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ON PAIRS OF GEOMETRIC FOLIATIONS ON A CROSS-CAP

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Abstract. We obtain the topological configurations of the lines of curvature, the asymptotic and characteristic curves on a cross-cap, in the domain of a parametrisation of this surface as well as on the surface itself.

1. Introduction. Given a surface patch parametrised by $r : U \subset \boldsymbol{R}^2 \to \boldsymbol{R}^3$, Whitney showed that r can have a stable singularity under smooth changes of coordinates in the source and target. A local model of this singularity is given by $(x, y) \mapsto (x, xy, y^2)$. The image of this map is a singular surface called a cross-cap or a surface with a pinch-point. (The zero set of the function $ZX^2 - Y^2 = 0$ is the union of a cross-cap together with a "handle" and is called a Whitney umbrella.)

Because the cross-cap is a stable singular surface in $\boldsymbol{R}^3$, it is natural to seek to understand its differential geometry. Work in this direction was carried out in [12, 20, 22, 40]. In [12] (see also [40]) the flat geometry of the cross-cap is investigated using singularity theory. It is shown for instance in [12] that there are generically two types of cross-caps, one labelled hyperbolic cross-cap where all non-singular points of the immersed surface are hyperbolic, and the other labelled elliptic cross-cap where the parabolic set consists of two smooth curves meeting tangentially at the singularity and partitions the surface into hyperbolic and elliptic regions. This classification turned out to be very useful when seeking to understand the projections of smooth two dimensional surfaces in $\boldsymbol{R}^4$ to 3-spaces [31].

We study in this paper pairs of geometric foliations of a cross-cap. There are three classical pairs of foliations defined on a smooth oriented surface M in $\boldsymbol{R}^3$. These are the lines of curvature and the asymptotic and characteristic curves. A line of curvature of M is a curve whose tangent line at each point is parallel to a principal direction. The lines of curvature are defined everywhere on the surface and form an orthogonal net away from umbilic points. Their configurations at umbilics were drawn by Darboux, but a rigorous proof is given in [34] and [5] (see also [30] for related results). The study of the behaviour of these foliations in a neighbourhood of a closed orbit is also carried out in [34].

An asymptotic curve of M is a curve whose tangent line at each point is parallel to an asymptotic direction. The asymptotic curves are defined in the closure of the hyperbolic region of the surface. They form a family of cusps at a generic parabolic point. Their configurations

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at a cusp of Gauss are given in [1, 2, 29] and a more general approach for studying the singularities of their equation at such points is given in [14, 15, 30, 38]. Global properties of these foliations including the study of their cycles are given in [23].

Characteristic directions are defined in the closure of the elliptic region. At elliptic points there is a unique pair of conjugate directions for which the included angle is extremal ([19]). These directions are called the characteristic directions and their integral curves are called the characteristic curves. Their study is carried out independently in [11] and [21]. In [21] they are labelled harmonic mean curvature lines and are defined as curves along which the normal curvature is K/H, where K is the Gauss curvature and H is the mean curvature.

When the surface is given in a parametrised form, in the domain of the parametrisation, the above three foliations are the solution curves of some binary differential equations (BDEs), also called quadratic differential equations. BDEs are implicit differential equations that can be written, in a local chart, in the form

$$a(x, y)dy^2 + 2b(x, y)dxdy + c(x, y)dx^2 = 0\,, \tag{1}$$

where the coefficients a, b, c are smooth functions (here smooth means C^∞). A BDE defines no directions where $\delta = (b^2 - ac)(x, y) < 0$, two directions in the region where $\delta > 0$, and a double direction on the set $\Delta = \{\delta = 0\}$ provided that the coefficients of the equation do not all vanish at a given point. At such points, every direction is a solution. The set Δ is called the *discriminant* of the equation. BDEs are studied, using various approaches, in [5, 6, 8, 9, 10, 13–18, 25–28, 30, 35, 36, 38]. The solutions of (1) determine a pair of foliations $\mathcal{F}_i$, $i = 1, 2$, in the region $\delta > 0$. In this paper, the configuration of the solutions of (1) refere to the triple $\{\Delta, \mathcal{F}_1, \mathcal{F}_2\}$. In all the figures, we draw one foliation in black and the other in grey, and the discriminant in thick black.

In this paper we obtain the local topological configurations of the lines of curvature and of the asymptotic and characteristic curves of a cross-cap. We do this in two steps. Given a local parametrisation $r : \boldsymbol{R}^2, 0 \to \boldsymbol{R}^3, 0$ of the surface, we first obtain the configurations of the pairs of foliations in the domain. These are given by BDEs with coefficients all vanishing at the origin. We obtain in Section 3 a topological classification of BDEs with coefficients vanishing at the origin and whose discriminant has the same $\mathcal{K}$-singularities as those of the geometric foliations on the cross-cap. The topological models are obtained by extendending Guíñez's blowing-up technique [25, 26, 27] to cover the cases where the discriminant is not an isolated point.

Mapping the foliations to the surface is the second step. This is trivial for smooth surfaces as the parametrisation is a diffeomorphism from the domain to the image. However, this is not the case for the cross-cap. Here we need to analyse how the leaves of the foliations in the domain intersect the double point curve $\mathcal{D}$. There is an involution σ on $\mathcal{D}$ that interchanges points with the same image under r. We show in Section 2 that if a leaf intersects $\mathcal{D}$ in two points, then generically these are not mapped to the same point by r. This allows us to draw the pairs foliations on the cross-cap (Section 2).

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2. Classical BDEs on a cross-cap. Let $g : \boldsymbol{R}^2, 0 \to \boldsymbol{R}^3, 0$ be a germ of a smooth mapping. If we allow smooth changes of coordinates in the source and target (i.e., consider the action of the Mather group $\mathcal{A}$), then f has a local stable singularity if and only if it is $\mathcal{A}$-equivalent to $f(x, y) = (x, xy, y^2)$. We shall follow the notation in [12] and define a cross-cap as the image of any map-germ $r : \boldsymbol{R}^2, 0 \to \boldsymbol{R}^3, 0$ that is $\mathcal{A}$-equivalent to f, and say that r parametrises the cross-cap.

Given a smooth surface M in $\boldsymbol{R}^3$ with a family of normals N, we have a Gauss map $N : M \to S^2$. At a point p, the map $-dN(p) : T_pM \to T_{N(p)}S^2$ can be thought of as an automorphism of T_pM. This is the classical shape operator S_p, or simply S. If M is parametrised by $\boldsymbol{r}(x, y)$ with shape operator S, the coefficients of the first fundamental form $\mathrm{I}_p : T_pM \times T_pM \to \boldsymbol{R}$, with $\mathrm{I}_p(u, v) = u \cdot v$, are given by

$$E = \boldsymbol{r}_x \cdot \boldsymbol{r}_x\,, \quad F = \boldsymbol{r}_x \cdot \boldsymbol{r}_y\,, \quad G = \boldsymbol{r}_y \cdot \boldsymbol{r}_y\,.$$

Those of the second fundamental form $\mathrm{II}_p : T_pM \times T_pM \to \boldsymbol{R}$, with $\mathrm{II}_p(u, u) = S_p(u) \cdot v$, are given by

$$\begin{aligned} l &= S(\boldsymbol{r}_x) \cdot \boldsymbol{r}_x = N \cdot \boldsymbol{r}_{xx}\,, \\ m &= S(\boldsymbol{r}_x) \cdot \boldsymbol{r}_y = N \cdot \boldsymbol{r}_{xy}\,, \\ n &= S(\boldsymbol{r}_y) \cdot \boldsymbol{r}_y = N \cdot \boldsymbol{r}_{yy}\,. \end{aligned}$$

When considering the cross-cap singularity, we run into a problem as there is no well defined normal to the surface at the singular point. Away from the cross-cap point, the unit normal N is given by $N = r_x \times r_y / ||r_x \times r_y||$. However, the equations of the principal, asymptotic and characteristic directions are homogeneous in l, m, n (see below), so we can multiply them by an appropriate power of $||r_x \times r_y||$, alternatively, replace l, m, n respectively in their equations, by

$$l_1 = (r_x \times r_y).r_{xx}\,, \quad m_1 = (r_x \times r_y).r_{xy}\,, \quad n_1 = (r_x \times r_y).r_{yy}\,. \tag{2}$$

The flat differential geometry of the cross-cap (i.e., the geometry captured by its contact with lines and planes) is explored in [12] and [40], using singularity theory. It is shown there that the surface can locally be parametrised (after smooth changes of coordinates in the source and isometries in the target) by

$$r(x, y) = (x, xy + p(y), \lambda x^2 + \mu xy + y^2 + q(x, y))\,, \tag{3}$$

where $p(y)$ and $q(x, y)$ are germs of functions with zero 2-jets and λ, μ are constants. We shall write

$$\begin{aligned} j^4 p(y) &= p_3 y^3 + p_4 y^4\,, \\ j^3 q(x, y) &= q_{30} x^3 + q_{31} x^2 y + q_{32} x y^2 + q_{33} y^3\,, \end{aligned}$$

where the notation $j^k g$ means the k-jet of the map g, that is, its Taylor polynomial of order k at the origin. We use the above parametrisation of the cross-cap when seeking the configurations of the integral curves of the BDEs of interest.

