/m'^2 = \dim R' + 1$ *and* $\dim R'/K' = \dim R'$ *for all minimal primes K' of R'.*

Let $p_1, \dots, p_r$ be all irreducible elements of P which are not lying in $f(Q)$. Let l be the number of minimal primes of R. Then, after renumbering $p_1, \dots, p_r$, P is of semi-stable type

$$(r, l, p_1, \dots, p_r, q_0, b_{l+1}, \dots, b_r)$$

over Q for some $q_0 \in Q \setminus \{0\}$ and $b_{l+1}, \dots, b_l \in \boldsymbol{N}$.

PROOF. Let us consider the natural homomorphism

$$H : Q \times \boldsymbol{N}^r \to P$$

given by $H(q, T) = f(q) + T \cdot p$. Since $f : Q \to P$ is integral, the system of congruence relations of H is generated by

$$\{I_\lambda \cdot p = f(q_\lambda) + J_\lambda \cdot p\}_{\lambda \in \Lambda},$$

where for each $\lambda \in \Lambda$, $q_\lambda \in Q$ and $I_\lambda, J_\lambda \in \boldsymbol{N}^r$ with $\mathrm{Supp}(I_\lambda) \cap \mathrm{Supp}(J_\lambda) = \emptyset$. Let $\phi : k[\![X_1, \dots, X_r]\!] \to R$ be the homomorphism arising from

$$k[\boldsymbol{N}^r] = k \otimes_{k[Q]} k[Q \times \boldsymbol{N}^r] \to k \otimes_{k[Q]} k[P].$$

Then, by Lemma 1.2, the kernel of ϕ is generated by

$$\{X^{I_\lambda} - \beta(q_\lambda) X^{J_\lambda}\}_{\lambda \in \Lambda},$$

where β is given by

$$\beta(q) = \begin{cases} 1 & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$

Let m be the maximal ideal of R. By Assumption (2), it is easy to see that R is reduced, $\dim_k m/m^2 = \dim R + 1$ and $\dim R/K = \dim R$ for all minimal primes K of R. Let M be the maximal ideal of $k[\![X_1, \dots, X_r]\!]$. Here p_i's are irreducible. Thus, $\deg(I_\lambda) \geq 2$ if $q_\lambda \neq 0$, and $\deg(I_\lambda) \geq 2$ and $\deg(J_\lambda) \geq 2$ if $q_\lambda = 0$. Hence, $\mathrm{Ker}(\phi) \subseteq M^2$. Therefore,

$$\dim_k m/m^2 = \dim_k M/(M^2 + \mathrm{Ker}(\phi)) = \dim_k M/M^2 = r,$$

which implies that $r = \dim R + 1$. Since R is reduced, $\mathrm{Ker}(\phi) = \sqrt{\mathrm{Ker}(\phi)}$. Thus, we have a decomposition

$$\mathrm{Ker}(\phi) = K_1 \cap \cdots \cap K_l$$

such that K_i are prime, $K_i \not\subseteq K_j$ for all $i \neq j$ and each K_i corresponds to a minimal prime of R. Note that $\dim k[\![X_1, \dots, X_r]\!]/K_i = r - 1$ for each i. Here $k[\![X_1, \dots, X_r]\!]$ is a UFD. Thus, each K_i's are generated by an irreducible element, so that we can see that there is $f \in k[\![X_1, \dots, X_r]\!]$ with $\mathrm{Ker}(\phi) = (f)$. Here we claim the following.

CLAIM 2.6.1. *There is $\lambda \in \Lambda$ with $q_\lambda \neq 0$.*

We assume the contrary. Let N be a submonoid of P generated by p_i's. Let us see that

$$f(q) + n = f(q') + n' \quad (q, q' \in Q, n, n' \in N) \quad \Longrightarrow \quad q = q',\ n = n'.$$

Since $f : Q \to P$ is integral, there are $q_1, q_2 \in Q$ and $n'' \in N$ such that $n = f(q_1) + n''$, $n' = f(q_2) + n''$ and $q + q_1 = q' + q_2$. If $q_\lambda = 0$ for all $\lambda \in \Lambda$, then $q_1 = q_2 = 0$. We can see $q_1 = q_2 = 0$. Thus, $n = n' = n''$ and $q = q'$. This observation shows us that $P = Q \times N$, which contradicts to our assumption.

By the claim above, $\mathrm{Ker}(\phi)$ contains an element of the form X^{I_λ}. Note that f is a factor of X^{I_λ}, R is reduced and R contains l minimal primes. Thus, after renumbering $p_1, \dots, p_r$, we can set $f = X_1 \cdots X_l = X^{\Delta_l}$. Next we claim the following.

CLAIM 2.6.2. *$q_\lambda \neq 0$ for all $\lambda \in \Lambda$.*

We assume that there is $\lambda \in \Lambda$ with $q_\lambda = 0$. Then, $X_1 \cdots X_l$ divides $X^{I_\lambda} - X^{J_\lambda}$. This is impossible because $\mathrm{Supp}(I_\lambda) \cap \mathrm{Supp}(J_\lambda) = \emptyset$.

By the claim above, we can see that N is isomorphic to $\boldsymbol{N}^r$. Moreover, $\mathrm{Ker}(\phi)$ is generated by $\{X^{I_\lambda}\}_{\lambda \in \Lambda}$. Thus, there is $\lambda \in \Lambda$ with $I_\lambda = \Delta_l$. Hence, we have a congruence relation $\Delta_l \cdot p = f(q_0) + B \cdot p$.

Finally, let us consider a relation

$$I \cdot p = f(q) + J \cdot p$$

with $q \neq 0$. Then, X^I is an element of $\mathrm{Ker}(\phi)$. Thus, $I(i) > 0$ for all $i = 1, \dots, l$. $\square$

3. Local structure theorem on a semistable variety. The purpose of this section is to prove the following local structure theorem of a smooth log structure on a semistable variety. Classification results of log structures on a semistable variety have been already obtained in several important cases. F. Kato studied the local description of a log structure on a log smooth and integral morphism with relative dimension one [4, Theorem 1.3 and (1.8)]. M. Olsson investigated the local description of a log structure on a log smooth, vertical and integral morphism [12, Theorem 2.7]. (A morphism $(\phi, h) : (X, M_X) \to (Y, M_Y)$ is said to be *vertical* if $\mathrm{Coker}(h : \phi^* M_Y \to M_X)$ is a sheaf of groups.) We consider local structures of a log structure on a log smooth and integral morphism on a semistable variety without the assumptions of dimension and verticalness.

THEOREM 3.1. *Let k be an algebraically closed field and M_k a fine log structure of* $\mathrm{Spec}(k)$. *Let X be a semistable variety over k and M_X a fine log structure of X. We assume that (X, M_X) is log smooth and integral over* $(\mathrm{Spec}(k), M_k)$. *For a closed point $x \in X$, let $(Q \to M_k, P \to M_{X,\bar{x}}, Q \to P)$ be a good chart of $(X, M_X) \to$* $(\mathrm{Spec}(k), M_k)$ *at x, that is, $Q \to \bar{M}_k$ and $P \to \bar{M}_{X,\bar{x}}$ are bijective homomorphisms of fine sharp monoids, $k \otimes_{k[Q]} k[P] \to \mathcal{O}_{X,\bar{x}}$ is smooth and the following diagram*

$$\begin{array}{ccc} Q & \longrightarrow & P \\ \downarrow & & \downarrow \\ M_k & \longrightarrow & M_{X,\bar{x}} \end{array}$$

is commutative. Then, we have the following:

(1) *If the multiplicity of X at x is equal to* 1, *that is, x is a regular point, then $Q \to P$ splits and $P \simeq Q \times \boldsymbol{N}^r$ for some r.*

(2) *If the multiplicity of X at x is equal to* 2, *then we have one of the following*:

(2.1) *If $Q \to P$ does not split, then P is of semistable type over Q.*

(2.2) *If $Q \to P$ splits, then* $\mathrm{char}(k) \neq 2$ *and there is a submonoid N of P such that $P \simeq Q \times N$ and N is isomorphic to the momoid arising from the monomials of $k[X_1, X_2, \dots, X_a]/(X_1^2 - X_2^2)$ for some $a \geq 2$. In particular, $\hat{\mathcal{O}}_{X,x}$ is canonically isomorphic to*

$$k[\![X_1, \dots, X_r]\!]/(X_1^2 - X_2^2)\,.$$

(3) *If the multiplicity of X at x is greater than or equal to* 3, *then $Q \to P$ does not split and P is of semistable type over Q.*

(4) *If x is a singular point of X and P^{gp} is torsion free*, *then $Q \to P$ does not split and P is of semistable type over Q.*

In particular, *if M_X is saturated*, *then*, *for all $x \in X$, P is a monoid of semistable type over Q.*

In order to prove the above theorem, we need several preparations. First, let us consider a log smooth monoid on a smooth variety.

PROPOSITION 3.2. *Let k be a field and $f : Q \to P$ an integral homomorphism of fine sharp monoids* (*note that Q might be* $\{0\}$). *Let R be the completion of $k \otimes_{k[Q]} k[P]$ at the origin and $R[\![T_1, \dots, T_e]\!]$ the ring of formal power series of e-variables over R*, *where the homomorphism $k[Q] \to k = k[Q]/M_Q$ is given by the origin M_Q of $k[Q]$. If $R[\![T_1, \dots, T_e]\!]$ is regular*, *then there are a nonnegative integer r and a homomorphism $g : \boldsymbol{N}^r \to P$ such that the homomorphism*

$$h : Q \times \boldsymbol{N}^r \to P$$

given by $h(q, x) = f(q) + g(x)$ is bijective.

PROOF. First of all, note that R is regular. Let $p_1, \dots, p_r$ be all irreducible elements of P which are not lying in $f(Q)$. Then, we have a homomorphism $g : \boldsymbol{N}^r \to P$ given by $g(n_1, \dots, n_r) = \sum_{i=1}^{r} n_i p_i$. Thus, we get $h : Q \times \boldsymbol{N}^r \to P$ as in the statement of our proposition. Clearly, h is surjective. Then, since $f : Q \to P$ is integral, the congruence relation is generated by a system

$$\{I_\lambda \cdot p = f(q_\lambda) + J_\lambda \cdot p\}_{\lambda \in \Lambda},$$

where $q_\lambda \in Q$ and $I_\lambda, J_\lambda \in \boldsymbol{N}^r$ with $\mathrm{Supp}(I_\lambda) \cap \mathrm{Supp}(J_\lambda) = \emptyset$ for each λ. Then, by Lemma 1.2, the kernel K of

$$k[\![X_1, \dots, X_r]\!] \to R$$

is generated by

$$\{X^{I_\lambda} - \beta(q_\lambda) X^{J_\lambda}\}_{\lambda \in \Lambda},$$

where β is given by

$$\beta(q) = \begin{cases} 1 & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$

Using the fact that p_i's are irreducible, we can see that $K \subset M^2$, where M is the maximal ideal of $k[\![X_1, \dots, X_r]\!]$. Let m be the maximal ideal of R. Then,

$$m/m^2 = M/(M^2 + K) = M/M^2.$$

Thus, $\dim_k m/m^2 = r$. On the other hand, if we have a congruence relation, then $K \neq \{0\}$. Thus, $\dim R < r$. Therefore, $K = \{0\}$, which means that h is injective. □

In order to proceed with our arguments, let us see elementary facts of the ring

$$k[\![X_1, \dots, X_n]\!]/(X^{I_0} - X^{J_0}).$$

PROPOSITION 3.3. *Let k be a field and $k[[X_1, \dots, X_n]]$ the ring of formal power series of n-variables over k. Let I_0 and J_0 be elements of $\boldsymbol{N}^n$ such that* $\mathrm{Supp}(I_0) \cap \mathrm{Supp}(J_0) = \emptyset$, *$I_0 \neq (0, \dots, 0)$ and $J_0 \neq (0, \dots, 0)$. Consider the ring*

$$R = k[[X_1, \dots, X_n]]/(X^{I_0} - X^{J_0}) .$$

The image of X^I on R is denoted by x^I. Then, we have the following:

(1) *The multiplication of X_i in R is injective.*

(2) *For $I, J \in \boldsymbol{N}^n$ and $h \in R$, if $x^I = x^J h$ and $I \not\geq J$, then either $I \geq I_0$ or $I \geq J_0$* (*cf. Convention and terminology* 2).

(3) *Let u and v be units of R. For $I, J \in \boldsymbol{N}^n$, if $x^I u = x^J v$, then $u = v$ and $x^I = x^J$.*

(4) *For $I, J \in \boldsymbol{N}^n$, set $I = I' + aI_0 + bJ_0$ and $J = J' + a'I_0 + b'J_0$ such that $a, b, a', b' \in \boldsymbol{N}$ and that $I' \not\geq I_0$, $I' \not\geq J_0$, $J' \not\geq I_0$ and $J' \not\geq J_0$. If $x^I = x^J$, then $I' = J'$ and $a + b = a' + b'$.*

(5) *If* $\gcd(I_0)$ *and* $\gcd(J_0)$ *are coprime, then $X^{I_0} - X^{J_0}$ is irreducible in $k[[X_1, \dots, X_n]]$* (*cf. Convention and terminology* 2).

PROOF. The proof of (1), (2), (3) and (4) is elementary, so we left it to the reader. □

We only give a proof of (5). The following proof is due to the referee.

LEMMA 3.4. *Let k be a field and f a polynomial in $k[X_1, \dots, X_n]$. Assume that there exists a non-trivial weight such that f is homogeneous with respect to it. Then f is irreducible in $k[[X_1, \dots, X_n]]$ if and only if f is irreducible in $k[X_1, \dots, X_n]$.*

PROOF. The "only if" direction is clear. To see the "if" direction, assume that there exists a decomposition $f = gh$ in $k[[X_1, \dots, X_n]]$ such that both g and h are not invertible elements in $k[[X_1, \dots, X_n]]$. By taking account of the (weighted) degrees of f, g and h, we can easily see that f is not irreducible in $k[X_1, \dots, X_n]$. Thus we conclude the claim. □

Return to the proof of (5). By an easy observation, we see that $X^{I_0} - X^{J_0}$ is irreducible in $k[X_1, \dots, X_n]$. Thus $X^{I_0} - X^{J_0}$ is irreducible in $k[[X_1, \dots, X_n]]$ by Lemma 3.4. □

COROLLARY 3.5. *We assume that k is algebraically closed. Let I_0 and J_0 be elements of $\boldsymbol{N}^n$ such that* $\deg(I_0) \geq 1$, $\deg(J_0) \geq 1$ *and* $\mathrm{Supp}(I_0) \cap \mathrm{Supp}(J_0) = \emptyset$. *We set* $g = \gcd(\gcd(I_0), \gcd(J_0))$, $I_0 = gI_0'$ *and* $J_0 = gJ_0'$. *Then,*

$$X^{I_0} - X^{J_0} = (X^{I_0'} - X^{J_0'})(X^{I_0'} - \zeta X^{J_0'}) \cdots (X^{I_0'} - \zeta^{g-1} X^{J_0'})$$

is the irreducible decomposition of $X^{I_0} - X^{J_0}$, where ζ is a g-th primitive root of the unity.

