NOTES ON FOURIER ANALYSIS (XX): ON THE RIESZ LOGARITHMIC SUMMABILITY OF THE DERIVED FOURIER SERIES.*)

By

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1. Let $f(x)$ be an integrable function with the period 2π and its Fourier series be

(1) $$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

If we differentiate the series term by term, we get

(2) $$\sum_{1}^{\infty} n(-a_n \sin nx + b_n \cos nx),$$

which is said the derived Fourier series of $f(x)$ and denote it by $S'[f]$.

The object of the present paper is to treat the Riesz logarithmic summability of (2).

Concerning the Fourier series Wang has proved the following theorems:

Theorem A. If

$$\lim_{t\to 0} \varphi(t) = s \ (R, \log n, \alpha) \quad (\alpha > 0),$$

then (1) is $(R, \log n, \alpha+\delta)$-summable to s at $t = x$, where δ is any positive number.

Theorem B. If (1) is $(R, \log n, \alpha)$-summable to sum s at $t = x$, then

$$\lim_{t\to 0} \varphi(t) = s(R, \log n, \alpha+1+\delta) \quad (\alpha > 0).$$

We prove analogus theorems concenning derived Fourier series (2), which reads as follows:

Theorem 1. If

$$\psi(t)/t = s \ (R, \log n, \alpha) \quad (\alpha > 0),$$

then (2) is $(R, \log n, \alpha+1+\delta)$-summable to sum s at $t = x$, where δ is any positive number.

Theorem 2. If (2) is $(R, \log n, \alpha)$-summable to sum s at $t = x$ $(\alpha > 1)$, then

$$\lim_{t\to 0} \psi(t)/t = s \ (R, \log n, \alpha+1+\delta)$$

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δ being any positive constant.

In these theorems we suppose that $\psi(t)/t$ is integrable in $(0\ 2\pi)$.

2. Let $D_\alpha(\varphi)$ be the α-th mean of (2). We have

$$D_\alpha(\omega)-s=-\frac{2}{\pi}\frac{\omega^2}{(\log\ \omega)^\alpha}\int_0^\infty L'_\alpha\ (\omega t)\psi\ (t)\,dt.$$

If we put $\psi(t)/t=g(t)$, then the α-th mean of $g(t)$ is

$$g_\alpha(t)=\frac{1}{\Gamma(\alpha)}\int_t^1\left(\log\ \frac{u}{t}\right)^{\alpha-1}\frac{g(u)}{u}\,du,$$

for positive α. And we put

$$g_\alpha^\beta(t)=\frac{1}{\Gamma(\beta)}\int_0^t g_\alpha(u)\ (t-u)^{\beta-1}\,du,$$

for positive β. Then we have for positive α

$$(3)\qquad D_\alpha(\omega)-s=-\frac{2}{\pi}\frac{\omega}{(\log\omega)^\alpha}\int_0^\infty g\ (t)\left\{\alpha\ L_{\alpha-1}(\omega t)-L_\alpha(\omega t)\right\}dt$$
$$=-\frac{\alpha}{\log\ \omega}R_{\alpha-1}(\omega)+R_\alpha(\omega),$$

where $R_\alpha\ (\omega)$ is the α-th Riesz logarithmic mean of the Fourier series of $g(t)$.

On the other hand

$$-\frac{\pi}{2}\ (D_\alpha(\omega)-s)\ =\frac{\omega^2}{(\log\ \omega)^\alpha}\int_0^\infty tg(t)\ L'_\alpha(\omega t)\ dt$$
$$=\alpha\ \frac{\omega}{(\log\ \omega)^\alpha}\int_0^\infty g\ (t)\ L_{\alpha-1}\ (\omega t)\ dt-\frac{\omega}{(\log\ \omega)^\alpha}\int_0^\infty g\ (t)\ L_\alpha(\omega t)\ dt.$$

Since $g(t)$ is periodic, it is equal to,

$$=\alpha\ \frac{\omega}{(\log\ \omega)^\alpha}\int_0^1 g\ (t)L_{\alpha-1}\ (\omega t)\ dt-\frac{\omega}{(\log\omega)^\alpha}\int_0^1 g(t)\ L_\alpha(\omega t)\ dt$$
$$+O\left(\frac{1}{\log\ \omega}\right)+\ o(1)$$

$$=\alpha\ \frac{\omega}{(\log\ \omega)^\alpha}\int_0^1 g\ (t)\ L_{\alpha-1}\ (\omega t)\ dt-\frac{\omega}{(\log\ \omega)^\alpha}\Big[\ g_1(t)\ tL_\alpha(\omega t)\ \Big]_0^1$$
$$+\frac{\omega}{(\log\ \omega)^\alpha}\int_0^1 g_1(t)\ \frac{d}{dt}\ (t\ L_\alpha(\omega t))\ dt+o(1)$$

$$=\alpha\ \frac{\omega}{(\log\ \omega)^\alpha}\int_0^1 g(t)\ L_{\alpha-1}(\omega t)\ dt+\frac{\omega}{(\log\ \omega)^\alpha}\int_0^1 g_1(t)L_{\alpha-1}(\omega t)dt+o(1)$$

$$=\alpha\Gamma(\alpha)\ \frac{\omega}{(\log\ \omega)^\alpha}\int_0^1 g_{\alpha-1}(t)\ L_0(\omega t)\ dt$$
$$+\alpha\ \Gamma(\alpha)\ \frac{\omega}{(\log\ \omega)^\alpha}\int_0^1 g_\alpha(t)L_0(\omega t)\ dt$$

$$=\Gamma(\alpha+1)\frac{1}{(\log\omega)^{\alpha}}\int_{0}^{1}(g_{\alpha-1}(t)+g_{\alpha}(t))\frac{\sin\omega t}{t}\,dt+o(1),$$

where $g(t)$ is continuated periodically. Thus we have proved

$$(4)\qquad D_{\alpha}(\omega)-s=-\frac{2}{\pi}\Gamma(\alpha+1)\frac{1}{(\log\omega)^{\alpha}}\int_{0}^{1}(g_{\alpha-1}(t)+g_{\alpha}(t))\frac{\sin\omega t}{t}\,dt.$$

for any $\alpha\geqq 1$.