REMARK 2.1. It is shown in [4] that the right framework for studying the singularities of the discriminant is via the action of some group $\mathcal{G}$ on families of symmetric matrices. A list

of all the $\mathcal{G}$-simple singularities of families of symmetric matrices is obtained in [4]. However, some of the singularities of the discriminant in this paper are not $\mathcal{G}$-simple. So we refere to these singularities by their $\mathcal{K}$-type. (See [41] for the singularity theory concepts.)

2.1. Asymptotic curves. The equation of the asymptotic directions of a smooth surface is given by

$$ndy^2 + 2mdydx + ldx^2 = 0\,.$$

For a cross-cap, we take the equation to be

$$n_1dy^2 + 2m_1dydx + l_1dx^2 = 0\,,$$

where n_1, l_1, m_1 are as in (2).

The configuration of asymptotic curves is affine invariant ([12]), so we can use affine changes of coordinates in the target and set the local parametrisation of the surface in the form

$$r(x, y) = (x, xy + p(y), y^2 + \varepsilon x^2 + q(x, y))\,, \quad \varepsilon = \pm 1\,.$$

The discriminant $\Delta = \delta^{-1}(0) = (m_1^2 - l_1n_1)^{-1}(0)$ is the parabolic set. The function δ has an A_1-singularity (i.e., it is $\mathcal{A}$-equivalent to $x^2 \pm y^2$). When $\varepsilon = -1$, the singularity is of type A_1^+ so the parabolic set is an isolated point. Then every non-singular point on the surface is hyperbolic, and the cross-cap is labelled *hyperbolic cross-cap* in [12]. In this case, West showed in [40] that the BDE of the asymptotic directions in the domain is topologically equivalent to $ydy^2 + 2xdxdy - ydx^2 = 0$ (Figure 1, left).

When $\varepsilon = +1$, the parabolic set has an A_1^--singularity in the domain (a pair of transverse curves). These are mapped to two smooth curves intersecting tangentially at the cross-cap point ([40]). This cross-cap is labelled parabolic cross-cap in [12]. We shall labell it here *elliptic cross-cap* and call, as in [31], a parabolic cross-cap the one whose discriminant has an A_2-singularity. (A change from an elliptic to a hyperbolic cross-cap occurs at a parabolic cross-cap.) The asymptotic curves are defined in the closure of the hyperbolic region. To determine their configurations, we proceed as follows.

The coefficients of the asymptotic BDE are given by

$$(a, b, c) = (x + M_1(x, y), -y + M_2(x, y), x + M_3(x, y))\,, \tag{4}$$

where M_i, $i = 1, 2, 3$, are smooth functions depending on $p(y)$ and $q(x, y)$ (in (3)), and

$$\begin{aligned} j^2M_1 &= q_{32}x^2 + 3q_{33}xy - 3p_3y^2\,,\\ j^2M_2 &= \frac{1}{2}q_{31}x^2 - \frac{3}{2}q_{33}y^2\,,\\ j^2M_3 &= 3q_{30}x^2 + q_{31}xy + 3p_3y^2\,. \end{aligned}$$

We can therefore apply the results in Section 3.1 and Theorem 3.1 to deduce the topological models of the asymptotic curves in the parameter space. (The change of variables $(x, y) \mapsto (y, x)$ is required to get the same normal forms as in Section 3.1.) The genericity conditions

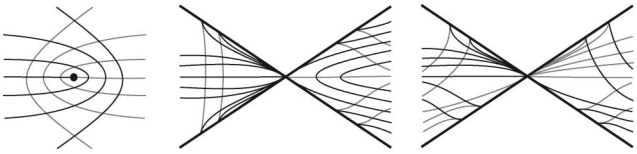

FIGURE 1. Configurations of the asymptotic curves in the domain: hyperbolic cross-cap left, elliptic cross-cap centre and right.

in Section 3.1 ($\Lambda_1 \neq 0$ and $\Lambda_2 \neq 0$) are now expressed in terms of the coefficients of the Taylor expansions of $p(y)$ and $q(x, y)$:

$$\Lambda_1 = -1/2(9q_{30} + 5q_{31} + q_{32} - 3q_{33} + 6p_3) \neq 0\,,$$
$$\Lambda_2 = 1/2(9q_{30} - 5q_{31} + q_{32} + 3q_{33} + 6p_3) \neq 0\,.$$

PROPOSITION 2.2. *The equation of the asymptotic curves in the domain of a parametrisation of a cross-cap is topologically equivalent to one of the following.*

1. *At a hyperbolic cross-cap* ([40]): $ydy^2 + 2xdxdy - ydx^2 = 0$ (*Figure* 1, *left*).
2. *At an elliptic cross-cap*:
   (i) $ydy^2 + 2(-x + y^2)dxdy + ydx^2 = 0$ (*Figure* 1, *centre*), *or*
   (ii) $ydy^2 + 2(-x + xy)dxdy + ydx^2 = 0$ (*Figure* 1, *right*).

The topological type is completely determined by the 3*-jet of the parametrisation of the surface.*

The configurations of the asymptotic curves in Figure 1 are in the parameter space. We need now to map them to the surface. When the surface is parametrised as in (3), the 3-jet, at the origin, of a parametrisation of the double point curve in the domain is given by $(-p_3y^2 - p_3(-\mu p_3 + q_{33})y^3, y)$. (We can take $\mu = 0$ when dealing with the asymptotic curves.) In particular, this curve is transverse to the two branches of the parabolic set when the later has an A_1^--singularity. We observe that the double point curve lives in the hyperbolic region of the surface (see [40]).

There is one separatrix in the case of a hyperbolic cross-cap and three at an elliptic cross-cap. (Here, a separatrix is a curve in the parameters space which is the blowing-down of a stable/unstable or centre manifold of the fields associated to the BDE in Section 3. This is an abuse of notation as these separatrices, in some cases, do not separate distinct sectors.) The 3-jet, at the origin, of a parametrisation of the unique separatrix at a hyperbolic cross-cap and of the separatrix transverse to the parabolic set at an elliptic cross-cap is given by $(-p_3y^2 + 1/5(3q_{33}p_3 - 8p_4)y^3, y)$. Therefore this separatrix and the double point curve have generically a 3-point contact at the origin. The image of the above separatrix under r has a cusp at the cross-cap point (see Figures 2 and 4).

Mapping the solution curves in the parameter space to the surface can be done without difficulties in the elliptic cross-cap case. In the parameter space, a solution curve of the asymptotic BDE intersects the double point curve in at most one point in a neighbourhood of

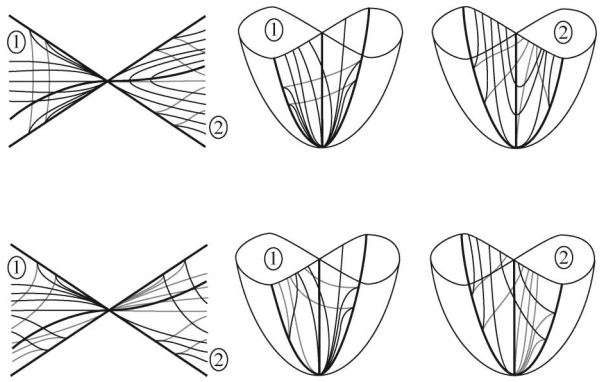

FIGURE 2. Configurations of the asymptotic curves at an elliptic cross-cap, in the domain left and on the surface right. The thick curves are the parabolic set and the double point curve.

the origin (Figure 2, left). One can then map, in the appropriate way, the configuration in each hyperbolic region in the domain to the surface, as shown in Figure 2.

PROPOSITION 2.3. *The configurations of the asymptotic curves at an elliptic cross-cap are as shown in Figure* 2.

A regular solution curve of the asymptotic BDE of a hyperbolic cross-cap intersects the double point curve at two points in a neighbourhood of the origin (Figure 3(a)). The question is whether or not these two points map to the same image on the surface.

There is an involution σ on the double point curve in the parameter space that interchanges two points with the same image on the surface. This involution is smooth in a neighbourhood of the origin.

The BDE of the asymptotic curves determines a pair of foliations $\mathcal{F}_i$, $i = 1, 2$, in the parameter space. In turn, each foliation determines an involution τ_i, $i = 1, 2$, on the double point curve which interchanges the two points of intersection of a leaf of the foliation with the double point curve. (We define $\tau_i(0) = 0$.)

The set $C_2^{\times 3}$ of germs of mappings $\boldsymbol{R}^2, 0 \to \boldsymbol{R}^3, 0$ is endowed with the Whitney topology. The subset $W_1 \subset C_2^{\times 3}$ of germs of parametrisations of hyperbolic cross-caps is given the induced topology.

THEOREM 2.4. *For an open and dense set of parametrisations of hyperbolic cross-caps, $\tau_i(p) \neq \sigma(p)$, $i = 1, 2$, for any point $p \neq (0, 0)$ on the double point curve in a neighbourhood of the origin. As a consequence, the configuration of the asymptotic curves at a hyperbolic cross-cap is as shown in Figure* 3.