PROOF. It is sufficient to show that $X^{I_0'} - \zeta^i X^{J_0'}$ is irreducible. Changing coordinates $X_1, \dots, X_n$ by $c_1 X_1, \dots, c_n X_n$, we can make $X^{I_0'} - X^{J_0'}$ of $X^{I_0'} - \zeta^i X^{J_0'}$. Thus, by (5) of Proposition 3.3, we have our corollary. □

COROLLARY 3.6. *We assume that k is algebraically closed. Let I_0 and J_0 be elements of $\boldsymbol{N}^n$ such that* $\deg(I_0) \geq 1$, $\deg(J_0) \geq 1$ *and* $\mathrm{Supp}(I_0) \cap \mathrm{Supp}(J_0) = \emptyset$. *If*

$$k[[X_1, \dots, X_n]]/(X^{I_0} - X^{J_0})$$

is isomorphic to a ring of the type $k[\![T_1, \dots, T_e]\!]/(T_1 \cdots T_l)$ $(l \geq 2)$, *then* $\mathrm{char}(k) \neq 2$ *and there are* $i, j \in \{1, \dots, n\}$ *such that* $i \neq j$ *and* $X^{I_0} - X^{J_0} = X_i^2 - X_j^2$.

PROOF. We set $g = \gcd(\gcd(I_0), \gcd(J_0))$, $I_0 = gI_0'$ and $J_0 = gJ_0'$. Then, by the above corollary,

$$X^{I_0} - X^{J_0} = (X^{I_0'} - X^{J_0'})(X^{I_0'} - \zeta X^{J_0'}) \cdots (X^{I_0'} - \zeta^{g-1} X^{J_0'})$$

is the irreducible decomposition of $X^{I_0} - X^{J_0}$, where ζ is a g-th primitive root of the unity. Since $k[\![X_1, \dots, X_n]\!]/(X^{I_0} - X^{J_0})$ is reduced, $\mathrm{char}(k)$ does not divide g. Here $k[\![T_1, \dots, T_n]\!]/(T_1 \cdots T_l)$ has l-minimal primes, so that $g = l$. Moreover, since every irreducible component is regular, either $X^{I_0'}$ or $X^{J_0'}$ is linear. Clearly, we may assume that $X^{I_0'}$ is linear, namely, $X^{I_0'} = X_i$ for some i. Let m be the maximal ideal of $k[\![X_1, \dots, X_n]\!]/(X^{I_0} - X^{J_0})$. Let V be the vector subspace of m/m^2 generated by $X_i - X^{J_0'}, X_i - \zeta X^{J_0'}, \dots, X_i - \zeta^{l-1} X^{J_0'}$. Then we must have $\dim_k V = l$, since

$$k[\![X_1, \dots, X_n]\!]/(X^{I_0} - X^{J_0}) \simeq k[\![T_1, \dots, T_n]\!]/(T_1 \cdots T_l)\,.$$

If $\deg(J_0') \geq 2$, then $\dim_k V = 1$. This contradict to the fact $l \geq 2$. Thus, $\deg(J_0') = 1$, so that $X^{J_0'} = X_j$ for some j. In this case, $\dim_k V \leq 2$. Therefore, $g = l = 2$. □

PROPOSITION 3.7. *Let k be a field, N a fine sharp monoid, and $k[\![N]\!]$ the completion of $k[N]$ at the origin. Let $\alpha : N \to k[\![N]\!]$ be the canonical homomorphism. Let $p_1, \dots, p_r$ be the irreducible elements of N and $h : \boldsymbol{N}^r \to N$ the natural homomorphism given by $h(a_1, \dots, a_r) = \sum_{i=1}^r a_i p_i$. Let $\phi : k[\![X_1, \dots, X_r]\!] \to k[\![N]\!]$ be the homomorphism induced by h. Let $R' = k[\![N]\!][\![X_1, \dots, X_e]\!]$ be the ring of formal power series of e-variables over $k[\![N]\!]$ and m' the maximal ideal of R'. We assume that R' is reduced,* $\dim_k m'/m'^2 = \dim R' + 1$ *and* $\dim R'/K' = \dim R'$ *for all minimal primes K' of R'. Then, we have the following*:

(1) *The kernel of ϕ is generated by an element of the form $X^{I_0} - X^{J_0}$ such that $I_0, J_0 \in \boldsymbol{N}^r$, $\deg(I_0) \geq 2$, $\deg(J_0) \geq 2$,* $\mathrm{Supp}(I_0) \cap \mathrm{Supp}(J_0) = \emptyset$ *and* $\gcd(\gcd(I_0), \gcd(J_0))$ *is not divisible by* $\mathrm{char}(k)$.

(2) *Renumbering of $p_1, \dots, p_r$, we assume that*

$$\mathrm{Supp}(I_0) \subseteq \{1, \dots, l\} \quad \textit{and} \quad \mathrm{Supp}(J_0) \subseteq \{l+1, \dots, r\}\,.$$

Let U (resp. V) be the submonoid of N generated by $p_1, \dots, p_l$ (resp. $p_{l+1}, \dots, p_r$). Then, $U \simeq \boldsymbol{N}^l$, $V \simeq \boldsymbol{N}^{r-l}$ and the natural homomorphism

$$U \times_{(I_0 \cdot p,\ J_0 \cdot p)} V \to N$$

is bijective (*cf. Convention and terminology* 4).

PROOF. (1) Let us consider all relations

$$\{I_\lambda \cdot p = J_\lambda \cdot p\}_{\lambda \in \Lambda}$$

in N, where $I_\lambda, J_\lambda \in \boldsymbol{N}^r$ and $\mathrm{Supp}(I_\lambda) \cap \mathrm{Supp}(J_\lambda) = \emptyset$ for all λ. Then, the kernel of ϕ is generated by

$$\{X^{I_\lambda} - X^{J_\lambda}\}_{\lambda \in \Lambda}\,.$$

Let m be the maximal ideal of $k[\![N]\!]$. Then, it is easy to see that $k[\![N]\!]$ is reduced, $\dim_k m/m^2 = \dim k[\![N]\!] + 1$ and $\dim k[\![N]\!]/K = \dim k[\![N]\!]$ for all minimal primes K of $k[\![N]\!]$. Let M be the maximal ideal of $k[\![X_1, \dots, X_r]\!]$. Since p_i's are irreducible, $\deg(I_\lambda) \geq 2$ and $\deg(J_\lambda) \geq 2$. Thus, $\mathrm{Ker}(\phi) \subseteq M^2$. Therefore,

$$m/m^2 = M/(\mathrm{Ker}(\phi) + M^2) = M/M^2 .$$

Then, in the same way as in the proof of Proposition 2.6, there is $f \in k[\![X_1, \dots, X_r]\!]$ with $\mathrm{Ker}(\phi) = (f)$. We set $X^{I_\lambda} - X^{J_\lambda} = f u_\lambda$ for all $\lambda \in \Lambda$. If u_λ is not a unit for any $\lambda \in \Lambda$, then $X^{I_\lambda} - X^{J_\lambda} \in f \cdot M$. Thus, there is $\lambda \in \Lambda$ such that u_λ is a unit. Hence we get (1).

(2) By using (4) of Proposition 3.3, it is easy to see that $U \simeq \boldsymbol{N}^l$ and $V \simeq \boldsymbol{N}^{r-l}$. Let $I, I', J, J' \in \boldsymbol{N}^r$ such that

$$\mathrm{Supp}(I),\ \ \mathrm{Supp}(I') \subseteq \{1, \dots, l\} \quad \text{and} \quad \mathrm{Supp}(J),\ \ \mathrm{Supp}(J') \subseteq \{l+1, \dots, r\} .$$

It is sufficient to see that if $I \cdot p + J \cdot p = I' \cdot p + J' \cdot p$, then $(I \cdot p,\ J \cdot p) \sim (I' \cdot p,\ J' \cdot p)$ in $U \times_{(I_0 \cdot p,\ J_0 \cdot p)} V$. We set $I = T + aI_0$, $I' = T' + a'I_0$, $J = S + bJ_0$ and $J' = S' + b'J_0$ such that $a, a', b, b' \in \boldsymbol{N}$ and $T \not\geq I_0$, $T' \not\geq I_0$, $S \not\geq J_0$ and $S' \not\geq J_0$. Then, by (4) of Proposition 3.3, we can see that $T + S = T' + S'$ and $a + b = a' + b'$. In particular, $T = T'$ and $S = S'$. Therefore, since $(I_0 \cdot p,\ 0) \sim (0,\ J_0 \cdot p)$,

$$\begin{aligned}(I \cdot p,\ J \cdot p) &= ((T + aI_0) \cdot p,\ (S + bJ_0) \cdot p) \sim (T \cdot p,\ (S + (a+b)J_0) \cdot p) \\ &= (T' \cdot p,\ (S' + (a'+b')J_0) \cdot p) \sim ((T' + a'I_0) \cdot p,\ (S' + bJ_0) \cdot p) \\ &= (I' \cdot p,\ J' \cdot p) .\end{aligned}$$ □

Let us start the proof of Theorem 3.1. This is a consequence of all results in §2 and §3. Indeed, if $x \notin \mathrm{Sing}(X)$, then our assertion holds by Proposition 3.2. Thus, we may assume that $x \in \mathrm{Sing}(X)$.

We assume that $Q \to P$ split, so that $P \simeq Q \times N$ for some N. Then,

$$k \otimes_{k[Q]} k[P] \simeq k[N] .$$

Since $k[N] \to \mathcal{O}_X$ is smooth, $k[\![N]\!][\![X_1, \dots, X_e]\!]$ is isomorphic to the ring of the type $k[\![T_1, \dots, T_n]\!]/(T_1 \cdots T_l)$. Thus, by Corollary 3.6 and Proposition 3.7, $\mathrm{char}(k) \neq 2$ and $l = 2$. Moreover, if P^{gp} is torsion free, then N^{gp} is torsion free. Thus, $k[\![N]\!]$ is an integral domain by Lemma 3.8 below. This is a contradiction. Therefore, if P^{gp} is torsion free, then $Q \to P$ does not split.

If $Q \to P$ does not split, then we get our assertion by Proposition 2.6. □

LEMMA 3.8. *Let T be a fine sharp monoid such that T^{gp} is torsion free. Then $k[T]$ and the completion $k[\![T]\!]$ at the origin are integral domains.*

PROOF. First of all, it is well known that if σ is a finitely generated cone in $\boldsymbol{Q}^n$ with $\sigma \cap (-\sigma) = \{0\}$, then there is an isomorphism $\phi : \boldsymbol{Q}^n \to \boldsymbol{Q}^n$ such that $\phi(\sigma) \subseteq \boldsymbol{Q}^n_{\geq 0}$. Thus, we can find an injective homomorphism $\psi : T^{\mathrm{gp}} \to \boldsymbol{Z}^n$ such that $\mathrm{Coker}(\psi)$ is finite and $\psi(T) \subseteq \boldsymbol{N}^n$, where $n = \mathrm{rk}(T^{\mathrm{gp}})$. Therefore, $k[T] \hookrightarrow k[\boldsymbol{N}^n] = k[X_1, \dots, X_n]$ and $k[\![T]\!] \hookrightarrow k[\![\boldsymbol{N}^n]\!] = k[\![X_1, \dots, X_n]\!]$. □

4. Rigidity of log morphisms. In this section, we consider a uniqueness problem of a log morphism for a fixed scheme morphism, which is one of main results of this paper.

THEOREM 4.1. *Let k be an algebraically closed field and M_k a fine log structure of* $\mathrm{Spec}(k)$. *Let X and Y be semistable varieties over k, and M_X and M_Y fine log structures of X and Y, respectively. We assume that (X, M_X) and (Y, M_Y) are log smooth and integral over* $(\mathrm{Spec}(k), M_k)$. *We set*

$$\mathrm{Supp}(M_Y/M_k) = \{y \in Y \mid M_k \times \mathcal{O}_{Y,\bar{y}}^{\times} \to M_{Y,\bar{y}} \textit{ is not surjective}\}.$$

Let $\phi : X \to Y$ be a morphism over k such that $\phi(X') \not\subseteq \mathrm{Supp}(M_Y/M_k)$ for any irreducible component X' of X. If $(\phi, h) : (X, M_X) \to (Y, M_Y)$ and $(\phi, h') : (X, M_X) \to (Y, M_Y)$ are morphisms of log schemes over $(\mathrm{Spec}(k), M_k)$, *then $h = h'$.*

PROOF. This is a local question. Let us take a fine sharp monoid Q with $M_k = Q \times k^{\times}$. Let x be a closed point of X and $y = f(x)$. Let us choose étale local neighborhoods U and V at x and y, respectively, with $f(U) \subseteq V$. Moreover, shrinking U and V enough, by Corollary 1.4, we may assume that there are good charts

$$(Q \to M_k, \pi : P \to M_X|_U, f : Q \to P)$$

and

$$(Q \to M_k, \pi' : P' \to M_Y|_V, f' : Q \to P')$$

of $(X, M_X) \to (\mathrm{Spec}(k), M_k)$ and $(Y, M_Y) \to (\mathrm{Spec}(k), M_k)$ at x and y, respectively. Let $\tilde{\pi} : P \times \mathcal{O}_{X,\bar{x}}^{\times} \to M_{X,\bar{x}}$ and $\tilde{\pi}' : P' \times \mathcal{O}_{Y,\bar{y}}^{\times} \to M_{Y,\bar{y}}$ be the natural homomorphisms induced by π and π'. Note that $\tilde{\pi}$ and $\tilde{\pi}'$ are isomorphisms. Let $H : P' \times \mathcal{O}_{Y,\bar{y}}^{\times} \to P \times \mathcal{O}_{X,\bar{x}}^{\times}$ and $H' : P' \times \mathcal{O}_{Y,\bar{y}}^{\times} \to P \times \mathcal{O}_{X,\bar{x}}^{\times}$ be homomorphisms of monoids such that the following diagrams are commutative:

$$\begin{array}{ccc} P' \times \mathcal{O}_{Y,\bar{y}}^{\times} & \xrightarrow{H} & P \times \mathcal{O}_{X,\bar{x}}^{\times} \\ {\scriptstyle \tilde{\pi}'}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ M_{Y,\bar{y}} & \xrightarrow{h} & M_{X,\bar{x}} \\ {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathcal{O}_{Y,\bar{y}} & \xrightarrow{\phi^*} & \mathcal{O}_{X,\bar{x}}, \end{array} \qquad \begin{array}{ccc} P' \times \mathcal{O}_{Y,\bar{y}}^{\times} & \xrightarrow{H'} & P \times \mathcal{O}_{X,\bar{x}}^{\times} \\ {\scriptstyle \tilde{\pi}'}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ M_{Y,\bar{y}} & \xrightarrow{h'} & M_{X,\bar{x}} \\ {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathcal{O}_{Y,\bar{y}} & \xrightarrow{\phi^*} & \mathcal{O}_{X,\bar{x}}. \end{array}$$

Here α and α' are the canonical homomorphism. By abuse of notation, $\alpha \cdot \tilde{\pi}$ and $\alpha' \cdot \tilde{\pi}'$ are also denoted by α and α'. Then, $\alpha(p, u) = \alpha(\pi(p)) \cdot u$ and $\alpha'(p', u') = \alpha'(\pi'(p')) \cdot u'$.