We will state two lemmas due to Mr. Wang:

Lemma 1. If the partial sum s_n of the Fourier series of $f(x)$ is of order $o(\log n)^{\alpha}$ $(\alpha>0)$, then

$$f(t)=o\left(t^{1+\delta}\left(\log\frac{1}{t}\right)^{\alpha}\right)\quad(C,\,1+\delta)$$

for any $\delta>0$.

Lemma 2. If for any $\delta>0$,

$$f(t)=o\,(t^{1+\delta}(\log 1/t)^{\alpha})\quad(C,\,1+\delta),$$

then

$$f(t)=o((\log 1/t)^{1+\alpha+\varepsilon})\quad(R,\,\log n,\,1+\alpha+\varepsilon)$$

for any $\varepsilon>\delta>0$.

3. Proof of Theorem 1. By the hypothesis $S[g]$ is $(R,\ \log n,\ \alpha+\delta)$ -summable to sum 0 at $t=x$. Hence $R_{\alpha+\delta}(\omega)=o(1)$ and $R_{\alpha+\varepsilon+1}(\omega)=o(1)$. From (3) we have $D_{1+\alpha+\delta}(\omega)-s=o(1)$, which is the required.

4. Proof of Theorem 2. By (3) and Lemma 1 we have

$$(5)\qquad \frac{1}{\Gamma(1+\delta)}\int_{0}^{t}(g_{\alpha-1}(u)+g_{\alpha}(u))(t-u)^{\delta}\,du=o(t^{1+\delta}(\log 1/t)^{\alpha}).$$

On the other hand we have

$$g_{\alpha-1}^{\delta+1}(t)=\frac{1}{\Gamma(\delta+1)}\int_{0}^{t}g_{\alpha-1}(u)\,(t-u)^{\delta}\,du$$

$$=\left[\frac{1}{\Gamma(\delta+1)}g_{\alpha}(u)u(t-u)^{\delta}\right]_{0}^{t}$$

$$-\frac{1}{\Gamma(\delta+1)}\int_{0}^{t}g_{\alpha}(u)(t-u)^{\delta}du+\frac{\delta}{\Gamma(\delta+1)}\int_{0}^{t}g_{\alpha}(u)u(t-u)^{\delta-1}\,du$$

$$=-g_{\alpha}^{\delta+1}(t)+\frac{\delta}{\Gamma(\delta+1)}t\int_{0}^{t}g_{\alpha}(u)(t-u)^{\delta-1}du-\frac{\delta}{\Gamma(\delta+1)}\int_{0}^{t}g_{\alpha}(u)(t-u)^{\delta}du$$

$$=-g_{\alpha}^{\delta+1}+tg_{\alpha}^{\delta}-\delta\,g_{\alpha}^{\delta+1}.$$

Hence by (5) we have

$$(6)\qquad o(t^{1+\delta}(\log\frac{1}{t})^{\alpha})=tg_{\alpha}^{\delta}-\delta g_{\alpha}^{\delta+1}.$$

We have also

$$g^{\delta}_{\alpha}(t)=\frac{1}{\Gamma(\delta)}\int_0^t g_\alpha(u)\,(t-u)^{\delta-1}\,du$$

$$=\frac{1}{\Gamma(\delta)}\,\frac{1}{\delta}\,\frac{d}{dt}\int_0^t g_\alpha(u)\,(t-u)^{\delta}\,du$$

$$=\frac{d}{dt}\,g^{\delta+1}_{\alpha}(t).$$

By (6)

$$t\frac{d}{dt}\,g^{\delta+1}_{\alpha}(t)-\delta g^{\delta+1}_{\alpha}(t)=o(t^{1+\delta}\,(\log 1/t)^{\alpha}),$$

or

$$\frac{d}{dt}\,(t^{-\delta}\,g^{\delta+1}_{\alpha}\,(t))=o(\log 1/t)^{\alpha},$$

$$t^{-\delta}\,g^{\delta+1}_{\alpha}\,(t)=\frac{1}{\Gamma(\delta+1)}\,\frac{1}{t^{\delta}}\int_0^t g_\alpha(u)\,(t-u)^{\delta}\,du=o(\int_0^t |g_\alpha(u)|\,du)=o(1).$$

Hence we have

$$t^{-\delta}g^{\delta+1}_{\alpha}(t)=\int_0^t o(\log 1/t)^{\alpha}\,dt=o(t\,\log 1/t)^{\alpha}),$$

$$g^{\delta+1}_{\alpha}(t)=o(t^{\delta+1}\log 1/t).$$

By Lemma 2

$$g_{\alpha+\varepsilon+1}(t)=o(\log 1/t)^{\alpha+1+\varepsilon}\text{ for any }\varepsilon>\delta>0.$$

Thus the theorem is proved.

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References

1) F. T. Wang, Tohoku Math. Journ., 40 (1935).
2) cf. T. Takahashi, ibidem, 38 (1933).
3) cf. T. Wang, loc. cit. and N. Matsuyama, Notes on Fourier Analysis (X): On the Riesz logarithmic summability of Fourier series, under the press.