PROOF. The equation of the asymptotic curves has a unique separatrix at a hyperbolic cross-cap (see for example [8]). This curve is smooth and for a surface parametrised as in (3),

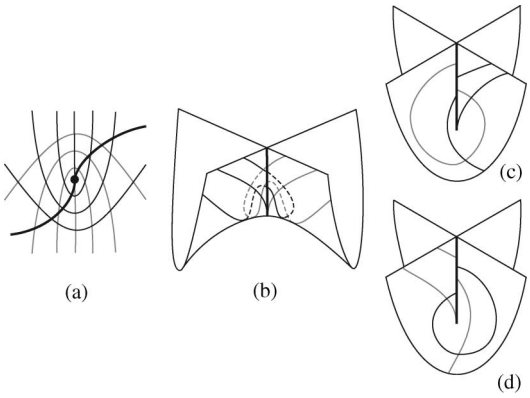

FIGURE 3. Configuration of the asymptotic curves at a hyperbolic cross-cap, in the domain (a) and on the surface (b), (c), (d) viewed from different directions. The thick curve is the double point curve.

it is given locally by the graph of a function $x = h(y)$. The function h satisfies the following identity

$$a(h(y), y) + 2b(h(y), y)h'(y) + c(h(y), y)h'(y)^2 \equiv 0\,,$$

where (a, b, c) are as in (4).

We seek changes coordinates in the form $x = X + f(Y)$, $y = Y$ so that the unique separatrix is along the Y-axis. The new BDE is given by

$$A(X, Y)dY^2 + 2B(X, Y)dXdY + C(X, Y)dX^2 = 0\,, \tag{5}$$

with

$$\begin{aligned} A(X, Y) &= a(X + f(Y), Y) + 2b(X + f(Y), Y)f'(Y) + c(X + f(Y), Y)f'(Y)^2\,,\\ B(X, Y) &= b(X + f(Y), Y) + c(X + f(Y), Y)f'(Y)\,,\\ C(X, Y) &= c(X + f(Y), Y)\,. \end{aligned}$$

The unique separatrix is along the Y-axis if and only if $A(0, Y) \equiv 0$. So we take $f(Y) = h(Y)$, with h as above. A calculation shows that $j^3 f(Y) = -p_3 Y^2 + 1/5(3q_{33}p_3 - 8p_4)Y^3$. In this new system of coordinates the double point curve is given by

$$X = \zeta(Y) = \frac{8}{5}(-q_{33}p_3 + p_4)Y^3 + \text{h.o.t}\,.$$

The horizontal direction is a solution of the BDE along a smooth curve $\mathcal{C}$ given by $C(X, Y) = 0$. A calculation shows that $\mathcal{C}$ is the graph of a function

$$X = -2p_3 Y^2 - \frac{3}{5}(q_{33}p_3 + 4p_4)Y^3 + \text{h.o.t}\,.$$

Given a point $(0,t)$ $(t \neq 0)$ on the Y-axis, there is a leaf of, say, $\mathcal{F}_1$ that passes transversally through $(0,t)$ for $t < 0$ and a smooth leaf of $\mathcal{F}_2$ that passes transversally through $(0,t)$ for $t > 0$. We shall consider only the foliation $\mathcal{F}_1$ as the approach is the same for $\mathcal{F}_2$.

Let γ_t denote the leaf of $\mathcal{F}_1$ passing through $(0,t)$, $t < 0$. This curve intersects the X-axis at two points. Denote by $U(t)$ the positive point.

The foliation $\mathcal{F}_1$ is given by the direction field parallel to the vector field $\xi_1 = a\partial/\partial x + (b + \sqrt{b^2 - ac})\partial/\partial y$. The polar blowing-up $x = \rho\cos\theta$, $y = \rho\sin\theta$ of ξ_1 yields a regular vector field η_1 for $(\theta,\rho) \in [-\pi/2, 0] \times [0, l)$, with l a small positive real number. So the map $k : -\pi/2 \times [0, l_1) \to 0 \times [0, l_2)$ determined by the flow of η_1 is smooth and $k'(0) \neq 0$ (here l_1 and l_2 are appropriately chosen small positive real numbers). Blowing-down yields $U(t) = k(t)$, so $U(t)$ depends smoothly on t and $U'(0) \neq 0$. Therefore $U(t) = t(u + L(t))$, for some non-zero scalar u and a smooth function L vanishing at $t = 0$.

In the new system of coordinates, the involution $\sigma = (\sigma_1, \sigma_2)$ takes a point $p_1 = (\zeta(Y), Y)$ to a $p_2 = (\sigma_1(p_1), \sigma_2(p_1))$, with $\sigma_2(p_1) = -Y(1 + \Psi(Y))$, for some smooth function Ψ vanishing at the origin. We want to show that $\sigma(p_1) \neq \tau_1(p_1)$ (Figure 4).

The double point curve intersect the leaf in consideration in two points p_1 and l (see Figure 4). For generic cross-caps, the double point curve has a genuine inflection at the origin ($p_4 - p_3 q_{33} \neq 0$). We assume, without loss of generality, that the double point curve is as in Figure 4, that is, $p_4 - p_3 q_{33} > 0$; the other case is similar.

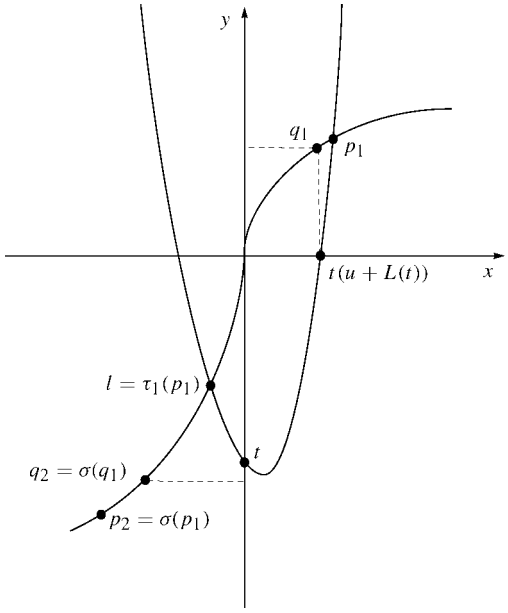

FIGURE 4. Involutions on the double point curve.

Consider the point $q_1 = (\zeta(Y_1), Y_1)$ on the double point curve with X-coordinate $U_1(t)$. Then $\sigma_2(q_1) > \sigma_2(p_1)$. We have $t(u + L(t)) = \zeta(Y_1)$ implies $Y_1 = t^{1/3}k(t)$, for some function k smooth off the origin, continuous at the origin and with $k(0) \neq 0$. (In the above setting, $k(0) < 0$, Figure 4.)

Now $\sigma_2(q_1) = -t^{1/3}k(t)(1 + \Psi(t^{1/3}k(t)))$, and therefore $\sigma_2(q_1) < t$ for t small. But as the graph of the leaf in consideration is strictly decreasing for $X < 0$, the Y-coordinate of $l = \tau_1(p_1)$ is bigger than t, hence l is distinct from q_2, and therefore l is distinct from p_2.

In the above calculations we assumed $p_3 \neq 0$ and $-q_{33}p_3 + p_4 \neq 0$ (we also need $\Lambda_1\Lambda_2 \neq 0$ for the topological models in the domain). So the subset of parametrisations of cross-caps satisfying these conditions is open and dense in W_1. □

2.2. Lines of curvature. The equation of the principal directions of a smooth surface is given by

$$(Fn - Gm)dy^2 + (En - Gl)dydx + (Em - Fl)dx^2 = 0\,.$$

For a cross-cap, we take the equation to be

$$(Fn_1 - Gm_1)dy^2 + (En_1 - Gl_1)dydx + (Em_1 - Fl_1)dx^2 = 0\,,$$

where n_1, l_1, m_1 are as in (2). When the surface is parametrised as in (3), the coefficients of the principal directions BDE at a cross-cap are given by

$$(a, b, c) = (M_1(x, y), x + M_2(x, y), -2y + M_3(x, y))\,,$$

where M_i, $i = 1, 2, 3$, are germs of smooth functions depending on p and q, with

$$\begin{aligned} j^3M_1 &= 4\lambda\mu x^3 + 4(1 + \mu^2 + 2\lambda)x^2y + 12\mu xy^2 + 8y^3\,,\\ j^2M_2 &= q_{32}x^2 + 3(q_{33} - \mu p_3)xy - 3p_3y^2\,,\\ j^2M_3 &= q_{31}x^2 + 3(\mu p_3 - q_{33})y^2\,. \end{aligned}$$

We can make changes of coordinates (see for example the proof of Proposition 3.2) in the source and write the 3-jet of the coefficients of the BDE in the form

$$(a, b, c) = \left(-4y^3, -\frac{1}{2}x + \frac{3}{2}p_3y^2 + \beta y^3, y\right)\,,$$

where β is a constant depending on the coefficients of the monomials in the 4-jet of the parametrisation of the surface. We can therefore use the results in Section 3.2 and Theorem 3.3 to deduce the following.

PROPOSITION 2.5. *The equation of the lines of curvature in the domain of a parametrisation of a cross-cap is topologically equivalent to*

$$-y^3dy^2 - xdxdy + ydx^2 = 0\,.$$

See Figure 5(*a*) *for illustration.*

REMARKS 2.6. 1. The result in Proposition 2.5 is also obtained in [22] by studying directly the equation of the lines of curvature.

2. Proposition 2.5 shows that, for any surface with a cross-cap point, the singularity of the BDE of its principal directions is locally an isolated point. Therefore, there is no sequence of umbilic points on the smooth part of the surface that converges to the cross-cap point. (I would like to thank Masaaki Umehara for asking the question that led to this remark.)