Note the following two claims.

CLAIM 4.1.1. *$H(0, u) = H'(0, u)$ for all $u \in \mathcal{O}_{Y,\bar{y}}^{\times}$.*

PROOF. It is obvious because $h(\mathcal{O}_{Y,\bar{y}}) \subseteq \mathcal{O}_{X,\bar{x}}^{\times}$. □

CLAIM 4.1.2. *$H(f'(q), 1) = H'(f'(q), 1)$ for all $q \in Q$.*

PROOF. Since $\pi : P \to M_X|_U$ and $\pi' : P' \to M_Y|_V$ are good charts at $\bar{x}$ and $\bar{y}$, respectively, and all homomorphisms are lying over M_k, thus our claim is clear. □

From now on, we consider the following four cases:

(A) $f : Q \to P$ splits and $f' : Q \to P'$ splits.

(B) $f : Q \to P$ does not split and $f' : Q \to P'$ splits.

(C) $f : Q \to P$ splits and $f' : Q \to P'$ does not split.

(D) $f : Q \to P$ does not split and $f' : Q \to P'$ does not split.

By Theorem 3.1, if $f : Q \to P$ (resp. $f' : Q \to P'$) splits, then $\bar{x}$ (resp. $\bar{y}$) is either a smooth point or a singular points étale locally isomorphic to

$$\operatorname{Spec} k[x_1, x_2, \dots, x_r]/(x_1^2 - x_2^2) \quad (\text{resp. } \operatorname{Spec} k[y_1, y_2, \dots, y_{r'}]/(y_1^2 - y_2^2)).$$

Moreover, if $f : Q \to P$ (resp. $f' : Q \to P'$) does not split, then P (resp. P') is of semistable type over Q.

For each case, let $U_1, \cdots, U_l$ and $V_1, \cdots, V_{l'}$ be all irreducible components of U and V, respectively. Here, since $\operatorname{Sing}(Y) \subseteq \operatorname{Supp}(M_Y/M_k)$ and $\phi(U_j) \not\subseteq \operatorname{Supp}(M_Y/M_k)$ for each j, there is a unique i with $\phi(U_j) \subseteq V_i$. We denote this i by $\sigma(j)$. Note that we have a map $\sigma : \{1, \dots, l\} \to \{1, \dots, l'\}$. In the following, we give irreducible elements $p_1, \dots, p_r \in P$ (resp. $p'_1, \dots, p'_{r'} \in P'$) for each case (A), (B), (C) and (D) such that P (resp. P') is generated by $f(Q)$ and $p_1, \dots, p_r$ (resp. $f'(Q')$ and $p'_1, \dots, p'_{r'}$). The last claim is the following

CLAIM 4.1.3. *$H(p'_i, 1) = H'(p'_i, 1)$ for all $i = 1, \cdots, r'$.*

For this purpose, we fix common notation for all cases. We denote $\alpha(p_j, 1)$ by x_j and $\alpha'(p'_i, 1)$ by y_i. Here we set

$$H(p'_i, 1) = (f(q_i) + I_i \cdot p,\ u_i) \quad \text{and} \quad H'(p'_i, 1) = (f(q'_i) + I'_i \cdot p,\ u'_i), \tag{4.1.4}$$

where $I_i, I'_i \in \boldsymbol{N}^r$, $q_i, q'_i \in Q$ and $u_i, u'_i \in \mathcal{O}^\times_{X,\bar{x}}$. Then, since $\alpha(H(p'_i, 1)) = \phi^*(\alpha'(p'_i, 1))$ and $\alpha(H'(p'_i, 1)) = \phi^*(\alpha'(p'_i, 1))$, we have

$$\phi^*(y_i) = \beta(q_i) \cdot x^{I_i} \cdot u_i = \beta(q'_i) \cdot x^{I'_i} \cdot u'_i. \tag{4.1.5}$$

Let us begin with Case A.

CASE A. In this case, there are submonoids N and N' of P and P', respectively, such that $P = f(Q) \times N$ and $P' = f'(Q) \times N'$. Let $p_1, \dots, p_r$ (resp. $p'_1, \dots, p'_{r'}$) be all irreducible elements of N (resp. N'). By Theorem 3.1,

$$\operatorname{Supp}(M_Y/M_k) = \{y_1 = 0\} \cup \cdots \cup \{y_{r'} = 0\}$$

around $\bar{y}$. Thus, we have

$$\phi^*(y_i)|_{U_j} = \beta(q_i) \cdot x^{I_i} \cdot u_i|_{U_j} = \beta(q'_i) \cdot x^{I'_i} \cdot u'_i|_{U_j} \neq 0$$

for all j. In particular, $q_i = q'_i = 0$ for all $i = 1, \dots, r'$. Therefore,

$$x^{I_i} \cdot u_i = x^{I'_i} \cdot u'_i$$

for all i. Thus, by (3) of Proposition 3.3, $u_i = u'_i$ and $x^{I_i} = x^{I'_i}$. Note that the natural homomorphism $k[N] \to \mathcal{O}_{X,\bar{x}}$ is injective. Hence, we get $I_i \cdot p = I'_i \cdot p$.

CASE B. In this case, there is a submonoid N' of P' such that $P' = f'(Q) \times N'$. Let $p'_1, \dots, p'_{r'}$ be all irreducible elements of N'. Moreover, by Proposition 2.6, P is of semistable type

$$(r, l, p_1, \dots, p_r, q_0, b_{l+1}, \dots, b_r)$$

over Q. Renumbering $U_1, \dots, U_l$, we may assume that U_j is defined by $x_j = 0$. By the same argument as in (Case A), we have

$$\phi^*(y_i)|_{U_j} = \beta(q_i) \cdot x^{I_i} \cdot u_i|_{U_j} = \beta(q'_i) \cdot x^{I'_i} \cdot u'_i|_{U_j} \neq 0$$

for all j. In particular, $q_i = q'_i = 0$ and $I_i(j) = I'_i(j) = 0$ for $j = 1, \dots, l$. Further, since $\mathcal{O}_{U_j,\bar{x}}$ is a UFD, we can see that $I_i = I'_i$. Moreover, $u_i|_{U_j} = u'_i|_{U_j}$ for all j. Thus, $u_i = u'_i$. Therefore, $H(p'_i, 1) = H'(p'_i, 1)$ for all $i = 1, \dots, r'$.

CASE C. There is a submonoid N of P such that $P = f(Q) \times N$. Let $p_1, \dots, p_r$ be all irreducible elements of N. Moreover, by Proposition 2.6, P' is of semistable type

$$(r', l', p'_1, \dots, p'_{r'}, q'_0, b'_{l+1}, \dots, b'_r)$$

over Q. Renumbering $V_1, \dots, V_{l'}$, we may assume that V_i is defined by $y_i = 0$. Note that

$$\mathrm{Supp}(M_Y/M_k) = \mathrm{Sing}(Y) \cup \{y_{l'+1} = 0\} \cup \cdots \cup \{y_{r'} = 0\}$$

around $\bar{y}$. Therefore, if $i \neq \sigma(j)$, then $\phi^*(y_i)|_{U_j} \neq 0$. Thus, we can see $q_i = q'_i = 0$ for $i \neq \sigma(j)$.

First, we consider the case where $\sigma(1) = \cdots = \sigma(l) = s$. Note that $s \leq l'$. Then, for $i \neq s$, $q_i = q'_i = 0$. Thus, $x^{I_i} \cdot u_i = x^{I'_i} \cdot u'_i$ for all $i \neq s$. Therefore, in the same way as in Case A, we can see

$$I_i \cdot p = I'_i \cdot p \quad \text{and} \quad u_i = u'_i$$

for all $i \neq s$. On the other hand, we have the relation $p'_1 + \cdots + p'_{l'} = f'(q'_0) + \sum_{i>l'} b'_i p'_i$. Therefore, we have $H(p'_s, 1) = H'(p'_s, 1)$.

Hence, we may assume that $\#(\sigma(\{1, \cdots, l\})) \geq 2$. In this case, we can conclude that $q_i = q'_i = 0$ for all i. Therefore, in the same way as in Case A, we can see

$$I_i \cdot p = I'_i \cdot p \quad \text{and} \quad u_i = u'_i$$

for all i.

CASE D. By Proposition 2.6, P and P' are of semistable type

$$(r, l, p_1, \dots, p_r, q_0, b_{l+1}, \dots, b_r) \quad \text{and} \quad (r', l', p'_1, \dots, p'_{r'}, q'_0, b'_{l'+1}, \dots, b'_{r'})$$

over Q. Renumbering $U_1, \dots, U_l$ and $V_1, \dots, V_{l'}$, we may assume that U_j is defined by $x_j = 0$ and V_i is defined by $y_i = 0$. Note that

$$\mathrm{Supp}(M_Y/M_k) = \mathrm{Sing}(Y) \cup \{y_{l'+1} = 0\} \cup \cdots \cup \{y_{r'} = 0\}$$

around $\bar{y}$. Therefore, if $i \neq \sigma(j)$, then $\phi^*(y_i)|_{U_j} \neq 0$. Thus, we can see $q_i = q'_i = 0$ and $I_i(j) = I'_i(j) = 0$. Moreover, since $\mathcal{O}_{U_j,\bar{x}}$ is a UFD, considering $\phi^*(y_i)|_{U_j}$, we can see that

$$I_i = I'_i \quad \text{and} \quad u_i|_{U_j} = u'_i|_{U_j} .$$

Gathering the above observation, we get the following: For all $i = 1, \dots, r'$ and $j = 1, \dots, l$ with $i \neq \sigma(j)$,

$$\begin{cases} q_i = q'_i = 0 \,, \\ I_i(j) = I'_i(j) = 0 \,, \\ I_i = I'_i \,, \\ u_i|_{U_j} = u'_i|_{U_j} \,. \end{cases} \tag{4.1.6}$$

Let us see that for all $i > l'$,

$$q_i = q'_i = 0 \,, \quad u_i = u'_i \,, \quad I_i = I'_i \,.$$

Note that if $i > l'$, then $i \neq \sigma(j)$ for all $j = 1, \dots, l$. Thus, we get $q_i = q'_i = 0$ and $I_i = I'_i$. Moreover, $u_i|_{U_j} = u'_i|_{U_j}$ for all $j = 1, \dots, l$. Hence, $u_i = u'_i$. Therefore,

$$H(p'_i, 1) = H'(p'_i, 1) \quad \text{for all } \; i > l' \,. \tag{4.1.7}$$

First, we consider the case where $\sigma(1) = \cdots = \sigma(l) = s$. Then, for $i \neq s$,

$$q_i = q'_i = 0 \,, \quad I_i = I'_i \,.$$

Moreover, for all $j = 1, \dots, l$ and $i \neq s$, $u_i|_{U_j} = u'_i|_{U_j}$. Therefore, $u_i = u'_i$ for $i \neq s$. Thus, $H(p'_i, 1) = H'(p'_i, 1)$ for all $i \neq s$. On the other hand, we have the relation $p'_1 + \cdots + p'_{l'} = f'(q'_0) + \sum_{i>l'} b'_i p'_i$. Therefore, we have $H(p'_s, 1) = H'(p'_s, 1)$.

Hence, we may assume that $\#(\sigma(\{1, \cdots, l\})) \geq 2$. In this case, we can conclude that

$$q_i = q'_i = 0 \,, \quad I_i = I'_i$$

for all i. Moreover, $u_i|_{U_j} = u'_i|_{U_j}$ if $i \neq \sigma(j)$. Since $p'_1 + \cdots + p'_{l'} = f'(q'_0) + \sum_{i>l'} b'_i p'_i$,

$$H(p'_1 + \cdots + p'_{l'}, 1) = H'(p'_1 + \cdots + p'_{l'}, 1) \,.$$

Thus, considering the $\mathcal{O}^{\times}_{X,\bar{x}}$-factor, we find

$$u_1 \cdots u_{l'} = u'_1 \cdots u'_{l'} \,.$$

Moreover, if we set $S_i = \{1, \dots, l\} \setminus \sigma^{-1}(i)$, then $S_i \cup S_{i'} = \{1, \dots, l\}$ for all $i \neq i'$. Further, if we set $v_i = u_i/u'_i$, then

$$v_1 \cdots v_{l'} = 1 \quad \text{and} \quad v_i|_{U_j} = 1 \quad \text{for all } \; j \in S_i \quad \text{and } \; i = 1, \dots, l' \,.$$

Therefore, using the following Lemma 4.2, we have $v_i = 1$ for $i = 1, \dots, l'$. Hence, we can see $H(p'_i, 1) = H'(p'_i, 1)$ for $i = 1, \dots, l'$. □

LEMMA 4.2. *Let k be a field, $R = k[\![X_1, \dots, X_n]\!]/(X_1 \cdots X_l)$ and $\Lambda = \{1, \dots, l\}$. Let $\pi_j : R \to R/X_i R$ be the canonical homomorphism for $j \in \Lambda$. Let $S_1, \dots, S_s$ be subsets of Λ with $S_i \cup S_{i'} = \Lambda$ for $i \neq i'$. Moreover, let $u_1, \dots, u_s$ be units in R. If $u_1 \cdots u_s = 1$ and, for each i, $\pi_j(u_i) = 1$ for all $j \in S_i$, then $u_1 = \cdots = u_s = 1$.*

PROOF. If $S_{i_0} = \emptyset$ for some i_0, then $S_i = \Lambda$ for all $i \neq i_0$. Thus, $u_i = 1$ for all $i \neq i_0$, since

$$\pi_1 \times \cdots \times \pi_l : R \to R/X_1R \times \cdots \times R/X_lR$$

is injective. Then, $u_{i_0} = 1$. Therefore, we may assume that $S_i \neq \emptyset$ for all i.

For a monomial $X_1^{a_1} \cdots X_n^{a_n}$, the support with respect to Λ is given by

$$\mathrm{Supp}_\Lambda(X_1^{a_1} \cdots X_n^{a_n}) = \{i \in \Lambda \mid a_i > 0\}\,.$$

For a subset S of Λ, let Γ_S be the set of formal sums of monomials $X_1^{a_1} \cdots X_n^{a_n}$ with $\mathrm{Supp}_\Lambda(X_1^{a_1} \cdots X_n^{a_n}) = S$. Note that $\Gamma_\emptyset = k[\![X_{l+1}, \dots, X_n]\!]$. Then,

$$k[\![X_1, \dots, X_n]\!] = \bigoplus_{S \subseteq \Lambda} \Gamma_S\,.$$

Moreover, the natural map $\bigoplus_{S \subsetneq \Lambda} \Gamma_S \to R$ is an isomorphism as k-vector spaces. We denote the image of Γ_S in R by $\bar{\Gamma}_S$. For $f_S \in \bar{\Gamma}_S$ and $f_{S'} \in \bar{\Gamma}_{S'}$, $f_S \cdot f_{S'} \in \bar{\Gamma}_{S \cup S'}$ if $S \cup S' \subsetneq \Lambda$, and $f_S \cdot f_{S'} = 0$ if $S \cup S' = \Lambda$.

Here we set $u_i = \sum_{S \subsetneq \Lambda} f_{i,S}$, where $f_{i,S} \in \bar{\Gamma}_S$. Then, for all $j \in S_i$,

$$\pi_j(u_i) = \sum_{j \notin S \subsetneq \Lambda} f_{i,S} = 1\,.$$

Thus, $f_{i,\emptyset} = 1$ and $f_{i,S} = 0$ for all $S \neq \emptyset$ with $j \notin S$. Therefore, setting

$$\Delta_i = \{S \subsetneq \Lambda \mid S_i \subseteq S\}\,,$$

we can write

$$u_i = 1 + \sum_{S \in \Delta_i} f_{i,S}\,.$$

Since $S_i \cup S_{i'} = \Lambda$ $(i \neq i')$, for $S \in \Delta_i$ and $S' \in \Delta_{i'}$ with $i \neq i'$, we can easily see (1) $S \cup S' = \Lambda$ and (2) $S \neq S'$. Thus, using the above (1), we obtain

$$u_1 \cdots u_s = 1 + \sum_{i=1}^{s} \sum_{S \in \Delta_i} f_{i,S}\,.$$

Moreover, using the above (2), we can find $f_{i,S} = 0$. Thus, we get $u_i = 1$ for all i. □

REMARK 4.3. If we do not assume the condition

"$\phi(X') \not\subseteq \mathrm{Supp}(M_Y/M_k)$ for any irreducible component X' of X"

in Theorem 4.1, then the assertion of the theorem does not hold in general. For example, let us consider $\boldsymbol{A}_k^1 = \mathrm{Spec}(k[X])$. Let M be a log structure associated with $\alpha : \boldsymbol{N} \times \boldsymbol{N} \to k[X]$ given by

$$\alpha(a, b) = \begin{cases} X^b & \text{if } a = 0\,, \\ 0 & \text{if } a \neq 0\,. \end{cases}$$

Further, let $f : \boldsymbol{N} \to \boldsymbol{N} \times \boldsymbol{N}$ be a homomorphism defined by $f(a) = (a, 0)$. Then, $(\boldsymbol{A}_k^1, M)$ is log smooth and integral over $(\mathrm{Spec}(k), \boldsymbol{N} \times k^\times)$. Here let us consider a morphism ϕ :

$\boldsymbol{A}_k^1 \to \boldsymbol{A}_k^1$ induced by a homomorphism $\psi : k[X] \to k[X]$ given by $\psi(X) = 0$. Then, $\phi(\boldsymbol{A}_k^1) = \mathrm{Supp}(M/\boldsymbol{N} \times k^\times)$. Moreover, we consider a homomorphism

$$h : \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{N} \times \boldsymbol{N}$$

defined by $h(1,0) = (1,0)$ and $h(0,1) = (a_0, b_0)$ $(a_0 > 0)$. Then, it is easy to see that the following diagrams are commutative:

$$\begin{array}{ccc} \boldsymbol{N} \times \boldsymbol{N} & \xrightarrow{h} & \boldsymbol{N} \times \boldsymbol{N} \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f} \\ \boldsymbol{N} & = & \boldsymbol{N}, \end{array} \qquad \begin{array}{ccc} \boldsymbol{N} \times \boldsymbol{N} & \xrightarrow{h} & \boldsymbol{N} \times \boldsymbol{N} \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ k[X] & \xrightarrow{\psi} & k[X]\,. \end{array}$$

Thus, $(\phi, h) : (\boldsymbol{A}_k^1, M) \to (\boldsymbol{A}_k^1, M)$ is a log morphism over $(\mathrm{Spec}(k), \boldsymbol{N})$. On the other hand, we have infinitely many choices of a_0 and b_0.

5. Log differential sheaves on a semistable variety. Here, let us consider a log differential module on a semistable variety.

PROPOSITION 5.1. *Let k be an algebraically closed field and M_k a fine log structure on* $\mathrm{Spec}(k)$. *Let X be a semistable variety over k and M_X a fine log structure of X. Assume that (X, M_X) is log smooth and integral over* $(\mathrm{Spec}(k), M_k)$. *Let $\nu : \tilde{X} \to X$ be the normalization of X and $M_{\tilde{X}}$ the underlying log structure of $\nu^*(M_X)$, that is, $M_{\tilde{X}} = \nu^*(M_X)^u$ (cf. see Convention and terminology* 7). *Then, $(\tilde{X}, M_{\tilde{X}})$ is log smooth over* $(\mathrm{Spec}(k), k^\times)$ *and* $\Omega^1_{\tilde{X}}(\log(M_{\tilde{X}}/k^\times))$ *is isomorphic to* $\nu^*\Omega^1_X(\log(M_X/M_k))$.

PROOF. First of all, there is a fine sharp monoid Q with $M_k = Q \times k^\times$. Let $\alpha : M_X \to \mathcal{O}_X$ and $\alpha' : \nu^*(M_X) \to \mathcal{O}_{\tilde{X}}$ be the canonical homomorphisms. For a closed point $x \in \tilde{X}$, let $(\pi_Q : Q \to M_k, \pi_P : P \to M_{X,\overline{\nu(x)}}, f : Q \to P)$ be a good chart of $(X, M_X) \to (\mathrm{Spec}(k), M_k)$ at $\nu(x)$. Here we have three cases:

(A) $\nu(x)$ is a smooth point of X.

(B) $\nu(x)$ is a singular point of X and $f : Q \to P$ splits.

(C) $\nu(x)$ is a singular point of X and $f : Q \to P$ does not split.

CLAIM 5.1.1. $(\tilde{X}, M_{\tilde{X}}) \to (\mathrm{Spec}(k), k^\times)$ *is log smooth at x.*

CASE A. In this case, $\nu(x) = x$. Then, by Theorem 3.1, $P = f(Q) \times \boldsymbol{N}^r$. Let e_i be the i-th standard basis of $\boldsymbol{N}^r$ and $T_i = 1 \otimes e_i$ in $k \otimes_{k[Q]} k[P]$. Then, $k[T_1, \dots, T_r]_{(T_1,\dots,T_r)} \to \mathcal{O}_{X,\bar{x}}$ is smooth. Therefore, adding indeterminates $T_{r+1}, \dots, T_n$, we see that

$$h : k[T_1, \dots, T_r, T_{r+1}, \dots, T_n]_{(T_1,\dots,T_n)} \to \mathcal{O}_{X,\bar{x}}$$

is étale. Set $t_i = \alpha(\pi_P(e_i))$ for $i = 1, \dots, r$. Then, $t_1, \dots, t_r$ form a part of local parameters of $\mathcal{O}_{X,\bar{x}}$, since $h(T_i) = t_i$ for $i = 1, \dots, r$ and h is étale. Moreover, $M_{\tilde{X},\bar{x}}$ is generated by $t_1, \dots, t_r$ and $\mathcal{O}^\times_{X,\bar{x}}$. Thus, we get our assertion.

CASE B. In this case, by Theorem 3.1, $\mathrm{char}(k) \neq 2$, $P = f(Q) \times N$ and N is a monoid such that

$$k[N] = k[T_1, \dots, T_r]/(T_1^2 - T_2^2)\,.$$

Moreover, after adding indeterminates $T_{r+1}, \dots, T_{n+1}$,

$$h : k[T_1, \dots, T_r, T_{r+1}, \dots, T_{n+1}]_{(T_1,\dots,T_{n+1})}/(T_1^2 - T_2^2) \to \mathcal{O}_{X,\overline{\nu(x)}}$$

is étale. Set $t_i = \alpha(\pi_P(\bar{T}_i))$ for $i = 1, \dots, r$. Changing the sign of $\pi_P(\bar{T}_2)$, we may assume that $\tilde{X}$ at x is the component corresponding to $t_1 = t_2$. Note that $h(\bar{T}_i) = t_i$ for $i = 1, \dots, r$. Thus, $M_{\tilde{X},\bar{x}}$ is generated by $t_2, \dots, t_r$ and $\mathcal{O}_{X,\bar{x}}^{\times}$, and $t_2, \dots, t_r$ form a part of local parameters of $\mathcal{O}_{\tilde{X},\bar{x}}$. This shows the assertion.

CASE C. In this case, by Theorem 3.1, P is of semistable type

$$(r, l, p_1, \dots, p_r, q_0, c_{l+1}, \dots, c_r)$$

over Q. Then, we have

$$k \otimes_{k[Q]} k[P] \simeq k[T_1, \dots, T_r]/(T_1 \cdots T_l)$$

via the correspondence $1 \otimes p_i \longleftrightarrow T_i$. After adding indeterminates $T_{r+1}, \dots, T_{n+1}$, we have

$$k[T_1, \dots, T_r, T_{r+1}, \dots, T_{n+1}]_{(T_1,\dots,T_{n+1})}/(T_1 \cdots T_l) \to \mathcal{O}_{X,\overline{\nu(x)}}$$

is étale. We denote $\alpha(\pi_P(p_i))$ by t_i for $i = 1, \dots, r$. Renumbering $p_1, \dots, p_r$, we may assume that the component $\tilde{X}$ at x is given by $t_1 = 0$. Note that $h(\bar{T}_i) = t_i$ for $i = 1, \dots, r$. Thus, $M_{\tilde{X},\bar{x}}$ is generated by $t_2, \dots, t_r$ and $\mathcal{O}_{X,\bar{x}}^{\times}$, and $t_2, \dots, t_r$ form a part of local parameters of $\mathcal{O}_{\tilde{X},\bar{x}}$. Hence, we get our assertion.

Next we claim the following:

CLAIM 5.1.2. *For $a \in M_{\tilde{X},\bar{x}}$, there is $b \in \nu^*(M_X)_{\bar{x}}$ with $\alpha'(b) = a$. Moreover, $b \otimes 1$ is uniquely determined in $\nu^*(M_X)_{\bar{x}}^{\mathrm{gp}} \otimes_{\mathbf{Z}} \mathcal{O}_{\tilde{X},\bar{x}}$.*

The existence of b is obvious, so that we consider only a uniqueness of b. We use the same notation as in Claim 5.1.1 for each case.

CASE A. Set $a = u \cdot t_1^{a_1} \cdots t_r^{a_r}$ ($u \in \mathcal{O}_{X,\bar{x}}^{\times}$ and $a_1, \dots, a_r \in \mathbf{N}$). In order to see the uniqueness of b, we set $b = (f(q), b_1, \dots, b_r, v)$ ($q \in Q$, $b_1, \dots, b_r \in \mathbf{N}$ and $v \in \mathcal{O}_{X,\bar{x}}^{\times}$). Then, $\alpha'(b) = \beta(q) \cdot v \cdot t_1^{b_1} \cdots t_r^{b_r}$, where β is given by

$$\beta(q) = \begin{cases} 1 & \text{if } q = 0\,, \\ 0 & \text{if } q \neq 0\,. \end{cases}$$

Thus, $q = 0$, $v = u$ and $(b_1, \dots, b_r) = (a_1, \dots, a_r)$.

CASE B. We can set $a = u \cdot t_2^{a_2} \cdots t_r^{a_r}$ ($u \in \mathcal{O}_{\tilde{X},\bar{x}}^{\times}$ and $a_2, \dots, a_r \in \mathbf{N}$). Moreover, we set $b = (f(q), \bar{T}_1^{b_1} \cdot \bar{T}_2^{b_2} \cdots \bar{T}_r^{b_r}, v)$ ($q \in Q$, $b_1, \dots, b_r \in \mathbf{N}$ and $v \in \mathcal{O}_{\tilde{X},\bar{x}}^{\times}$). Then, $\alpha'(b) = \beta(q) \cdot v \cdot t_2^{b_1+b_2} \cdot t_3^{b_3} \cdots t_r^{b_r}$. Thus,

$$q = 0\,, \quad v = u\,, \quad a_2 = b_1 + b_2 \quad \text{and} \quad (b_3, \dots, b_r) = (a_3, \dots, a_r)\,.$$

Therefore, for $b' = (f(q'), \bar{T}_1^{b'_1} \cdot \bar{T}_2^{b'_2} \cdots \bar{T}_r^{b'_r}, v')$, if $\alpha'(b) = \alpha'(b') = a$, then

$$b = b' + (0, (\bar{T}_2/\bar{T}_1)^c, 1)$$

in $\nu^*(M_X)^{\mathrm{gp}}_{\bar{x}}$ for some $c \in \boldsymbol{Z}$. Here $\mathrm{char}(k) \neq 2$ and $(\bar{T}_2/\bar{T}_1)^2 = 1$. Hence, $b \otimes 1 = b' \otimes 1$ in $\nu^*(M_X)^{\mathrm{gp}}_{\bar{x}} \otimes_{\boldsymbol{Z}} \mathcal{O}_{\tilde{X},\bar{x}}$.

CASE C. Set $a = u \cdot t_2^{a_2} \cdots t_r^{a_r}$ ($u \in \mathcal{O}^{\times}_{\tilde{X},\bar{x}}$ and $a_2, \ldots, a_r \in \boldsymbol{N}$). Let us see the uniqueness of b. Let us set $b = (f(q) + \sum_{i=1}^r b_i p_i, v)$ ($q \in Q$, $b_1, \ldots, b_r \in \boldsymbol{N}$ and $v \in \mathcal{O}^{\times}_{\tilde{X},\bar{x}}$). Then, $\alpha'(b) = \beta(q) \cdot v \cdot t_1^{b_1} \cdots t_r^{b_r}$. Thus, $q = 0$, $v = u$, $b_1 = 0$ and $(b_2, \ldots, b_r) = (a_2, \ldots, a_r)$.

By Claim 5.1.2, there is a natural homomorphism

$$\gamma : \Omega^1_{\tilde{X}}(\log(M_{\tilde{X}}/k^{\times})) \to \Omega^1_{\tilde{X}}(\log(\nu^*(M_X)/M_k)) .$$

Moreover, we have a natural homomorphism

$$\gamma' : \nu^*(\Omega^1_X(\log(M_X/M_k))) \to \Omega^1_{\tilde{X}}(\log(\nu^*(M_X)/M_k)) .$$

CLAIM 5.1.3. *γ and γ' are isomorphisms.*

CASE A. In this case, γ' is an isomorphism around x. Set $t_j = h(T_j)$ for $j = r+1, \ldots, n$. Then, $d\log(t_1), \ldots, d\log(t_r), dt_{r+1}, \ldots, dt_n$ form a basis of $\Omega^1_{\tilde{X},\bar{x}}(\log(M_{\tilde{X}}/k^{\times}))$. Moreover, $d\log(e_1), \ldots, d\log(e_r), dt_{r+1}, \ldots, dt_n$ form a basis of $\Omega^1_{\tilde{X},\bar{x}}(\log(\nu^*(M_X)/M_k))$. On the other hand, $\gamma(d\log(t_i)) = d\log(e_i)$ for $i = 1, \ldots, r$ and $\gamma(dt_j) = dt_j$ for $j = r+1, \ldots, n$. Thus, γ is an isomorphism around x.