We investigate now how the configuration of lines of curvature in the domain is mapped to the cross-cap. We first observe that there are three separatrices in this case. When the surface is parametrised as in (3), one separatrix has a horizontal tangent and is given by $y = -(1/2)q_{31}x^2 +$ h.o.t. and the remaining two have a vertical tangent and are given by $x = \alpha_i y^2 +$h.o.t., $i = 1, 2$, where α_i are the roots of the quadratic equation $\alpha^2+3p_3\alpha-2 = 0$. These last two separatrices are tangent to the double point curve given by $x = -p_3y^2 +$ h.o.t. We observe that the double point curve is between the two separatrices.

The equation of the principal curves determines a pair of foliations $\mathcal{F}_i$, $i = 1, 2$, in the parameter space. In turn, each foliation determines an involution τ_i, $i = 1, 2$, on the double point curve which interchanges the two points of intersection of a smooth leave with the double point curve, see Figure 5 (a). (We define $\tau_i(0) = 0$.)

THEOREM 2.7. *We have $\tau_i(p) \neq \sigma(p)$, $i = 1, 2$, for $p \neq (0, 0)$ on the double point curve in a neighbourhood of the origin. As a consequence, the configuration of the lines of curvature on a cross-cap is as in Figure* 5.

PROOF. The double point curve is given by $x = h(y)$, for some smooth function h with a zero 1-jet. We re-parametrise the surface by taking $x = X + h(Y)$, $y = Y$. In the new coordinate system (that we still denote by (x, y)), the double point curve is along the y-axis. We denote by $(\bar{a}, \bar{b}, \bar{c})$ the coefficients of the lines of curvature BDE in this new coordinates

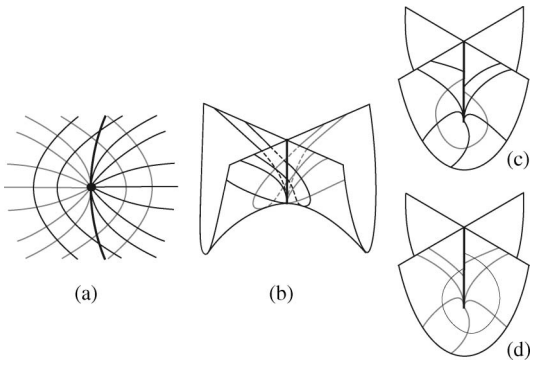

FIGURE 5. Configuration of the lines of curvature at a cross-cap, in the domain (a) and on the surface (b), (c), (d) viewed from different directions. The thick curve is the double point curve.

system. We have

$$j^2\bar{a}(x,y) = -4p_3xy\,,$$
$$j^2\bar{b}(x,y) = x + q_{32}x^2 + (-3\mu p_3 + 3q_{33})xy\,,$$
$$j^2\bar{c}(x,y) = -2y + q_{31}x^2 + (3\mu p_3 - 3q_{33})y^2\,.$$

Given a point $(0,t)$, $t \neq 0$, on the Y-axis, there is a smooth leaf say of $\mathcal{F}_1$ that passes through this point and another of $\mathcal{F}_2$. We shall consider the foliation $\mathcal{F}_1$ as the approach is the same for $\mathcal{F}_2$.

We consider the polar blowing up $x = \rho^2\cos\theta$, $y = \rho\sin\theta$ of the direction field parallel to $\xi_1 = \bar{a}\partial/\partial x + (\bar{b} + \sqrt{\bar{b}^2 - \bar{a}\bar{c}})\partial/\partial y$, which is tangent $\mathcal{F}_1$. The resulting field η_1 is regular for $(\theta,\rho) \in [-\pi/2,\pi/2]\times[0,l)$, with l a small positive real number. So the map $k : \pi/2 \times [0,l_1) \to -\pi/2 \times [0,l_2)$ determined by the flow of η_1 is smooth and $k'(0) \neq 0$. Blowing down yields $\tau_1(t) = -k(t)$.

The involution σ on the double point curve is given by $\sigma(t) = t(-1+\Psi(t))$ for some smooth function Ψ with $\Psi(0) = 0$. We shall show that $\tau_1'(0) \neq \sigma'(0)$.

We seek changes of coordinates of the form $x = X + Yf(X)$, $y = Y$, so that the direction determined ξ_1 is vertical on the X-axis. The new BDE is given by

$$A(X,Y)dY^2 + 2B(X,Y)dXdY + C(X,Y)dX^2 = 0\,, \tag{6}$$

with

$$A(X,Y) = \bar{a}(X+Yf(X),Y) + 2\bar{b}(X+Yf(X),Y)f(X) + \bar{c}(X+Yf(X),Y)f(X)^2\,,$$
$$B(X,Y) = (1+Yf'(X))(\bar{b}(X+Yf(X),Y) + f'(X)\bar{c}(X+Yf(X),Y))\,,$$
$$C(X,Y) = (1+Yf'(X))^2\bar{c}(X+Yf(X),Y)\,.$$

So we need the coefficient of dY^2 to vanish when $Y = 0$, that is,

$$A(X,0) + 2B(X,0)f(X) + C(X,0)f^2(X) \equiv 0\,.$$

We can factor out X, and since $(B(X,0)/X)(0,0) \neq 0$, it follows (by the implicit function theorem) that there exists a germ of a smooth function f that solves the above identity. A calculation shows that

$$j^3f(X) = -2\lambda\mu X^2 + \left(\left(\frac{1}{2}\mu^2 - 2\lambda + \frac{1}{2}\right)q_{31} - 3\mu q_{30}q_{31}\right)X^3\,.$$

We observe that the change of coordinates $x = X + Yf(X)$, $y = Y$ preserves the Y-axis, that is, the double point curve in the domain.

It follows from the above setting that the leaf of $\mathcal{F}_1$ through $(0,t)$ is the graph of a smooth function $X = G_t(Y)$, with say $G_t(0) = 0$. We can find the Taylor expansion of $G_t(Y)$ in Y for t fixed by substituting X and dX in the BDE (6). The coefficient α_1 and α_2 of $Y - t$ and

$(Y-t)^2$ respectively are smooth functions given in the form

$$\alpha_1(t) = t\big(-2\sqrt{p_3^2+1}+\tilde{\alpha_1}(t)\big)$$
$$\alpha_2(t) = -\frac{1}{2}\big(p_3+3\sqrt{p_3^2+1}\big)+\tilde{\beta}(t)$$

with $\tilde{\alpha_i}(0)=0$, $i=1,2$. We can show, by induction, that the coefficient of $(Y-t)^k$, for $k\geq 3$, in the Taylor expansion of G_t, is of the form $t^{2-k}\alpha_k(t)$ for some smooth function α_k.

The point $\tau_1(t)$ is the solution of $G_t(Y)=0$ which is distinct from $Y=t$, that is, the solution of

$$\alpha_1(t)+\alpha_2(t)(Y-t)+O_2(Y-t)=0\,.$$

Therefore $\tau_1'(0)=1-\alpha_1'(0)/\alpha_2(0)=(p_3-\sqrt{p_3^2+1})/(p_3+3\sqrt{p_3^2+1})$. We have $\tau_1'(0)\neq -1=\sigma'(0)$ and hence $\tau_1(t)\neq\sigma(t)$ for t near the origin and $t\neq 0$. □

2.3. Characteristic curves. The equation of the characteristic directions of a smooth surface is given by

$$\begin{aligned}&(2m(Gm-Fn)-n(Gl-En))dy^2\\&+2(m(Gl+En)-2Fln)dydx\\&+(l(Gl-En)-2m(Fl-Em))dx^2=0\,,\end{aligned}$$

(see for example [19, 7, 11]). These are the analogue of the asymptotic directions in the elliptic region. (The above BDE has no solutions in the hyperbolic region.) Characteristic/harmonic curves are studied in [19, 32, 33], and more recently in [7, 11, 21].

For a cross-cap, we take the equation to be

$$\begin{aligned}&(2m_1(Gm_1-Fn_1)-n_1(Gl_1-En_1))dy^2\\&+2(m_1(Gl_1+En_1)-2Fl_1n_1)dydx\\&+(l_1(Gl_1-En_1)-2m_1(Fl_1-Em_1))dx^2=0\,.\end{aligned}$$

where n_1, l_1, m_1 are as in (2). There are no characteristic curves on a hyperbolic cross-cap as all regular points are hyperbolic. We analyse the situation at an elliptic cross-cap.

When the surface is parametrised as in (3), the coefficients of the characteristic directions BDE at a cross-cap are given by

$$(a,b,c)=(x^2+A(x,y),-xy+B(x,y),-\lambda x^2+2y^2+C(x,y))\,,$$

where A, B, C are smooth functions depending on p and q (in (3)).

We can make changes of coordinates in the source and write the 4-jet of the coefficients (a,b,c) of the BDE in the form

$$(x^2+(36p_3^2+8)y^4,-xy+b_3(x,y)+b_4(x,y),-\lambda x^2+2y^2+c_1x^2y+c_2y^3+c_3xy^3+c_4y^4)\,,$$

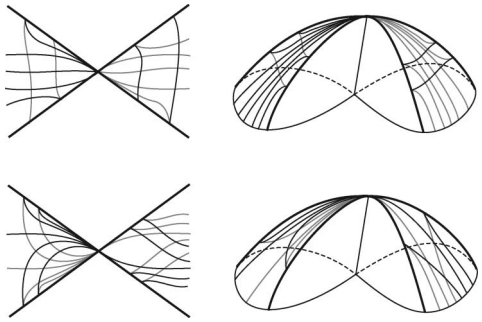

FIGURE 6. Configurations of the characteristic curves in the domain left and on the surface right at an elliptic cross-cap. The thick curves are the parabolic curves.

with

$$c_1 = -(-6\lambda^2 p_3 + 3\lambda q_{32} + 3q_{30})/\lambda\,,$$
$$c_2 = -(3\lambda q_{33} - 3\lambda\mu p_3 + 3q_{31})/\lambda\,,$$

and b_i, $i = 3, 4$, are homogeneous polynomial of degree i. The coefficients of y^3 in b_3 is given by $-6p_3$. So the discriminant has always an $X_{1,2}$-singularity (which is $\mathcal{K}$-equivalent to $x^4 + ax^2y^2 - y^6$ with $a < 0$).