CASE B. Set $t_j = h(\bar{T}_j)$ for $j = r+1, \ldots, n+1$. Then,

$$d\log(t_2), \ldots, d\log(t_r), dt_{r+1}, \ldots, dt_{n+1}$$

form a basis of $\Omega^1_{\tilde{X},\bar{x}}(\log(M_{\tilde{X}}/k^{\times}))$. Moreover, $\gamma(d\log(t_i)) = d\log(\bar{T}_i)$ for $i = 2, \ldots, r$ and $\gamma(dt_j) = dt_j$ for $j = r+1, \ldots, n+1$. Let N' be the submonoid of N generated by $\bar{T}_2, \ldots, \bar{T}_r$. Then, we can see that $N^{\mathrm{gp}} = N'^{\mathrm{gp}} \times \langle \bar{T}_1/\bar{T}_2 \rangle$, $(\bar{T}_1/\bar{T}_2)^2 = 1$ and $N' \simeq \boldsymbol{N}^{r-1}$. Thus, if we set $M' = f(Q) \times N' \times \mathcal{O}^{\times}_{\tilde{X},\bar{x}}$, then the natural homomorphism

$$\Omega^1_{\tilde{X},\bar{x}}(\log(M'/M_k)) \to \Omega^1_{\tilde{X},\bar{x}}(\log(\nu^*(M_X)/M_k))$$

is an isomorphism because $\mathrm{char}(k) \neq 2$. Moreover, M' is log smooth over M_k. Therefore, $\Omega^1_{\tilde{X},\bar{x}}(\log(\nu^*(M_X)/M_k))$ is a free $\mathcal{O}_{\tilde{X},\bar{x}}$-module whose basis is given by

$$d\log(\bar{T}_2), \ldots, d\log(\bar{T}_r), d\log(t_{r+1}), \ldots, d\log(t_{n+1}) .$$

Thus, γ is an isomorphism. On the other hand, we can choose

$$d\log(\bar{T}_2), \ldots, d\log(\bar{T}_r), d\log(t_{r+1}), \ldots, d\log(t_{n+1})$$

as a basis of $\nu^*\Omega^1_X(\log(M_X/M_k))_{\bar{x}}$. Thus, γ' is also an isomorphism.

CASE C. Set $t_j = h(\bar{T}_j)$ for $j = r+1, \dots, n+1$. Then,

$$d\log(t_2), \dots, d\log(t_r), dt_{r+1}, \dots, dt_{n+1}$$

form a basis of $\Omega^1_{\tilde{X},\bar{x}}(\log(M_{\tilde{X}}/k^{\times}))$. Moreover, $\gamma(d\log(t_i)) = d\log(p_i)$ for $i = 2, \dots, r$ and $\gamma(dt_j) = dt_j$ for $j = r+1, \dots, n+1$. Let P' be the submonoid of P generated by $f(Q)$ and $p_2, \dots, p_r$. Then, since

$$p_1 = -(p_2 + \cdots + p_l) + f(q_0) + \sum_{i>l} c_i p_i\,,$$

we have $P'^{\mathrm{gp}} = P^{\mathrm{gp}}$. Thus, if we set $M' = P' \times \mathcal{O}^{\times}_{\tilde{X},\bar{x}}$, then the natural homomorphism

$$\Omega^1_{\tilde{X},\bar{x}}(\log(M'/M_k)) \to \Omega^1_{\tilde{X},\bar{x}}(\log(\nu^*(M_X)/M_k))$$

is an isomorphism. Moreover, since $P' = f(Q) \times \boldsymbol{N}^{r-1}$, we can see M' is log smooth over M_k. Therefore, $\Omega^1_{\tilde{X},\bar{x}}(\log(\nu^*(M_X)/M_k))$ is a free $\mathcal{O}_{\tilde{X},\bar{x}}$-module whose basis is given by

$$d\log(p_2), \dots, d\log(p_r), d\log(t_{r+1}), \dots, d\log(t_{n+1})\,.$$

Thus, γ is an isomorphism. On the other hand,

$$d\log(p_2), \dots, d\log(p_r), d\log(t_{r+1}), \dots, d\log(t_{n+1})$$

is a basis of $\nu^*\Omega^1_X(\log(M_X/M_k))_{\bar{x}}$. Thus, γ' is also an isomorphism. □

6. Geometric preliminaries.

6.1. Relative rational maps. Let k be an algebraically closed field, X and Y proper algebraic varieties over k, and T a reduced algebraic scheme over k. Let $\Phi : X \times_k T \dashrightarrow Y \times_k T$ be a relative rational map over T. Recall that this means that there is a dense open set U of $X \times_k T$ such that Φ is defined over U, $\Phi : U \to Y \times_k T$ is a morphism over T and for all $t \in T$, $U \cap (X \times \{t\}) \neq \emptyset$. In this subsection, we prove the following proposition.

PROPOSITION 6.1.1. *Let k, X, Y, T and $\Phi : U \to Y \times_k T$ be as above. Then the following holds.*

(1) *$\{t \in T \mid \Phi|_{X\times\{t\}}$ is dominant$\}$ is closed.*

(2) *$\{t \in T \mid \Phi|_{X\times\{t\}}$ is separably dominant$\}$ is locally closed.*

(3) *Assume that X is normal. Let D_X and D_Y be reduced divisors on X and Y, respectively. For a rational map $\phi : X \dashrightarrow Y$, we denote by X_ϕ the maximal open set over which ϕ is defined. Then,*

$$\{t \in T \mid (\Phi|_{X\times\{t\}})^{-1}(D_Y) \subseteq D_X \text{ on } X_{\Phi|_{X\times\{t\}}}\}$$

is constructible.

(4) *Let Z be a subvariety of Y. Then, $\{t \in T \mid \Phi|_{X\times\{t\}}(X) \subseteq Z\}$ is closed.*

(5) *Let $h : F \to G$ be a homomorphism of locally free sheaves on $X \times_k T$ such that $h_t : F_t \to G_t$ is not zero for every $t \in T$. Then,*

$$\{t \in T \mid \text{the image of } h_t : F_t \to G_t \text{ is rank one}\}$$

is closed.

PROOF. (1) Let Z be the closure of $\Phi(U)$ and $p : Z \to T$ the projection induced by $Y \times_k T \to T$. Since Z is proper over T, it is well known that the function $T \to \mathbf{Z}$ given by $t \mapsto \dim Z_t$ is upper semicontinuous. Moreover, $\dim Z_t \leq \dim Y$ and the equality holds if and only if $Z_t = Y$. Thus, we get (1).

(2) By virtue of (1), we may assume that $\Phi|_{X\times\{t\}}$ is dominant for all $t \in T$. In this case, we note that it is open. Indeed, this can be easily checked by Lemma 6.1.2 and the following fact: Let L be a finitely generated field over a field K. Then, by [8, 4.4.2], $\dim_L \Omega^1_{L/K} \geq \operatorname{tr.deg}_K(L)$ and the equality holds if and only if L is separable over K.

(3) First assume that T is normal. We may assume that U is maximal. Then, since $X \times_k T$ is normal, by applying Zariski main theorem to each fibre, we see that $\operatorname{codim}(X \times \{t\} \setminus U) \geq 2$ for all $t \in T$. Thus, $(\Phi|_{X\times\{t\}})^{-1}(D_Y) \subseteq D_X$ on $X_{\Phi|_{X\times\{t\}}}$ if and only if $(\Phi|_{(X\times\{t\})\cap U})^{-1}(D_Y) \subseteq D_X$. Here we set $W = \Phi^{-1}(D_Y \times_k T) \setminus D_X \times_k T$ on U. Let $q : W \to T$ be the projection induced by $X \times_k T \to T$. Then, $t \notin q(W)$ if and only if $(\Phi|_{(X\times\{t\})\cap U})^{-1}(D_Y) \subseteq D_X$, which proves our assertion by Chevalley's lemma.

Next we consider the general case. Let $\pi : \tilde{T} \to T$ be the normalization of T. Then,

$$\{t \in T \mid (\Phi|_{X\times\{t\}})^{-1}(D_Y) \subseteq D_X \text{ on } X_{\Phi|_{X\times\{t\}}}\}$$
$$= \pi(\{\tilde{t} \in \tilde{T} \mid (\Phi|_{X\times\{\tilde{t}\}})^{-1}(D_Y) \subseteq D_X \text{ on } X_{\Phi|_{X\times\{\tilde{t}\}}}\})\,.$$

Thus, we get (3).

(4) Let W be the Zariski closure of $\Phi^{-1}(Z \times_k T)$. Then, $\Phi|_{X\times\{t\}}(X) \subseteq Z$ if and only if $X \times \{t\} = W_t$. Since W is proper over T, it is well known that the function $T_1 \to \mathbf{Z}$ given by $t \mapsto \dim W_t$ is upper semicontinuous. Moreover, $\dim W_t \leq \dim X$ and the equality holds if and only if $W_t = X$. Thus, we obtain (4).

(5) Let K be the function field of X. Let us consider homomorphisms $F \otimes_k K \to G \otimes_k K$. Since $h_t \neq 0$ for all $t \in T$, we have (5) by Lemma 6.1.2. □

LEMMA 6.1.2. *Let $K[X_1, \dots, X_r]$ be the r-variable polynomial ring over a field K and k an algebraically closed subfield of K. Let I be an ideal of $k[X_1, \dots, X_r]$ and $A(X_1, \dots, X_r)$ an $n \times m$-matrix whose entries are elements of*

$$K[X_1, \dots, X_r]/IK[X_1, \dots, X_r]\,.$$

Then the function given by

$$k^r \supseteq V(I) \ni (t_1, \dots, t_r) \mapsto \operatorname{rk} A(t_1, \dots, t_r) \in \mathbf{Z}$$

is lower semi-continuous, where

$$V(I) = \{(x_1, \dots, x_r) \in k^r \mid f(x_1, \dots, x_r) = 0 \text{ for all } f \in I\}\,.$$

PROOF. Clearly, we may assume that $I = \{0\}$. Considering minors of the matrix $A(X_1, \dots, X_r)$, it is sufficient to see the following claim:

CLAIM 6.1.2.1. *For $f_1, \dots, f_l \in K[X_1, \dots, X_r]$, the set*

$$\{(x_1, \dots, x_r) \in k^r \mid f_1(x_1, \dots, x_r) = \cdots = f_l(x_1, \dots, x_r) = 0\}$$

is closed.

Replacing K by a field generated by coefficients of $f_1, \dots, f_l$ over k, we may assume that K is finitely generated over k. Since k is algebraically closed, K is separated over k. Thus, there are $T_1, \dots, T_s$ of K such that $T_1, \dots, T_s$ are algebraically independent over k and K is a finite separable extension over $k(T_1, \dots, T_s)$. By taking the Galois closure of K over $k(T_1, \dots, T_s)$, we may assume that K is a Galois extension over $k(T_1, \dots, T_s)$. For $f = \sum_I a_I X^I \in K[X_1, \dots, X_r]$ and $\sigma \in \mathrm{Gal}(K/k(T_1, \dots, T_s))$, we denote $\sum_I \sigma(a_I) X^I$ by f^σ. Here, we set

$$F_i = \prod_{\sigma \in \mathrm{Gal}(K/k(T_1,\dots,T_s))} f_i^\sigma$$

for $i = 1, \dots, l$. Then, $F_1, \dots, F_l \in k(T_1, \dots, T_l)[X_1, \dots, X_r]$ and, for $(x_1, \dots, x_r) \in k^r$,

$$F_i(x_1, \dots, x_r) = 0 \iff f_i(x_1, \dots, x_r) = 0$$

for $i = 1, \dots, l$. Indeed, if $F_i(x_1, \dots, x_r) = 0$, then $f_i^\sigma(x_1, \dots, x_r) = 0$ for some $\sigma \in \mathrm{Gal}(K/k(T_1, \dots, T_s))$, which implies that

$$0 = \sigma^{-1}(f_i^\sigma(x_1, \dots, x_r)) = f_i(x_1, \dots, x_r).$$

By the above observation, we may assume that $K = k(T_1, \dots, T_s)$. By multiplying some $\phi(T_1, \dots, T_r) \in k[T_1, \dots, T_s]$ to f_i, we may further assume that

$$f_1, \dots, f_l \in k[T_1, \dots, T_s][X_1, \dots, X_r].$$

We set

$$f_i = \sum_J c_{i,J} T^J \qquad (c_{i,J} \in k[X_1, \dots, X_r])$$

for $i = 1, \dots, l$. Then, for $(x_1, \dots, x_r) \in k^r$,

$$f_i(x_1, \dots, x_r) = 0 \iff c_{i,J}(x_1, \dots, x_r) = 0 \quad \text{for all } J.$$

Thus,

$$\begin{aligned} &\{(x_1, \dots, x_r) \in k^r \mid f_i(x_1, \dots, x_r) = 0 \ \text{for all } i\} \\ &\qquad\qquad = \{(x_1, \dots, x_r) \in k^r \mid c_{i,J}(x_1, \dots, x_r) = 0 \ \text{for all } i, J\}. \end{aligned}$$

Therefore, we get the claim. □

6.2. Geometric trick for finiteness. Let k be an algebraically closed field. Let X be a proper normal variety over k and Y a proper algebraic variety over k. Let E be a vector bundle on X and H a line bundle on Y. We assume that there is a dense open set Y_0 of Y such that $H^0(Y, H) \otimes_k \mathcal{O}_Y \to H$ is surjective over Y_0. Let $\phi : X \dashrightarrow Y$ be a dominant rational map over k. Let X_ϕ be the maximal open set of X over which ϕ is defined. We also assume that there is a non-trivial homomorphism $\theta : \phi^*(H) \to E|_{X_\phi}$. Then, since $\operatorname{codim}(X \setminus X_\phi) \geq 2$, we have a sequence of homomorphisms

$$H^0(Y, H) \to H^0(X_\phi, \phi^*(H)) \to H^0(X_\phi, E) = H^0(X, E)\,.$$

We denote the composition of the above homomorphisms by

$$\beta(\phi, \theta) : H^0(Y, H) \to H^0(X, E)\,.$$

Then we have the following.

LEMMA 6.2.1. *Let L be the image of*

$$H^0(Y, H) \otimes_k \mathcal{O}_X \xrightarrow{\beta(\phi,\theta)\otimes_k \mathrm{id}} H^0(X, E) \otimes_k \mathcal{O}_X \longrightarrow E\,.$$

Then the rank of L is one and the rational map

$$\phi' : X \dashrightarrow \boldsymbol{P}(H^0(Y, H))$$

induced by $H^0(Y, H) \otimes_k \mathcal{O}_X \to L$ is the composition of rational maps

$$X \overset{\phi}{\dashrightarrow} Y \overset{\phi_{|H|}}{\dashrightarrow} \boldsymbol{P}(H^0(Y, H))\,,$$

namely, $\phi' = \phi_{|H|} \cdot \phi$.

PROOF. Considering the following commutative diagram

$$\begin{tikzcd} H^0(Y, H) \otimes_k \mathcal{O}_{X_\phi} \arrow[r, "\beta(\phi,\theta)\otimes_k \mathrm{id}"] \arrow[d] & H^0(X, E) \otimes_k \mathcal{O}_{X_\phi} \arrow[d] \\ \phi^*(H) \arrow[r, "\theta"] & E|_{X_\phi}\,, \end{tikzcd}$$

we can see that θ gives rise to an isomorphism

$$\phi^*(H)|_{X_\phi \cap \phi^{-1}(Y_0)} \xrightarrow{\sim} L|_{X_\phi \cap \phi^{-1}(Y_0)}\,.$$

Moreover, the rational map $X_\phi \dashrightarrow \boldsymbol{P}(H^0(Y, H))$ given by $H^0(Y, H) \otimes_k \mathcal{O}_{X_\phi} \to \phi^*(H)$ is $\phi_{|H|} \cdot \phi$. Thus, the rational map $\phi' : X \dashrightarrow \boldsymbol{P}(H^0(Y, H))$ induced by $H^0(Y, H) \otimes_k \mathcal{O}_X \to L$ is nothing more than the composition of rational maps

$$X \overset{\phi}{\dashrightarrow} Y \overset{\phi_{|H|}}{\dashrightarrow} \boldsymbol{P}(H^0(Y, H))\,. \qquad \square$$

From now on, we assume that H is very big, that is, the morphism $Y_0 \to \boldsymbol{P}(H^0(Y, H))$ induced by $H^0(Y, H) \otimes_k \mathcal{O}_{Y_0} \to H|_{Y_0}$ is a birational morphism onto its image. Let $\mathcal{C}$ be a subset of $\mathrm{Rat}_k(X, Y)$ (the set of all rational maps of X into Y over k). We assume that for all $\phi \in \mathcal{C}$,

(1) ϕ is a dominant rational map, and

(2) we can attach a non-trivial homomorphism $\theta_\phi : \phi^*(H) \to E|_{X_\phi}$ to ϕ, where X_ϕ is the maximal Zariski open set of X over which ϕ is defined.

As before, we have a homomorphism

$$\beta(\phi, \theta_\phi) : H^0(Y, H) \to H^0(X, E)\,.$$

We denote the class of $\beta(\phi, \theta_\phi)$ in $\boldsymbol{P}(\mathrm{Hom}_k(H^0(Y, H), H^0(X, E))^\vee)$ by $\gamma(\phi, \theta_\phi)$. By abuse of notation, we often use $\gamma(\phi)$ instead of $\gamma(\phi, \theta_\phi)$.

LEMMA 6.2.2. *Let ϕ, ψ be rational maps in $\mathcal{C}$ and let $\theta_\phi : \phi^*(H) \to E|_{X_\phi}$ and $\theta_\psi : \psi^*(H) \to E|_{X_\psi}$ be non-trivial homomorphisms. Then $\gamma(\phi, \theta_\phi) = \gamma(\psi, \theta_\psi)$ implies $\phi = \psi$.*

PROOF. By our assumption, there is $a \in k^\times$ with $a\beta(\phi) = \beta(\psi)$. Hence we have the following commutative diagram:

$$\begin{array}{ccccc}
H^0(Y, H) \otimes_k \mathcal{O}_X & \xrightarrow{\beta(\phi,\theta_\phi)\otimes_k \mathrm{id}} & H^0(X, E) \otimes_k \mathcal{O}_X & \longrightarrow & E \\
\| & & \downarrow{\scriptstyle \times a} & & \downarrow{\scriptstyle \times a} \\
H^0(Y, H) \otimes_k \mathcal{O}_X & \xrightarrow{\beta(\psi,\theta_\psi)\otimes_k \mathrm{id}} & H^0(X, E) \otimes_k \mathcal{O}_X & \longrightarrow & E\,.
\end{array}$$

Let L_ϕ (resp. L_ψ) be the image of $H^0(Y, H) \otimes_k \mathcal{O}_X \to E$ in terms of $\beta(\phi, \theta_\phi)$ (resp. $\beta(\psi, \theta_\psi)$). Then, the above diagram gives rise to a commutative diagram

$$\begin{array}{ccc}
H^0(Y, H) \otimes_k \mathcal{O}_X & \longrightarrow & L_\phi \\
\| & & \downarrow{\scriptstyle \times a} \\
H^0(Y, H) \otimes_k \mathcal{O}_X & \longrightarrow & L_\psi\,.
\end{array}$$

Let $\phi' : X \dashrightarrow \boldsymbol{P}(H^0(Y, H))$ and $\psi' : X \dashrightarrow \boldsymbol{P}(H^0(Y, H))$ be the rational maps induced by $H^0(Y, H) \otimes_k \mathcal{O}_X \to L_\phi$ and $H^0(Y, H) \otimes_k \mathcal{O}_X \to L_\psi$, respectively. Then, by the above diagram, we can see $\phi' = \psi'$. Hence we get our lemma by Lemma 6.2.1. □

Next we show the following.

PROPOSITION 6.2.3. *Let T be a connected proper normal variety over k, and let*

$$\Phi : X \times_k T \dashrightarrow Y \times_k T$$

be a relative rational map over T (cf. Convention and terminology 8). Let $f : X \times_k T \to T$ and $g : Y \times_k T \to T$ be the projections to the second factor, respectively, and let $p : X \times_k T \to X$ and $q : Y \times_k T \to Y$ be the projections to the first factor, respectively. Assume that there exist a dense open subset T_0 of T and a non-trivial homomorphism $\Theta : \Phi^*(q^*(H)) \to p^*(E)|_U$ such that, for all $t \in T_0$, $\Phi|_{X\times\{t\}} \in \mathcal{C}$ and the class of $\beta(\Phi_t, \Theta_t)$ in $\boldsymbol{P}(\mathrm{Hom}_k(H^0(Y, H), H^0(X, E))^\vee)$ is $\gamma(\Phi_t, \Theta_t)$, where U is the maximal open set over which Φ is defined. Then there is $\phi \in \mathcal{C}$ such that $\Phi = \phi \times \mathrm{id}_T$.

PROOF. Since $X \times_k T$ is normal, we may assume that $\mathrm{codim}((X \times_k T) \setminus U) \geq 2$. Here we have a homomorphism

$$H^0(Y, H) \otimes_k \mathcal{O}_T = g_*(q^*(H)) \to (f|_U)_*(\Phi^*(q^*(H))) \xrightarrow{\Theta} (f|_U)_*(p^*(E))\,.$$

We claim that the natural homomorphism $f_*(p^*(E)) \to (f|_U)_*(p^*(E))$ is an isomorphism. Indeed, if W is an open set of T, then

$$(f|_U)_*(p^*(E))(W) = H^0(U \cap (X \times_k W), p^*(E)).$$

Note that $\operatorname{codim}((X \times_k W) \setminus U \cap (X \times_k W)) \geq 2$. Thus, $H^0(U \cap (X \times_k W), p^*(E)) = H^0(X \times_k W, p^*(E))$. Hence we get a homomorphism

$$\beta : H^0(Y, H) \otimes_k \mathcal{O}_T \to H^0(X, E) \otimes \mathcal{O}_T.$$

Here, T is proper and irreducible. Hence there is $\beta_0 \in \operatorname{Hom}_k(H^0(Y, H), H^0(X, E))$ such that $\beta = \beta_0 \otimes \mathrm{id}$. This means that $\beta(\Phi_t, \Theta_t) = \beta_0$. Thus, by Lemma 6.2.2, there is $\phi \in \mathcal{C}$ such that $\Phi_t = \phi$ for all $t \in T_0$. Therefore, we get our proposition. $\square$

Finally, let us see the following proposition.

PROPOSITION 6.2.4. *There exist a closed subset T of*

$$\boldsymbol{P}(\operatorname{Hom}_k(H^0(Y, H), H^0(X, E))^\vee)$$

and a relative rational map $\Phi : X \times_k T \dashrightarrow Y \times_k T$ over T such that if we consider $\gamma : \mathcal{C} \to \boldsymbol{P}(\operatorname{Hom}_k(H^0(Y, H), H^0(X, E))^\vee)$, then $\gamma(\mathcal{C}) \subseteq T$ and $\Phi|_{X \times \{\gamma(\phi)\}} = \phi$.

PROOF. We set $P = \boldsymbol{P}(\operatorname{Hom}_k(H^0(Y, H), H^0(X, E))^\vee)$. Then, there is the canonical homomorphism

$$\operatorname{Hom}_k(H^0(Y, H), H^0(X, E))^\vee \otimes_k \mathcal{O}_P \to \mathcal{O}_P(1),$$

which gives rise to a universal homomorphism

$$\beta : H^0(Y, H) \otimes_k \mathcal{O}_P(-1) \to H^0(X, E) \otimes_k \mathcal{O}_P,$$

that is, for all $t \in P$, the class of

$$\beta_t : H^0(Y, H) \otimes_k (\mathcal{O}_P(-1) \otimes \kappa(t)) \to H^0(X, E)$$

in P coincides with t, where $\kappa(t)$ is the residue field of $\mathcal{O}_P$ at t. Here we consider the composition of homomorphisms

$$h : H^0(Y, H) \otimes_k \mathcal{O}_P(-1) \otimes_k \mathcal{O}_X \xrightarrow{\beta \otimes \mathrm{id}} H^0(X, E) \otimes_k \mathcal{O}_P \otimes_k \mathcal{O}_X \to E \otimes_k \mathcal{O}_P$$

on $X \times_k P$. Then, by (5) of Proposition 6.1.1, if T_1 is the set of all $t \in P$ such that the image of h_t is of rank 1, then T_1 is closed. Let L be the image of

$$h|_{T_1} : H^0(Y, H) \otimes_k \mathcal{O}_{T_1}(-1) \otimes_k \mathcal{O}_X \to E \otimes_k \mathcal{O}_{T_1}.$$

Then we have the surjective homomorphism

$$H^0(Y, H) \otimes_k \mathcal{O}_{X \times_k T_1} \to L \otimes_{\mathcal{O}_{X \times_k T_1}} \mathcal{O}_{X \times_k T_1}(1).$$

Let U_1 be the maximal Zariski open set of $X \times_k T_1$ such that L is invertible over U_1. Here, note that for all $t \in T_1$, $U_1 \cap (X \times_k \{t_1\}) \neq \emptyset$. Thus, we get a relative rational map

$$\Phi_1 : X \times_k T_1 \dashrightarrow \boldsymbol{P}(H^0(Y, H)) \times_k T_1$$

over T_1 (cf. Convention and terminology 8). Let Y_1 be the closure of the image of $\phi_{|H|}(Y)$. By (4) of Proposition 6.1.1, the set

$$T = \{t \in T_1 \mid (\Phi_1)_t(X) \subseteq Y_1\}$$

is closed. Hence we obtain a relative rational map

$$\Phi_2 : X \times_k T \dashrightarrow Y_1 \times_k T ,$$

which gives rise to a relative rational map

$$\Phi : X \times_k T \dashrightarrow Y \times_k T .$$

By our construction, this rational map has the following properties: For all $t \in T$, let $\beta_t : H^0(Y, H) \to H^0(X, E)$ be the homomorphism modulo $k^\times$ corresponding to $t \in P$, and let L_t be the image of

$$H^0(Y, H) \otimes \mathcal{O}_X \to H^0(X, E) \otimes \mathcal{O}_X \to E .$$

Here, the rank of L_t is one. Thus, we have a rational map $\phi_t : X \dashrightarrow \boldsymbol{P}(H^0(Y, H))$ induced by $H^0(Y, H) \otimes \mathcal{O}_X \to L_t$. Then, $\phi_t(X) \subseteq Y_1$ and the following diagram is commutative:

$$\begin{tikzcd} X \arrow[rr, "\Phi|_{X\times\{t\}}"] \arrow[dr, "\phi_t"'] & & Y \arrow[dl, "\phi_{|H|}"] \\ & Y_1 . & \end{tikzcd}$$

Therefore, by Lemma 6.2.1, $\Phi : X \times_k T \dashrightarrow Y \times_k T$ is our desired relative rational map. $\square$

7. Finiteness theorem over the trivial log structure. Let k be an algebraically closed field and let X and Y be proper normal algebraic varieties over k. Let D_X and D_Y be reduced Weil divisors on X and Y, respectively. Let M_X and M_Y be fine log structures of X and Y, respectively, such that

$$M_X = j_{X*}(\mathcal{O}^\times_{X\setminus D_X}) \cap \mathcal{O}_X \quad \text{and} \quad M_Y \subseteq j_{Y*}(\mathcal{O}^\times_{Y\setminus D_Y}) \cap \mathcal{O}_Y ,$$

where j_X and j_Y are natural inclusion maps $X \setminus D_X \hookrightarrow X$ and $Y \setminus D_Y \hookrightarrow Y$, respectively. Then, for a rational map $\phi : X \dashrightarrow Y$, ϕ extends to $(X, M_X) \to (Y, M_Y)$ if $\phi^{-1}(D_Y) \subseteq D_X$. Throughout this section we assume that (X, M_X) and (Y, M_Y) are log smooth over $(\mathrm{Spec}(k), k^\times)$.

Note that if X is smooth over k, then the log smoothness of (X, M_X) over $(\mathrm{Spec}(k), k^\times)$ guarantees that $M_X = j_{X*}(\mathcal{O}^\times_{X\setminus D_X}) \cap \mathcal{O}_X$ for $D_X = \mathrm{Supp}(M_X/\mathcal{O}^\times_X)$ (cf. Theorem 3.1). In this case, if we assume further that Y is smooth and of general type over k, then the finiteness theorem (cf. Theorem 7.1) for such (X, M_X) and $(Y, \mathcal{O}_Y^{\times})$ follows from the finiteness theorem in [2].

Suppose that (Y, M_Y) is of log general type over $(\mathrm{Spec}(k), k^{\times})$, namely, $\det \Omega_Y^1(\log(M_Y/k^{\times}))$ is big. Then there is a positive integer m such that $\det \Omega_Y^1(\log(M_Y/k^{\times}))^{\otimes m}$ is very big. Here we set

$$H = \det \Omega_Y^1(\log(M_Y/k^{\times}))^{\otimes m} \quad \text{and} \quad E = \mathrm{Sym}^m(\textstyle\bigwedge^{\dim Y} \Omega_X^1(\log(M_X/k^{\times})))\,.$$

Then, if $\phi : (X, M_X) \dashrightarrow (Y, M_Y)$ is a rational map, we have a natural homomorphism

$$\theta_\phi : \phi^*(H) \to E|_{X_\phi}\,,$$

where X_ϕ is the maximal open set over which ϕ is defined. Moreover, if ϕ is separably dominant, then θ_ϕ is non-trivial. Let $\mathrm{SDRat}((X, M_X), (Y, M_Y))$ be the set of separably dominant rational maps $(X, M_X) \dashrightarrow (Y, M_Y)$ over $(\mathrm{Spec}(k), k^{\times})$.

THEOREM 7.1. $\mathrm{SDRat}((X, M_X), (Y, M_Y))$ *is finite.*

PROOF. First we need the following lemma.