The constants in Section 3.3 that determine the topological type of the BDE are $\Lambda_1 = -(1/2)(\lambda c_2 + \sqrt{\lambda}c_1)$ and $\Lambda_2 = -(1/2)(\lambda c_2 - \sqrt{\lambda}c_1)$ with c_1 and c_2 as above.

We can now use the results in Theorem 3.6 and deduce the following.

PROPOSITION 2.8. *The equation of the characteristic curves in the domain of a parametrisation of an elliptic cross-cap is topologically equivalent to one of the following normal forms.*

(i) $(x^2 + y^4, -xy, -x^2 + 2y^2 + y^3)$ *if* $\Lambda_1\Lambda_2 > 0$ (*Figure* 6, *top*), *or*

(ii) $(x^2 + y^4, -xy, -x^2 + 2y^2 + xy^2)$ *if* $\Lambda_1\Lambda_2 < 0$ (*Figure* 6, *bottom*).

The topological type is completely determined by the 3*-jet of the parametrisation of the surface.*

Mapping the solution curves in the domain to the surface can be done without difficulties in this case. As observed before, the double point curves lies in the hyperbolic region of the surface. So one only need to map each sector of the configuration in the parameter space to the surface in the appropriate way (as shown in Figure 6).

3. Topological normal forms of BDEs. The three foliations in the previous section are solution curves of binary differential equations (BDEs), also called quadratic differential equations. These are implicit differential equations that can be written, in a local chart, as in (1).

One approach for dealing with the qualitative study of BDEs that define at most two directions in the plane is given in [38] (see also [14]). It consists of lifting the bi-valued direction field defined in the plane to a single field ξ on the surface $\tilde{M} = F^{-1}(0)$ in $\boldsymbol{R}^3$. (The vector field ξ is determined by the restriction of the standard contact form $dy - pdx$ in $\boldsymbol{R}^3$ to the surface.)

When the coefficients a, b, c all vanish, say at the origin, all directions are solutions at this point. One way to proceed is given in [5] where the associated surface

$$M = \left\{(x, y, [\alpha : \beta]) \in \boldsymbol{R}^2, 0 \times \boldsymbol{R}P^1 \mid a\beta^2 + 2b\alpha\beta + c\alpha^2 = 0\right\}$$

to the BDE is considered. The discriminant function $\delta = b^2 - ac$ plays a key role. When δ has a Morse singularity the surface M is smooth and the projection $\pi : M \to \boldsymbol{R}^2, 0$ is a double cover of the set $\{(x, y) \mid \delta(x, y) > 0\}$ ([8]). The bi-valued direction field defined by the BDE lifts to a single field ξ on M and extends smoothly to $\pi^{-1}(0)$. Note that $0 \times \boldsymbol{R}P^1 \subset \pi^{-1}(\Delta)$ and is an integral curve of ξ.

Consider the affine chart $p = \beta/\alpha$ (we also consider the chart $q = \alpha/\beta$), and set

$$F(x, y, p) = a(x, y)p^2 + 2b(x, y)p + c(x, y)\,.$$

Then the lifted direction filed is parallel to the vector field

$$\xi = F_p \frac{\partial}{\partial x} + pF_p \frac{\partial}{\partial y} - (F_x + pF_y)\frac{\partial}{\partial p}\,.$$

If we write $j^1a = a_1x + a_2y$, $j^1b = b_1x + b_2y$, $j^1c = c_1x + c_2y$, the singularities of ξ on the exceptional fibre are given by the roots of the cubic

$$\begin{aligned}\phi(p) &= (F_x + pF_y)(0, 0, p)\\ &= a_2p^3 + (2b_2 + a_1)p^2 + (2b_1 + c_2)p + c_1\,.\end{aligned}$$

The eigenvalues of the linear part of ξ at a singularity are $-\phi'(p)$ and $\alpha_1(p)$ with

$$\alpha_1(p) = 2(a_2p^2 + (b_2 + a_1) + b_1)\,.$$

It is shown in [8] (see also [24]) that when M is smooth, we can change coordinates and write the 1-jet of the coefficients (a, b, c) of the form $(y, b_1x + b_2y, \pm y)$. There are special curves in the (b_1, b_2)-plane that bound open regions where the configuration of the BDE is topologically constant, and the models for these configurations are given in [8]. The topological configurations on generic points of these special curve are also determined in [36].

Another way to proceed in the study of BDEs is to consider a blowing-up of the singularity. This is done in [34] for the lines of curvature BDE on a smooth surface. Guíñez [25] used this technique on BDEs whose discriminant is an isolated point (labelled there positive quadratic equations). However, we show here and in [36] that Guíñez's method can be extended to cover general BDEs. We follow this approach to obtain topological models of BDEs whose discriminants have the same $\mathcal{K}$-singularity type as those of the asymptotic, characteristic and principal BDEs of the cross-cap.

Following the notation in [25], let ω denote the BDE with coefficients (a, b, c) and $f_i(w)$, $i = 1, 2$, the foliation associated to ω which is tangent to the vector field

$$\xi_i(\omega) = a\frac{\partial}{\partial u} + \big(-b + (-1)^i\sqrt{b^2 - ac}\big)\frac{\partial}{\partial v}\,.$$

If ψ is a diffeomorphism and $\lambda(x, y)$ is a non-vanishing real valued function, then ([25]) for $k = 1, 2$,

1. $\psi(f_k(w)) = f_k(\psi^*(\omega))$, if ψ is orientation preserving;
2. $\psi(f_k(w)) = f_{3-k}(\psi^*(\omega))$, if ψ is orientation reserving;
3. $f_k(\lambda w) = f_k(\omega)$, if $\lambda(x, y)$ is positive;
4. $f_k(\lambda w) = f_{3-k}(\omega)$, if $\lambda(x, y)$ is negative.

3.1. Discriminant with an A_1^--singularity. We study here BDEs with a discriminant having a Morse singularity at the origin and where the quadratic α_1 and the cubic ϕ have two common roots (see above for notation). The last condition is equivalent to two roots of ϕ being at the points of intersection of the lift of the branches of the discriminant with the exceptional fibre. (The case where α_1 and ϕ have one common root is dealt with in [36].)

When $j^1(a, b, c) = (y, b_1x + b_2y, y)$, α_1 and ϕ have a common root if and only if $b_1 = \pm b_2 - 1$. So we have two common roots when $(b_1, b_2) = (-1, 0)$. (At these points the lifted field ξ on M has generically a saddle-node singularity.) We write

$$\omega = (a, b, c) = (y + M_1(x, y), -x + M_2(x, y), y + M_3(x, y))\,,$$

where M_i are smooth functions with zero 1-jets at the origin. We set

$$\begin{aligned} A(x, y) &= j^2M_1 = a_0x^2 + a_1xy + a_2y^2\,,\\ B(x, y) &= j^2M_2 = b_0x^2 + b_1xy + b_2y^2\,,\\ C(x, y) &= j^2M_3 = c_0x^2 + c_1xy + c_2y^2\,. \end{aligned}$$

In order to obtain the configurations of the integral curves of these BDEs, we consider the directional blowing-up $x = u$, $y = uv$. (We also consider the blowing-up $x = uv$, $y = v$, but this does not yield extra information.) Then the new BDE is given by $\omega_0 = (u, v)^*\omega = \bar{a}dv^2 + 2\bar{b}dudv + \bar{c}du^2$ with

$$\begin{aligned} \bar{a} &= u^2(uv + M_1(u, uv))\,,\\ \bar{b} &= uv(uv + M_1(u, uv)) + u(-u + M_2(u, uv))\,,\\ \bar{c} &= v^2(uv + M_1(u, uv)) + 2v(-u + M_2(u, uv)) + uv + M_3(u, uv)\,. \end{aligned}$$

We can write $\omega_0 = u(u^2A_1, uB_1, C_1)$ with

$$\begin{aligned} A_1 &= v + uN_1(u, v)\\ B_1 &= v^2 - 1 + u(vN_1(u, v) + N_2(u, v))\\ C_1 &= v(v^2 - 1) + u(v^2N_1(u, v) + 2vN_2(u, v) + N_3(u, v)) \end{aligned}$$

and $M_i(u, uv) = u^2N_i(u, v)$, $i = 1, 2, 3$.

The quadratic form ω_1 with coefficients (u^2A_1, uB_1, C_1) is a product of two 1-forms, and to these 1-forms are associated the vector fields

$$X_i = u^2A_1\frac{\partial}{\partial u} + \big(-uB_1 + (-1)^i\sqrt{u^2(B_1^2 - A_1C_1)}\big)\frac{\partial}{\partial v}\,, \quad i = 1, 2\,.$$

We can also write ω_1 as the product of two 1-forms associated to the vector fields

$$Z_i = \big(-uB_1 + (-1)^i\sqrt{u^2(B_1^2 - A_1C_1)}\big)\frac{\partial}{\partial u} + C_1\frac{\partial}{\partial v}\,, \quad i = 1, 2\,.$$

We can factor out a term u in X_i and consider the vector fields

$$Y_i = uA_1\frac{\partial}{\partial u} + \big(-B_1 + (-1)^i\sqrt{B_1^2 - A_1C_1}\big)\frac{\partial}{\partial v}\,, \quad i = 1, 2\,.$$

The blowing-up transformation is orientation preserving if u is positive and orientation reserving if u is negative. As we have factored out u twice, it follows that Y_1 is tangent to the foliation associated to $f_1(w)$ if u is positive and to that associated to $f_2(w)$ if u is negative; while Y_2 is tangent to the foliation associated to $f_2(w)$ if u is positive and to that associated to $f_1(w)$ if u is negative (see [25] and the statement before Section 3.1).

We study the vector fields Y_i in a neighbourhood of the exceptional fibre $u = 0$. The fields Y_i are only defined in the regions where $B_1^2 - A_1C_1 \geq 0$. On $u = 0$, this means that

$$(v + 1)(v - 1) \leq 0\,.$$

(This is distinct from the cases treated by Guíñez, where Y_i are defined on the whole exceptional fibre.) We observe that the above segment of the exceptional fibre is an integral curve of both fields Y_i, $i = 1, 2$.

The singularities of Y_1 on $u = 0$ occur when $(-B_1 - \sqrt{B_1^2 - A_1C_1})(0, v) = 0$, that is, when $-(v^2 - 1) - \sqrt{1 - v^2} = 0$. Equivalently, when

$$\begin{cases} v^2(v^2 - 1) = v\phi(v) = 0 \quad \text{and} \\ v^2 - 1 \leq 0\,, \end{cases}$$

where ϕ is the cubic in Section 3. So Y_1 has singularities at $v = \pm 1$ and $v = 0$.

At $v = 0$, we have $B_1(0, 0) = -1 < 0$, so that

$$\begin{aligned} -B_1 - \sqrt{B_1^2 - A_1C_1} &= -B_1 + B_1\sqrt{1 - A_1C_1/B_1^2} \\ &= -\frac{A_1C_1}{2B_1} + A_1^2g(u, v) \end{aligned}$$

for some germ of a smooth function g with a zero 1-jet at the origin. Therefore Y_1 is singular along the curve $A_1(u, v) = 0$. We consider the vector field $\tilde{Y}_1 = Y_1/A_1$. Then $\tilde{Y}_1$ has a saddle singularity at the origin.

The singularities at $v = \pm 1$ occur at the points of intersection of the exceptional fibre with the branches of the blown-up discriminant. Consider the situation at $v = +1$. We change variables and set $s = u, t^2 = B_1^2 - A_1C_1$, with $t \geq 0$. The 2-jet of the vector field $(s, t)^*Y_1$

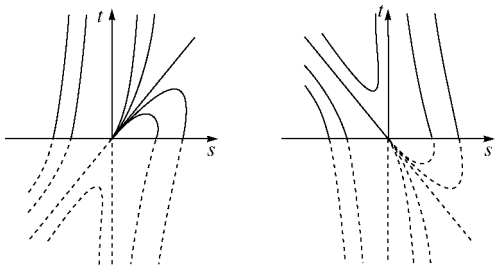

FIGURE 7. Integral curves of $(s,t)^*Y_i$ ($t \geq 0$), $i = 1$ left, and $i = 2$ right.

is equivalent to $(\Lambda_1 s + t)\partial/\partial t + st\partial/\partial s$, where Λ_1 is given by

$$\Lambda_1 = -\frac{1}{2}(A(1,1) + 4B(1,1) + 3C(1,1))\,.$$

The singularity of $(s,t)^*Y_1$ is a saddle-node provided $\Lambda_1 \neq 0$, and its integral curves (up to a reflection with respect to the vertical axis, depending on the sign of Λ_1) are as in Figure 7 left, and therefore those of Y_1 are as in Figure 8 top. Observe that the centre manifold of $(s,t)^*Y_1$ is transverse to the t-axis.

We proceed similarly at $v = -1$, change variables and set $s = u, t^2 = B_1^2 - A_1C_1$, with $t \geq 0$. The 2-jet of the vector field $(s,t)^*Y_1$ is equivalent to $(\Lambda_2 s + t)\partial/\partial t + st\partial/\partial s$, where Λ_2 is given by

$$\Lambda_2 = \frac{1}{2}(A(1,-1) - 4B(1,-1) + 3C(1,-1))\,.$$

The singularity of $(s,t)^*Y_1$ is a saddle-node provided $\Lambda_2 \neq 0$, and its integral curves (up to a reflection with respect to the vertical axis, depending on the sign of Λ_2) are as in Figure 7 left, and therefore those of Y_1 are as in Figure 8 top.

We have two possibilities (up to a reflection with respect to the vertical axis) for the configuration of the integral curves of Y_1 in a neighbourhood of the exceptional fibre depending on the sign of $\Lambda_1\Lambda_2$, see Figure 8 top.

The vector field Y_2 is singular on $u = 0$ when $(-B_1 + \sqrt{B_1^2 - A_1C_1})_{(0,v)=0}$, that is, when $v = \pm 1$. Similar calculations to those above for Y_1 show that the 2-jet of the vector field $(s,t)^*Y_2$ is equivalent to $(\Lambda_1 s - t)\partial/\partial t + st\partial/\partial s$ at $v = 1$ and $(\Lambda_2 s - t)\partial/\partial t + st\partial/\partial s$ at $v = -1$, with Λ_1 and Λ_2 as above. We observe that the configurations of the integral curves of $(s,t)^*Y_2$ can be deduced from those of $(s,t)^*Y_1$ by the change of variable $t \mapsto -t$. So at both points $v = \pm 1$, the integral curves of $(s,t)^*Y_1$ and $(s,t)^*Y_2$ are (up to a reflection of both figures with respect to the vertical axis) as in Figure 7 (left for $(s,t)^*Y_1$ and right for $(s,t)^*Y_2$). The configurations (two possibilities) of the integral curves of Y_2 are as in Figure 8 top. Blowing-down yields the configuration of the integral curves of the original BDE. Consequently, we have two distinct types of configurations depending on the sign of $\Lambda_1\Lambda_2$. One can show that any two configurations of the same type are topologically equivalent. This

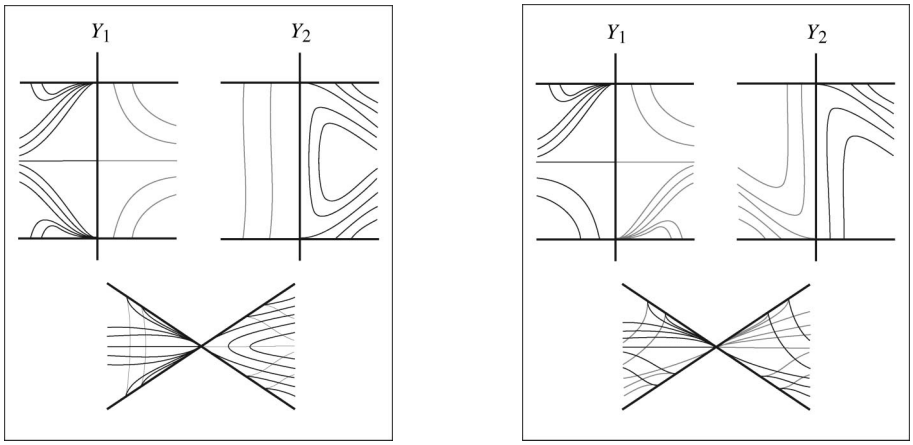

FIGURE 8. Configurations of the integral curves of the BDEs when α_1 and ϕ have two common roots and their blown up models: $\Lambda_1\Lambda_2 > 0$ left, and $\Lambda_1\Lambda_2 < 0$ right.

can be done by choosing an appropriate neighbourhood of the origin and by sliding along integral curves (see for example [6, 35, 37]).

We have thus the following result.

THEOREM 3.1. *Suppose that the quadratic α_1 and the cubic ϕ have two common roots and the lifted field ξ has genuine saddle-node singularities there. Then the BDE is topologically equivalent to one of the following normal forms.*

(i) $ydy^2 + 2(-x + y^2)dxdy + ydx^2 = 0$ (*Figure* 8, *left*), *or*

(ii) $ydy^2 + 2(-x + xy)dxdy + ydx^2 = 0$ (*Figure* 8, *right*).

The topological models are completely determined by the 2*-jets of the BDE.*

3.2. Discriminant with an A_3-singularity. We study here BDEs where the 1-jet of the coefficients is given by $(0, b_0x, y)$, $b_0 \neq 0$. We start by reducing the k-jet of the BDE $\omega = (a, b, c)$ to a normal form (see also [10]).