LEMMA 7.2. *Let T be a smooth proper curve over k and $\Phi : X \times_k T \dashrightarrow Y \times_k T$ a relative rational map over T (cf. Convention and terminology 8). If there is a non-empty open set T_0 of T such that for all $t \in T_0$, Φ_t is separably dominant and $\Phi_t^{-1}(D_Y) \subseteq D_X$, then there is a rational map $\phi : X \dashrightarrow Y$ with $\Phi = \phi \times \mathrm{id}_T$.*

PROOF. First of all, by Proposition 6.1.1, for all $t \in T$, $\Phi|_{X\times\{t\}} : X \dashrightarrow Y$ is dominant. Let us take a effective divisor D on X such that

$$\Phi|_{X\times\{t\}}^{-1}(D_Y) \subseteq D_X \cup D$$

for all $t \in T \setminus T_0$. By using de-Jong's alteration [1], we see that there are a smooth proper variety X' and a separable and generically finite morphism $\mu : X' \to X$ such that $\mu^{-1}(D_X \cup D)$ is a normal crossing divisor on X'. Let $D_{X'} = \mu^{-1}(D_X \cup D)$ and $M_{X'} = j_{X'*}(\mathcal{O}_{X'\setminus D_{X'}}^{\times}) \cap \mathcal{O}_{X'}$, where $j_{X'} : X' \setminus D_{X'} \to X'$ is the natural inclusion map. Then, $(X', M_{X'})$ is log smooth over $(\mathrm{Spec}(k), k^{\times})$. We set $\Phi' = \Phi \cdot (\mu \times \mathrm{id}_T)$. Then, for all $t \in T$, $\Phi'|_{X\times\{t\}}^{-1}(D_Y) \subseteq D_{X'}$. Moreover, for all $t \in T_0$, $\Phi'|_{X\times\{t\}}$ is separably dominant. Thus, in order to prove our lemma, we may assume that for all $t \in T$, $\Phi|_{X\times\{t\}}^{-1}(D_Y) \subseteq D_X$.

Let $f : X \times_k T \to T$ and $g : Y \times_k T \to T$ be the projections to the second factor, respectively, and let $p : X \times_k T \to X$ and $q : Y \times_k T \to Y$ be the projections to the first factor, respectively. Let U be the maximal open set over which Φ is defined. Then we have a rational map $(X \times_k T, p^*(M_X)) \dashrightarrow (Y \times_k T, q^*(M_Y))$ and $(X \times_k T, p^*(M_X))$ and $(Y \times_k T, q^*(M_Y))$ are log smooth over $(T, \mathcal{O}_T^{\times})$. Thus, there is a non-trivial homomorphism

$$\Theta : \Phi^*(q^*(H)) \to p^*(E)|_U\,.$$

Therefore, we get our lemma by Proposition 6.2.3. □

Let us go back to the proof of Theorem 7.1. If $\phi \in \mathrm{SDRat}((X, M_X), (Y, M_Y))$, then we have the non-trivial homomorphism

$$\theta_\phi : \phi^*(H) \to E|_{X_\phi} .$$

Thus, by Proposition 6.2.4, there is a closed subset T of

$$\boldsymbol{P}(\mathrm{Hom}_k(H^0(Y, H), H^0(X, E))^\vee)$$

and a relative rational map $\Phi : X \times_k T \dashrightarrow Y \times_k T$ over T such that if we consider

$$\gamma : \mathrm{SDRat}((X, M_X), (Y, M_Y)) \to \boldsymbol{P}(\mathrm{Hom}_k(H^0(Y, H), H^0(X, E))^\vee) ,$$

then

$$\gamma(\mathrm{SDRat}((X, M_X), (Y, M_Y))) \subseteq T$$

and $\Phi|_{X\times\{\gamma(\phi)\}} = \phi$. Note that γ is injective by Lemma 6.2.2. Let T_1 be the set of all $t \in T$ such that $\Phi|_{X\times\{t\}}$ is separably dominant and $\Phi|_{X\times\{t\}}^{-1}(D_Y) \subseteq D_X$. Then, by Proposition 6.1.1, T_1 is constructible. Let T_2 be the Zariski closure of T_1. If $\dim T_2 = 0$, then we have done, so that we assume that $\dim T_2 > 0$. Then there is a proper smooth curve C and $\pi : C \to T_2$ such that the generic point of C goes to T_1 via π. Moreover, we have a rational map $\Psi : X \times_k C \dashrightarrow Y \times_k C$ induced by $X \times_k T_2 \dashrightarrow Y \times_k T_2$. By our construction, there is an open set C_0 of C such that for all $t \in C_0$, $\Psi|_{X\times_k C_0}$ is separably dominant and $\Psi|_{X\times\{t\}}^{-1}(D_Y) \subseteq D_X$. Thus, by Lemma 7.2, there is a rational map $\psi : X \dashrightarrow Y$ with $\Psi = \psi \times \mathrm{id}$. We choose $x_1, x_2 \in C$ with $\pi(x_1) \neq \pi(x_2)$ and $\pi(x_1), \pi(x_2) \in T_1$. Then we have $\phi_1, \phi_2 \in \mathrm{SDRat}((X, M_X), (Y, M_Y))$ with $\gamma(\phi_1) = \pi(x_1)$ and $\gamma(\phi_2) = \pi(x_2)$. Since γ is injective, $\phi_1 \neq \phi_2$. On the other hand, we have

$$\psi = \Psi|_{X\times_k\{x_i\}} = \Phi|_{X\times_k\{\pi(x_i)\}} = \phi_i$$

for each i. This is a contradiction. □

8. The proof of the finiteness theorem. In this section, let us consider the proof of the finiteness theorem in general.

THEOREM 8.1. *Let k be an algebraically closed field and M_k a fine log structure of* Spec(k). *Let X and Y be proper semistable varieties over k, and let M_X and M_Y be fine log structures of X and Y, respectively. Assume that (X, M_X) and (Y, M_Y) are integral and smooth over* (Spec(k), M_k). *If (Y, M_Y) is of log general type over* (Spec(k), M_k), *then the set of all separably dominant rational maps $(X, M_X) \dashrightarrow (Y, M_Y)$ over* (Spec(k), M_k) *defined in codimension one is finite* (*see Convention and terminology* 8).

PROOF. First we prove the following lemma.

LEMMA 8.2. *Let Y be a semistable variety over k and H a line bundle on Y. Let Y' be an irreducible component of the normalization of Y and $\mu : Y' \to Y$ the natural morphism. If H is big, then $\mu^*(H)$ is big.*

PROOF. Let m be a positive integer m such that $H^{\otimes m}$ is very big. Let V be the image of $H^0(Y, H^{\otimes m}) \to H^0(Y', \mu^*(H^{\otimes m}))$. Then, we have the following diagram:

$$\begin{tikzcd} Y' \arrow[r, "\mu"] \arrow[dr, dashed] \arrow[ddr, dashed] & Y \arrow[r, dashed] & \boldsymbol{P}(H^0(Y, H^{\otimes m})) \\ & & \boldsymbol{P}(V) \arrow[u, hook] \\ & \boldsymbol{P}(H^0(Y', f^*(H^{\otimes m}))) \arrow[ur, dashed] & \end{tikzcd}$$

Let Y_1 and Y_2 be the image of $Y' \dashrightarrow \boldsymbol{P}(V)$ and $Y' \dashrightarrow \boldsymbol{P}(H^0(Y', \mu^*(H^{\otimes m})))$ respectively. Then,

$$k(Y') = k(Y_1) \subseteq k(Y_2) \subseteq k(Y') .$$

Thus, $Y' \dashrightarrow Y_2$ is birational. □

Let us go back to the proof of Theorem 8.1. Let $X_1, \dots, X_r$ and $Y_1, \dots, Y_s$ be irreducible components of the normalizations of X and Y, respectively. Moreover, let $f_i : X_i \to X$ and $g_j : Y_j \to Y$ be the canonical morphisms. We set $M_{X_i} = f_i^*(M_X)^u$ and $M_{Y_j} = g_j^*(M_Y)^u$ (cf. Convention and terminology 7). Then, by Proposition 5.1, (X_i, M_{X_i}) and (Y_j, M_{Y_j}) are integral and log smooth over $(\mathrm{Spec}(k), k^\times)$. Further, by Proposition 5.1 again, we see that

$$\Omega^1_{X_i}(\log(M_{X_i})) = f_i^*(\Omega^1_X(\log(M_X/M_k)))$$

and

$$\Omega^1_{Y_j}(\log(M_{Y_j})) = g_j^*(\Omega^1_Y(\log(M_Y/M_k))) .$$

Thus, by the above lemma, (Y_j, M_{Y_j}) is of log general type over $(\mathrm{Spec}(k), k^\times)$ for every j. We denote the set of all separably dominant rational maps $(X, M_X) \dashrightarrow (Y, M_Y)$ defined in codimension one over $(\mathrm{Spec}(k), M_k)$ by

$$\mathrm{SDRat}((X, M_X), (Y, M_Y)) .$$

Moreover, the set of all separably dominant rational maps $(X_i, M_{X_i}) \dashrightarrow (Y_j, M_{Y_j})$ over $(\mathrm{Spec}(k), k^\times)$ is denoted by

$$\mathrm{SDRat}((X_i, M_{X_i}), (Y_j, M_{Y_j})) .$$

Then, we have a natural map

$$\Psi : \mathrm{SDRat}((X, M_X), (Y, M_Y)) \longrightarrow \coprod_{\sigma \in S(r,s)} \prod_{i=1}^{r} \mathrm{SDRat}((X_i, M_{X_i}), (Y_{\sigma(i)}, M_{Y_{\sigma(i)}}))$$

as follows. Here $S(r, s)$ is the set of all maps from $\{1, \dots, r\}$ to $\{1, \dots, s\}$. Let $(\phi, h) \in \mathrm{SDRat}((X, M_X), (Y, M_Y))$. Then, for each i, there is a unique $\sigma(i)$ such that the Zariski closure of $\phi(X_i)$ is $Y_{\sigma(i)}$. Then we have $(\phi|_{X_i}, h_i) : (X_i, M_{X_i}) \to (Y_{\sigma(i)}, M_{Y_{\sigma(i)}})$ (cf. Convention and terminology 7). By Theorem 7.1,

$$\mathrm{SDRat}((X_i, M_{X_i}), (Y_j, M_{Y_j}))$$

is finite for every i, j. Therefore, it is sufficient to see that Ψ is injective. Let us pick $(\phi, h), (\phi', h') \in \mathrm{SDRat}((X, M_X), (Y, M_Y))$ with $\Psi(\phi) = \Psi(\phi')$. Then, clearly, $\phi = \phi'$. Thus, by Theorem 4.1, we have $h = h'$. $\square$

Appendix. In this appendix, we recall several results, which are well-known for researchers of log geometry. It is however difficult to find literatures, so that for the reader's convenience, we prove them here. Actually, we consider two propositions concerning the existence of a good chart of a smooth log morphism (cf. [10]).

PROPOSITION A.1. *Let $(\phi, h) : (X, M_X) \to (Y, M_Y)$ be a morphism of log schemes with fine log structures. Let $x \in X$ and $y = \phi(x)$. Assume the following*:

(1) *The homomorphism $\bar{h}_x : \bar{M}_{Y,\bar{y}} \to \bar{M}_{X,\bar{x}}$ induced by $h_x : M_{Y,\bar{y}} \to M_{X,\bar{x}}$ is injective and the torsion part of* $\mathrm{Coker}(\bar{h}_x^{\mathrm{gp}} : \bar{M}_{Y,\bar{y}}^{\mathrm{gp}} \to \bar{M}_{X,\bar{x}}^{\mathrm{gp}})$ *is a finite group of order invertible in $\mathcal{O}_{X,\bar{x}}$.*

(2) *There is a splitting homomorphism $s_y : \bar{M}_{Y,\bar{y}} \to M_{Y,\bar{y}}$ of the natural homomorphism $p_y : M_{Y,\bar{y}} \to \bar{M}_{Y,\bar{y}}$, that is, $p_y \cdot s_y = \mathrm{id}_{\bar{M}_{Y,\bar{y}}}$.*

Then there is a splitting homomorphism $s_x : \bar{M}_{X,\bar{x}} \to M_{X,\bar{x}}$ of the natural homomorphism $p_x : M_{X,\bar{x}} \to \bar{M}_{X,\bar{x}}$ such that $p_x \cdot s_x = \mathrm{id}_{\bar{M}_{X,\bar{x}}}$ and the following diagram is commutative:

$$\begin{array}{ccc}
\bar{M}_{Y,\bar{y}} & \xrightarrow{\bar{h}_x} & \bar{M}_{X,\bar{x}} \\
{\scriptstyle s_x}\downarrow & & \downarrow{\scriptstyle s_y} \\
M_{Y,\bar{y}} & \xrightarrow{h_x} & M_{X,\bar{x}}\,.
\end{array}$$

PROOF. First of all, note that $\mathrm{Coker}(\mathcal{O}_{X,\bar{x}}^{\times} \to \phi^*(M_Y)_{\bar{x}}) = \bar{M}_{Y,\bar{y}}$. Moreover,

$$s'_y : \bar{M}_{Y,\bar{y}} \xrightarrow{s_y} M_{Y,\bar{y}} \to \phi^*(M_Y)_{\bar{x}}$$

gives rise to a splitting homomorphism of $\phi^*(M_Y)_{\bar{x}} \to \bar{M}_{Y,\bar{y}}$.

Let us consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{X,\bar{x}}^{\times} & \longrightarrow & \phi^*(M_Y)_{\bar{x}}^{\mathrm{gp}} & \longrightarrow & \bar{M}_{Y,\bar{y}}^{\mathrm{gp}} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{X,\bar{x}}^{\times} & \longrightarrow & M_{X,\bar{x}}^{\mathrm{gp}} & \longrightarrow & \bar{M}_{X,\bar{x}}^{\mathrm{gp}} & \longrightarrow & 0\,,
\end{array}$$

which gives rise to

$$\begin{tikzcd}
\mathrm{Hom}(\bar{M}^{\mathrm{gp}}_{X,\bar{x}}, M^{\mathrm{gp}}_{X,\bar{x}}) \arrow[r] \arrow[d] & \mathrm{Hom}(\bar{M}^{\mathrm{gp}}_{X,\bar{x}}, \bar{M}^{\mathrm{gp}}_{X,\bar{x}}) \arrow[r, "\delta_1"] \arrow[d, "\gamma_1"] & \mathrm{Ext}^1(\bar{M}^{\mathrm{gp}}_{X,\bar{x}}, \mathcal{O}^{\times}_{X,\bar{x}}) \arrow[d, "\lambda"] \\
\mathrm{Hom}(\bar{M}^{\mathrm{gp}}_{Y,\bar{y}}, M^{\mathrm{gp}}_{X,\bar{x}}) \arrow[r] & \mathrm{Hom}(\bar{M}^{\mathrm{gp}}_{Y,\bar{y}}, \bar{M}^{\mathrm{gp}}_{X,\bar{x}}) \arrow[r, "\delta_2"] & \mathrm{Ext}^1(\bar{M}^{\mathrm{gp}}_{Y,\bar{y}}, \mathcal{O}^{\times}_{X,\bar{x}}) \arrow[d, equal] \\
\mathrm{Hom}(\bar{M}^{\mathrm{gp}}_{Y,\bar{y}}, \phi^*(M_Y)^{\mathrm{gp}}_{\bar{x}}) \arrow[r] \arrow[u] & \mathrm{Hom}(\bar{M}^{\mathrm{gp}}_{Y,\bar{y}}, \bar{M}^{\mathrm{gp}}_{Y,\bar{y}}) \arrow[r, "\delta_3"] \arrow[u, "\gamma_2"] & \mathrm{Ext}^1(\bar{M}^{\mathrm{gp}}_{Y,\bar{y}}, \mathcal{O}^{\times}_{X,\bar{x}})\,.
\end{tikzcd}$$