PROPOSITION 3.2. *The k-jet ($k \geq 2$) of a BDE ω with $j^1\omega = (0, b_0x, y)$ can be reduced, for b_0 distinct from a finite set of values, by smooth changes of coordinates and multiplication by a non-zero polynomial to*

$$a_k(y)dy^2 + 2(b_0x + b_k(y))dxdy + ydx^2,$$

where a_k and b_k are polynomials with zero 1*-jets.*

PROOF. We write $\omega = (a(x, y), b_0x + b(x, y), y + c(x, y))$, and make smooth changes of coordinates in the form

$$x = X + p(X, Y), \quad y = Y + q(X, Y),$$

where p and q are germs of homogeneous polynomials of degree k in X, Y. We also multiply the new BDE by $1 + r(X, Y)$, where r is a germ of homogeneous polynomial of degree $k-1$ in X, Y. The homogeneous part of degree k of the coefficients of the new BDE are given by

$$\begin{aligned} A_k &= a_k + 2b_0 X p_y \,, \\ B_k &= b_0 p + b_0 X(p_X + q_Y) + Y p_Y + b_0 X r + b_k \,, \\ C_k &= 2b_0 X q_X + 2Y p_X + q_k + Y r + c_k \,, \end{aligned}$$

where all the polynomials are evaluated at (X, Y). It is clear that we can eliminate all terms divisible by X in A_k by choosing an appropriate polynomial p. To reduce further B_k and C_k to the required forms, we need to show that the system in q and r

$$\begin{aligned} b_0 X q_Y + b_0 X r &= \bar{b}_k \,, \\ 2b_0 X q_X + q_k + Y r &= \bar{c}_k \end{aligned}$$

has a solution, where $\bar{b}_k$ is the polynomial $-(b_0 p + b_0 X p_X + Y p_Y + b_k)$ with the term Y^k removed and $\bar{c}_k = -(2Y p_X + c_k)$ (p chosen as above). The above system has a solution if a certain matrix has a non-zero determinant. This is the case if $b_0 \neq (i-1)/(2(k-i))$, $1 \leq i \leq k-1$. We observe that the system has always a solution when $b_0 = -1/2$, as is the case in Section 2.2. □

In Section 2.2 the discriminant has an A_3-singularity. So we can take $j^3\omega = (a_3 y^3, b_0 x + b_2 y^2 + b_3 y^3, y)$, with $a_3 \neq 0$. We write

$$\omega = (a_3 y^3 + M_1(x, y), b_0 x + b_2 y^2 + M_2(x, y), y + M_3(x, y)) \,,$$

where the germs M_1 and M_3 have zero 3-jets and M_2 has zero 2-jet.

We consider now the following quasi-homogeneous blowing-ups:

y-direction: $x = uv^2,\ y = v$,

x-direction: $x = u^2,\ y = uv$ and $x = -u^2,\ y = uv$, with $u \geq 0$.

The blowing-up in the y-direction does not give any singularities at the origin, and so it is enough to work with the blowing-ups in the x-direction.

The new BDE $\bar{\omega}$ obtained by considering the blowing-up $x = \varepsilon u^2,\ y = uv$ ($\varepsilon = \pm 1$) has coefficients

$$\begin{aligned} \bar{a} &= u^2 a(\varepsilon u^2, uv) \,, \\ \bar{b} &= uv a(\varepsilon u^2, uv) + 2\varepsilon u^2 b(\varepsilon u^2, uv) \,, \\ \bar{c} &= v^2 a(\varepsilon u^2, uv) + 4\varepsilon uv b(\varepsilon u^2, uv) + 4u^2 c(\varepsilon u^2, uv) \,. \end{aligned}$$

We write $\bar{\omega} = u^3(u^2 A_1, u B_1, C_1)$ with

$$\begin{aligned} A_1 &= a_3 v^3 + u N_1(u, v) \,, \\ B_1 &= 2b_0 + 2\varepsilon b_2 v^2 + a_3 v^4 + uv N_1(u, v) + 2\varepsilon u N_2(u, v) \,, \\ C_1 &= v(4(b_0 + 1) + 4\varepsilon b_2 v^2 + a_3 v^4) + uv^2 N_1(u, v) + 4\varepsilon uv N_2(u, v) + 4u^3 N_3(u, v) \,, \end{aligned}$$

where $M_i(\varepsilon u^2, v) = u^4 N_i(u, v)$ for $i = 1, 3$, and $M_2(\varepsilon u^2, v) = u^3 N_2(u, v)$.

The quadratic form $\omega_1 = (u^2A_1, uB_1, C_1)$ is a product of two 1-forms, and to these 1-forms are associated the vector fields

$$Z_i = \big(-uB_1 + (-1)^i \sqrt{u^2(B_1^2 - A_1C_1)}\,\big)\frac{\partial}{\partial u} + C_1\frac{\partial}{\partial v}\,, \quad i = 1, 2\,.$$

The blowing-up transformation is orientation preserving if $\varepsilon = +1$ and orientation reserving if $\varepsilon = -1$. Furthermore, as $u \geq 0$, it follows that Z_1 is tangent to the foliation associated to $f_1(w)$ if $\varepsilon = +1$ and to that associated to $f_2(w)$ if $\varepsilon = -1$; while Z_2 is tangent to the foliation associated to $f_2(w)$ if $\varepsilon = +1$ and to that associated to $f_1(w)$ if $\varepsilon = -1$ (see [25] and the statement before Section 3.1).

We can set $a_3 = \pm 1$ by a scalar change of coordinates and treat now the cases $a_3 = 1$ and $a_3 = -1$ separately.

(i) *The case* $j^3w = (-y^3, b_0x + b_2y^2 + b_3y^3, y)$. The vector fields Z_i are defined along the whole exceptional fibre. The number and type of their singularities along this fibre depend only on the pair (b_0, b_2). (Reflecting with respect to the origin shows that the type of the BDE associated to the pair $(b_0, -b_2)$ is the same as that associated to (b_0, b_2).) There are three curves in the (b_0, b_2)-plane where the number or the type of the singularities changes. These are the parabola $1 + b_0 + b_2^2 = 0$, and the lines $b_0 = -1$ and $b_0 = 0$ (see Figure 9, left). In the open regions determined by these curves, the configurations of the foliations associated to Z_i, $i = 1, 2$, are constant and are as in Figure 10. (The calculations are similar to those in the previous section and are omitted here.)

(ii) *The case* $j^3w = (y^3, b_0x + b_2y^2 + b_3y^3, y)$. Here the vector fields Z_i are defined in a neighbourhood of the exceptional fibre where $B_1^2 - A_1C_1 \geq 0$. On the criminant curve $B_1^2 - A_1C_1 = 0$, one can show that the integral curves of Z_i form a family of regular curves ending transversally at this curve (the exceptional fibre being a common integral curve of both fields).

The number and type of the singularities of Z_i on the exceptional fibre depend only on the pair (b_0, b_2). There are five curves in the (b_0, b_2)-plane where the number or type of

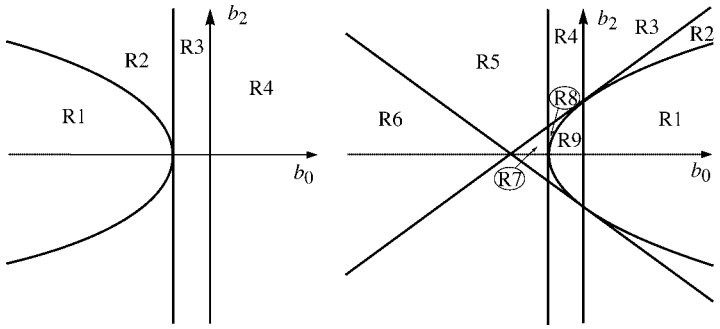

FIGURE 9. Partition of the (b_0, b_2)-plane, $\varepsilon = -1$ left and $\varepsilon = +1$ right.

singularity changes. These are: the parabola $1 + b_0 - b_2^2 = 0$, and the lines $b_0 = -1$, $b_0 = 0$, $2 + b_0 - 2b_2 = 0$, $2 + b_0 + 2b_2 = 0$ (see Figure 9, right). In the open regions determined by these curves, the configurations of the foliations associated to Z_i, $i = 1, 2$, are constant and are as in Figure 11.

We have then the following result.

THEOREM 3.3. *Suppose that* $j^3\omega = (\pm y^3, b_0x + b_2y^2 + b_3y^3, y)$. *Then the topological type of the BDE* ω *is constant in the open regions in the* (b_0, b_2)*-plane in Figure* 9. *The configurations of the integral curves of the BDE in these regions are as shown in Figures* 10 *and* 11.

REMARK 3.4. Normal forms for the BDEs in Theorem 3.3 can be taken in the form $(\pm y^3, b_0x + b_2y^2, y)$ with (b_0, b_2) any fixed value in the open regions in Figure 9.

3.3. $X_{1,2}$-singularity. We study in this section certain BDEs with a discriminant having an $X_{1,2}$-singularity $\mathcal{K}$-equivalent to $x^4 + \lambda x^2y^2 - y^6$ with $\lambda < 0$. The characteristic BDE in Section 2.3 has a 2-jet $(x^2, -xy, -ax^2 + 2y^2)$, $a > 0$.

PROPOSITION 3.5. *The* 4*-jet of a BDE with a* 2*-jet* $(x^2, -xy, -ax^2 + 2y^2)$ *and whose discriminant has an* $X_{1,2}$*-singularity can be reduced, by smooth changes of coordinates when*

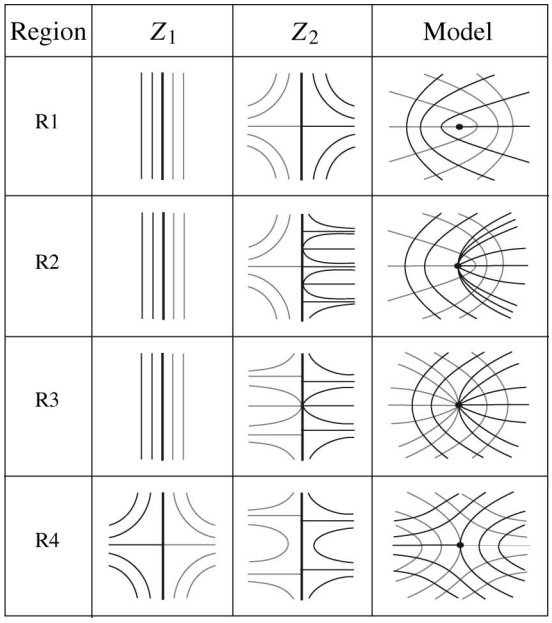

FIGURE 10. Configurations of the integral curves of $(-y^3, b_0x + b_2y^2 + b_3y^3, y)$ and their associated directional blowing-up models.

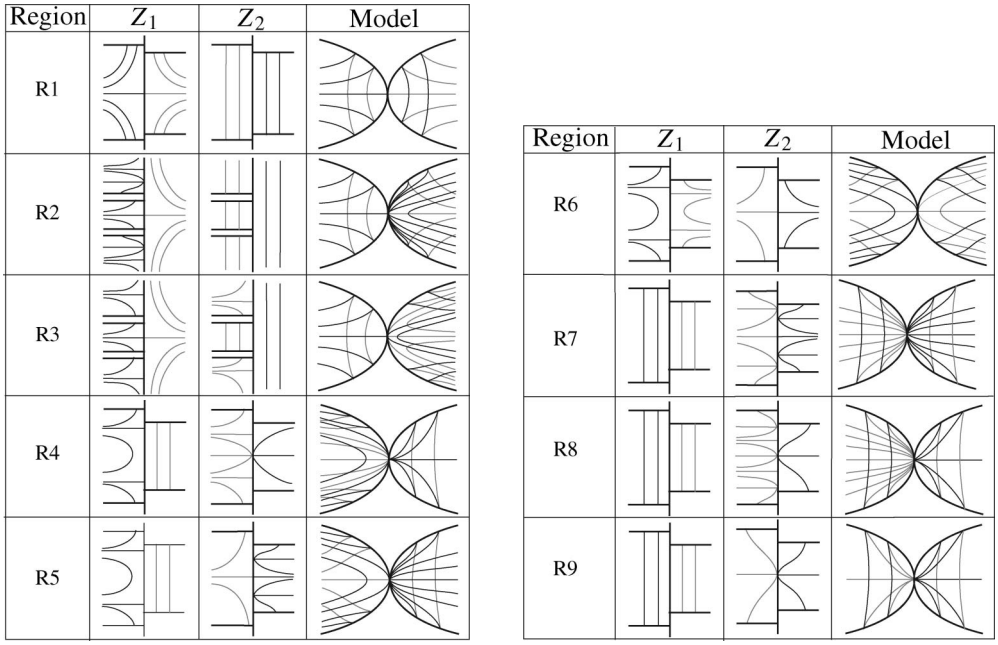

FIGURE 11. Configurations of the integral curves of $(y^3, b_0x + b_2y^2 + b_3y^3, y)$ and their associated directional blowing-up models.

$a \neq 0$, to

$$(x^2 + dy^4, -xy + b_3(x, y) + b_4(x, y), -ax^2 + 2y^2 + c_1xy^2 + c_2y^3 + c_2xy^3 + c_4y^4),$$

where b_3 (resp. b_4) is a homogeneous polynomial of degree 3 (*resp.* 4) *and d and c_i, $i =$* 1, 2, 3, 4, *are constants.*

The proof is similar to that of Proposition 3.2 and is omitted.

When $a < 0$, the discriminant has an $X_{1,2}$-singularity which is $\mathcal{K}$-equivalent to $x^4 + \lambda x^2y^2 - y^6$ with $\lambda < 0$ if $d - b_{33}^2 > 0$, where b_{33} is the coefficients of the term y^3 in b_3.

We consider the blowing-up $x = u, y = uv$. (We also consider the blowing-up $x = uv, y = v$, but this does not yield any extra information.) We can write the coefficients of the new BDE in the form $(\bar{a}, \bar{b}, \bar{c}) = u^2(u^2A_1, u^2B_1, C_1)$ with

$$\begin{aligned} A_1 =& 1 + u^2N_1(u, v), \\ B_1 =& b_3(1, v) + uN_2(u, v), \\ C_1 =& -a + v^2 + uN_3(u, v), \end{aligned}$$

where $N_i(u, v)$, $i = 1, 2, 3$, are smooth functions along the exceptional fibre.

We consider the vector fields

$$Z_i = \big(-u^2 B_1 + (-1)^i \sqrt{u^2(u^2 B_1^2 - A_1 C_1)}\,\big) \frac{\partial}{\partial u} + C_1 \frac{\partial}{\partial v}\,, \quad i = 1, 2\,.$$

The discriminant lifts to two smooth curves that are generically transverse to the exceptional fibre at $-a + v^2 = 0$. The fields Z_i, $i = 1, 2$, are defined in a neighbourhood of the segment $-a + v^2 \geq 0$ and are regular along $-a + v^2 > 0$.

At $v = \sqrt{a}$, we change variables and set $s = u, t^2 = u^2 B_1^2 - A_1 C_1$, with $t \geq 0$. It follows that $C_1 = (uB_1 - t)(uB_1 + t)/A_1$ and $v = g(s, t)$. The ODE associated to $(s, t)^* Z_i$ is

$$((s\bar{B}_1 - t)(s\bar{B}_1 + t)/\bar{A}_1)ds - (-s^2 \bar{B}_1 + (-1)^i |s| t)(g_s ds + g_t dt) = 0\,,$$

where $\bar{A}_1 = A_1(s, g(s, t))$ and $\bar{B}_1 = B_1(s, g(s, t))$. We can factor out $(s\bar{B}_1 - t)$ or $(s\bar{B}_1 + t)$, depending on i and the sign of s (i.e., u). So we need to consider the vector fields

$$(s\bar{B}_1 + \mathrm{sign}(s)(-1)^i t + s\bar{A}_1 g_s) \frac{\partial}{\partial t} + s\bar{A}_1 g_t \frac{\partial}{\partial s},\ s \geq 0\,, \quad i = 1, 2\,.$$

As the blowing-up transformation is orientation preserving if u (i.e., s) is positive and orientation reserving if u is negative, we can drop $\mathrm{sign}(s)$ above and deal with the vector fields

$$W_i = (s\bar{B}_1 + (-1)^i t + s\bar{A}_1 g_s) \frac{\partial}{\partial t} + s\bar{A}_1 g_t \frac{\partial}{\partial s}\,, \quad i = 1, 2\,.$$

Their foliations are associated to those of $f_i(w)$, $i = 1, 2$ (see [25] and the statement before Section 3.1).

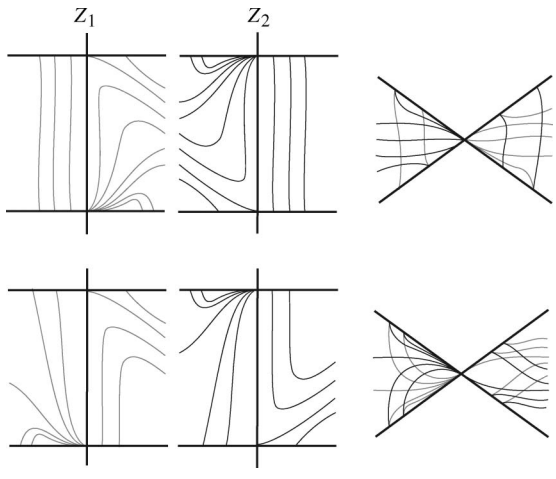

FIGURE 12. Configurations of the integral curves of a BDE at an $X_{1,2}$-singularity of the discriminant and their associated directional blowing-up models: $\Lambda_1 \Lambda_2 > 0$ top, $\Lambda_1 \Lambda_2 < 0$ bottom.

The 2-jet of W_i is equivalent to $(\Lambda_1 s + (-1)^i t)\partial/\partial t + (1/\sqrt{a})st\partial/\partial s$, where $\Lambda_1 = -(1/2)(ac_2 + \sqrt{a}c_1)$. The singularity is a saddle-node provided $\Lambda_1 \neq 0$.

At $v = -\sqrt{a}$, similar changes of coordinates as above show that the 2-jet of W_i, $i = 1, 2$, is equivalent to $(\Lambda_2 s - (-1)^i t)\partial/\partial t + (1/\sqrt{a})st\partial/\partial s$, with $\Lambda_2 = -(1/2)(ac_2 - \sqrt{a}c_1)$. The singularity is a saddle-node provided $\Lambda_2 \neq 0$.

We have two possible generic configurations for the integral curves of the BDE depending on the sign of $\Lambda_1\Lambda_2$ (see Figure 12).

THEOREM 3.6. *Suppose that the BDE has a 2-jet equivalent to* $(x^2, -xy, -ax^2 + 2y^2)$, *with* $a > 0$, *and its discriminant has an* $X_{1,2}$*-singularity of type* $x^4 + \lambda x^2 y^2 - y^6$, *with* $\lambda < 0$. *Then it is topologically equivalent to one of the following normal forms.*

(i) $(x^2 + y^4, -xy, -x^2 + 2y^2 + y^3)$ (*Figure* 12, *left*),

(ii) $(x^2 + y^4, -xy, -x^2 + 2y^2 + xy^2)$ (*Figure* 12, *right*).

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