By using the diagram

$$\begin{tikzcd}
\bar{M}^{\mathrm{gp}}_{Y,\bar{y}} \arrow[r, "\bar{h}^{\mathrm{gp}}_x"] \arrow[d, equal] & \bar{M}^{\mathrm{gp}}_{X,\bar{x}} \arrow[d, equal] \\
\bar{M}^{\mathrm{gp}}_{Y,\bar{y}} \arrow[r, "\bar{h}^{\mathrm{gp}}_x"] & \bar{M}^{\mathrm{gp}}_{X,\bar{x}}\,,
\end{tikzcd}$$

it follows that $\gamma_1(\mathrm{id}_{\bar{M}^{\mathrm{gp}}_{X,\bar{x}}}) = \bar{h}^{\mathrm{gp}}_x$ and $\gamma_2(\mathrm{id}_{\bar{M}^{\mathrm{gp}}_{Y,\bar{y}}}) = \bar{h}^{\mathrm{gp}}_x$. Note that the exact sequence

$$0 \to \mathcal{O}^{\times}_{X,\bar{x}} \to \phi^*(M_Y)^{\mathrm{gp}}_{\bar{x}} \to \bar{M}^{\mathrm{gp}}_{Y,\bar{y}} \to 0$$

splits by s'^{gp}_y. Thus,

$$\lambda(\delta_1(\mathrm{id}_{\bar{M}^{\mathrm{gp}}_{X,\bar{x}}})) = \delta_2(\gamma_1(\mathrm{id}_{\bar{M}^{\mathrm{gp}}_{X,\bar{x}}})) = \delta_2(\gamma_2(\mathrm{id}_{\bar{M}^{\mathrm{gp}}_{Y,\bar{y}}})) = \delta_3(\mathrm{id}_{\bar{M}^{\mathrm{gp}}_{Y,\bar{y}}}) = 0\,.$$

On the other hand, by our assumption, we have that

$$\mathrm{Ext}^1(\bar{M}_{X,\bar{x}}/\bar{M}_{Y,\bar{y}},\ \mathcal{O}_{X,\bar{x}}) = 0\,.$$

Thus, we obtain that λ is injective. Therefore, $\delta_1(\mathrm{id}_{\bar{M}^{\mathrm{gp}}_{X,\bar{x}}}) = 0$. Hence, we have a splitting homomorphism $s : \bar{M}^{\mathrm{gp}}_{X,\bar{x}} \to M^{\mathrm{gp}}_{X,\bar{x}}$ of $M^{\mathrm{gp}}_{X,\bar{x}} \to \bar{M}_{X,\bar{x}}$.

Here we claim that $s(\bar{M}_{X,\bar{x}}) \subseteq M_{X,\bar{x}}$. Indeed, let us choose $a \in \bar{M}_{X,\bar{x}}$. Then there is $b \in M_{X,\bar{x}}$ with $p_x(b) = a$. Since $p_x(s(a)) = a$, there is $c \in \mathcal{O}^{\times}_{X,\bar{x}}$ such that $s(a) = b + c$ in $M^{\mathrm{gp}}_{X,\bar{x}}$. Here $b, c \in M_{X,\bar{x}}$, which implies $s(a) \in M_{X,\bar{x}}$.

Therefore, we get a diagram

$$\begin{tikzcd}
\bar{M}_{Y,\bar{y}} \arrow[r, "\bar{h}_x"] \arrow[d, "s_y"'] & \bar{M}_{X,\bar{x}} \arrow[d, "s"] \\
M_{Y,\bar{y}} \arrow[r, "h_x"] & M_{X,\bar{x}}\,.
\end{tikzcd}$$

It should be noted that the above diagram is not necessarily commutative. By our assumption, for all $a \in \bar{M}_{Y,\bar{y}}$, there is a unique $u \in \mathcal{O}^{\times}_{X,\bar{x}}$ such that $s(\bar{h}_x(a)) + u = h_x(s_y(a))$. We denote this u by $\mu(a)$. Thus, we have a homomorphism $\mu^{\mathrm{gp}} : \bar{M}^{\mathrm{gp}}_{Y,\bar{y}} \to \mathcal{O}^{\times}_{X,\bar{x}}$. Here we consider an exact sequence

$$0 \to \bar{M}^{\mathrm{gp}}_{Y,\bar{y}} \to \bar{M}^{\mathrm{gp}}_{X,\bar{x}} \to \bar{M}^{\mathrm{gp}}_{X,\bar{x}}/\bar{M}^{\mathrm{gp}}_{Y,\bar{y}} \to 0\,,$$

which gives rise to

$$\mathrm{Hom}(\bar{M}^{\mathrm{gp}}_{X,\bar{x}}, \mathcal{O}^{\times}_{X,\bar{x}}) \to \mathrm{Hom}(\bar{M}^{\mathrm{gp}}_{Y,\bar{y}}, \mathcal{O}^{\times}_{X,\bar{x}}) \to \mathrm{Ext}^1(\bar{M}^{\mathrm{gp}}_{X,\bar{x}}/\bar{M}^{\mathrm{gp}}_{Y,\bar{y}}, \mathcal{O}^{\times}_{X,\bar{x}}) = \{0\}\,.$$

Thus, there is $\nu \in \mathrm{Hom}(\bar{M}^{\mathrm{gp}}_{X,\bar{x}}, \mathcal{O}^{\times}_{X,\bar{x}})$ with $\nu \cdot \bar{h}^{\mathrm{gp}}_x = \mu^{\mathrm{gp}}$. Here we set $s_x = s + \nu$. Then it holds that

$$s_x(\bar{h}_x(a)) = s(\bar{h}_x(a)) + \nu(\bar{h}_x(a)) = s(\bar{h}_x(a)) + \mu(a) = h_x(s_y(a))\,.$$

Thus, we get our desired s_x. □

PROPOSITION A.2. *Let $(\phi, h) : (X, M_X) \to (Y, M_Y)$ be a smooth morphism of log schemes with fine log structures. Fix $x \in X$ and $y = \phi(x)$. Assume that there are* (a) *étale neighborhoods U and V of x and y, respectively,* (b) *charts $\pi_P : P \to M_X|_U$ and $\pi_Q : Q \to M_Y|_V$, and* (c) *a homomorphism $f : Q \to P$ with the following properties*:

(1) $\phi(U) \subseteq V$,

(2) *The induced homomorphisms $P \to \bar{M}_{X,\bar{x}}$ and $Q \to \bar{M}_{Y,\bar{y}}$ are bijective.*

(3) *The following diagram is commutative*:

$$\begin{array}{ccc} Q & \xrightarrow{f} & P \\ {\scriptstyle \pi_Q}\big\downarrow & & \big\downarrow{\scriptstyle \pi_P} \\ M_Y|_V & \xrightarrow{h} & M_X|_U\,. \end{array}$$

Then the canonical morphism $g : X \to Y \times_{\mathrm{Spec}(\mathbf{Z}[Q])} \mathrm{Spec}(\mathbf{Z}[P])$ is smooth around x in the classical sense.

PROOF. We consider the natural homomorphism

$$\alpha : \mathrm{Coker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}}) \otimes_{\mathbf{Z}} \mathcal{O}_{X,\bar{x}} \to \Omega^1_{X/Y,\bar{x}}(\log(M_X/M_Y))\,.$$

First we note the following.

CLAIM A.2.1. *α is injective and gives rise to a direct summand of*

$$\Omega^1_{X/Y,\bar{x}}(\log(M_X/M_Y))\,.$$

In the same way as in [5, (3.13)], we can construct a chart $\pi_{P'} : P' \to M_{X,\bar{x}}$ and an injective homomorphism $f' : Q \to P'$ with the following properties:

(i) The torsion part of $\mathrm{Coker}(Q^{\mathrm{gp}} \to P'^{\mathrm{gp}})$ is a finite group of order invertible in $\mathcal{O}_{X,\bar{x}}$.

(ii) The following diagram is commutative:

$$\begin{array}{ccc} Q & \xrightarrow{f'} & P' \\ {\scriptstyle \pi_Q}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{P'}} \\ M_{Y,\bar{y}} & \longrightarrow & M_{X,\bar{x}}\,. \end{array}$$

(iii) The natural homomorphism

$$\alpha' : \mathrm{Coker}(Q^{\mathrm{gp}} \to P'^{\mathrm{gp}}) \otimes_{\mathbf{Z}} \mathcal{O}_{X,\bar{x}} \to \Omega^1_{X/Y,\bar{x}}(\log(M_X/M_Y))$$

is an isomorphism. Moreover, there are $t_1, \ldots, t_r \in P'$ such that a subgroup generated by $t_1, \ldots, t_r$ in $\mathrm{Coker}(Q^{\mathrm{gp}} \to P'^{\mathrm{gp}})$ is a free group of rank r and its index in $\mathrm{Coker}(Q^{\mathrm{gp}} \to P'^{\mathrm{gp}})$ is invertible in $\mathcal{O}_{X,\bar{x}}$. In particular,

$$d\log(\pi_{P'}(t_1)), \ldots, d\log(\pi_{P'}(t_r))$$

form a free basis of $\Omega^1_{X/Y,\bar{x}}(\log(M_X/M_Y))$.

Considering the commutative diagram

$$\begin{array}{ccccc} Q & \xrightarrow{\sim} & \bar{M}_{Y,\bar{y}} & \xleftarrow{\sim} & Q \\ {\scriptstyle f'}\downarrow & & {\scriptstyle \bar{h}_x}\downarrow & & \downarrow{\scriptstyle f} \\ P' & \longrightarrow & \bar{M}_{X,\bar{x}} & \xleftarrow{\sim} & P, \end{array}$$

we have a surjective homomorphism $\lambda : P' \to P$ with $\lambda \cdot f' = f$. Thus, we obtain the natural surjective homomorphism

$$\beta : \mathrm{Coker}(Q^{\mathrm{gp}} \to P'^{\mathrm{gp}}) \otimes_{\mathbf{Z}} \mathcal{O}_{X,\bar{x}} \to \mathrm{Coker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}}) \otimes_{\mathbf{Z}} \mathcal{O}_{X,\bar{x}}\,.$$

Hence we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Coker}(Q^{\mathrm{gp}} \to P'^{\mathrm{gp}}) \otimes_{\mathbf{Z}} \mathcal{O}_{X,\bar{x}} \arrow[r, "\sim", "\alpha'"'] \arrow[d, "\beta"] & \Omega^1_{X/Y,\bar{x}}(\log(M_X/M_Y)) \\ \mathrm{Coker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}}) \otimes_{\mathbf{Z}} \mathcal{O}_{X,\bar{x}}. \arrow[ur, "\alpha"'] & \end{tikzcd}$$

In order to see the claim, it is sufficient to see that $\gamma = \beta \cdot \alpha'^{-1} \cdot \alpha$ is an automorphism on $\mathrm{Coker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}}) \otimes_{\mathbf{Z}} \mathcal{O}_{X,\bar{x}}$, because $(\beta \cdot \alpha'^{-1}) \cdot (\alpha \cdot \gamma^{-1}) = \mathrm{id}$. Here we set $\pi_{P'}(t_i) = p_i u_i$ ($p_i \in P$, $u_i \in \mathcal{O}^{\times}_{X,\bar{x}}$) for $i = 1, \ldots, r$. Let us consider the natural surjective homomorphism

$$\theta : \Omega^1_{X/Y,\bar{x}}(\log(M_X/M_Y)) \otimes_{\mathbf{Z}} \kappa(\bar{x}) \to \mathrm{Coker}(\bar{M}^{\mathrm{gp}}_{Y,\bar{y}} \to \bar{M}^{\mathrm{gp}}_{X,\bar{x}}) \otimes_{\mathbf{Z}} \kappa(\bar{x}) \simeq \mathrm{Coker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}}) \otimes_{\mathbf{Z}} \kappa(\bar{x})$$

given by $d\log(a) \mapsto a \otimes 1$ as in [5, (3.13)]. This is nothing more than $(\beta \cdot \alpha'^{-1}) \otimes \kappa(\bar{x})$. Indeed, we see that

$$\begin{cases} (\beta \cdot \alpha'^{-1})(d\log(\pi_{P'}(t_i))) = \beta(t_i) = p_i\,, \\ \theta(d\log(\pi_{P'}(t_i))) = t_i = p_i \mod \mathcal{O}^{\times}_{X,\bar{x}}\,. \end{cases}$$

On the other hand, we have the natural map

$$\alpha \otimes \kappa(\bar{x}) : \mathrm{Coker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}}) \otimes_{\mathbf{Z}} \kappa(\bar{x}) \to \Omega^1_{X/Y,\bar{x}}(\log(M_X/M_Y)) \otimes_{\mathbf{Z}} \kappa(\bar{x})$$

given by $a \otimes 1 \mapsto d\log(a)$, which is a section of θ. Therefore, $\gamma \otimes \kappa(\bar{x}) = \mathrm{id}$. Thus, by Nakayama's lemma, γ is surjective, so that γ is an isomorphism by [7, Theorem 2.4].

Set $X' = Y \times_{\mathrm{Spec}(\mathbf{Z}[Q])} \mathrm{Spec}(\mathbf{Z}[P])$. Let $\psi : X' \to \mathrm{Spec}(\mathbf{Z}[P])$ be the canonical morphism and M_P the canonical log structure on $\mathrm{Spec}(\mathbf{Z}[P])$. Set $M_{X'} = \psi^*(M_P)$. Let o the origin of $\mathrm{Spec}(\mathbf{Z}[P])$ and $x' = (y, o) \in X'$. Then, $M_{X',\bar{x}'} = \mathcal{O}^\times_{X',\bar{x}'} \times P$. Here, $\Omega^1_{X'/Y,\bar{x}'}$ is generated by $\{d(1 \otimes x)\}_{x \in \mathbf{Z}[P]_{\bar{o}}}$. Thus, there is a natural surjective homomorphism

$$\mathrm{Coker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}}) \otimes_{\mathbf{Z}} \mathcal{O}_{X',\bar{x}'} \to \Omega^1_{X'/Y,\bar{x}'}(\log(M_{X'}/M_Y))\,.$$

Therefore, we have a surjective homomorphism

$$\mathrm{Coker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}}) \otimes_{\mathbf{Z}} \mathcal{O}_{X,\bar{x}} \to g^*(\Omega^1_{X'/Y,\bar{x}'}(\log(M_{X'}/M_Y)))\,.$$

Thus, by the claim,

$$g^*(\Omega^1_{X'/Y,\bar{x}'}(\log(M_{X'}/M_Y))) \to \Omega^1_{X/Y,\bar{x}}(\log(M_X/M_Y))$$

is injective and $g^*(\Omega^1_{X'/Y,\bar{x}'}(\log(M_{X'}/M_Y)))$ is a direct summand of

$$\Omega^1_{X/Y,\bar{x}}(\log(M_X/M_Y))\,.$$

Therefore, by [5, Proposition (3.12)], g is a smooth log morphism. Moreover, note that $g^*(M_{X'}) = M_X$. Thus, g is smooth in the classical sense. □

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