NOTES ON BANACH SPACE (VIII): A GENERALIZATION OF SILOV'S THEOREM.*)

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The set of all real-valued continuous functions defined on a compact Hausdorff space S forms a commutative Banach algebra R with respect to usual addition, product, scalar multiplication and the norm:

$$|x| = \sup_s |x(s)|.$$

Therefore, it is possible to introduce some notions of algebra with some modifications. For example, we mean by an *ideal* I a closed algebraic ideal in R and by a principal ideal $[x]$ the closure of the set of all elements xy where y runs through R. Moreover, we say, R is a principal ideal ring if and only if all its ideals are principal.

Under the above definitions, G. Silov [3] proved that R is a principal ideal ring if S is a compact metric space. In this note, we prove the converse theorem, which reads as follows:

Theorem. Banach algebra of real-valued continous functions on a compact Hausdorff space is a principal ideal ring if and only if the space is completely normal.

By a completely normal space S we mean a T_1-space satisfying one of the following three equivalent conditions:[1)]

1. S is normal and every its closed set is a G_δ-set.

2. For any closed set F of S, there exists a continuous function $x(s)$ in R such as

$$F = \{s \mid x(s) = 0\}.$$

3. For any two closed sets F and F' mutually disjoint, there is a non-negative continuous function $x(s)$ in R such that $|x| = 1$,

$$F = \{s \mid x(s) = 0\} \text{ and } F' = \{s \mid x(s) = 1\}.$$

The proof of the sufficiency of the theorem is almost similar to that of G. Silov in the case of a compact metric space S. But, for the sake of completeness, we give it in full. Firstly, we will prove the following lemma.

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1) The term is due to A. Komatu[2], where the equivalence of the conditions is also proved.

Lemma 1. If S is completely normal and compact, then a principal ideal $[x]$ coincides with the set of all elements $y(s)$ vanishing on s whenever $x(s)$ vanishes.

Proof: Let X be the set of vanishing points of $x(s)$. Then evidently $y \varepsilon [x]$ implies $y(s)=0$ for each $s \varepsilon X$. Therefore, it is sufficient to show the converse. By the definition of the complete normality, there exists a decreasing sequence of open sets G_i converging X. Let F_i be the complements of G_i respectively. Then they are closed and disjoint with X. Therefore, by the normality, there exist two open sets G_i' and G_i'' satisfying

$$X \leqq G_i'', \ F_i \leqq G_i' \text{ and } G_i' \frown G_i'' = \theta$$

for each i. Let F_i' be the complements of G_i', then F_i' is closed and disjoint with F_i satisfying $F_i' \geqq G_i' \geqq X$. Therefore, by the condition 3 of the complete normality, there exists a sequence of elements x_i in R such that $0 \leqq x_i \leqq y$,

$$x_i(s) = y(s) \text{ for } s \varepsilon F_i \text{ and } x_i(s) = 0 \text{ for } s \varepsilon F_i,$$

for we can assume without loss of generality that y is non-negative. Evidently $x_i(s)$ converges to $x(s)$ for each s. Hence, if we put

$$y_i = \bigvee_{j=1}^{i} x_j,$$

then y_i converges monotonically to y. Thus, by a theorem of Dini, it converges uniformly, that is, strongly with respect to the norm of R. Furthermore, let us suppose that

$$z_i(s) = y_i(s)/x(s) \text{ for } s \varepsilon S - F_i' \text{ and } z_i(s) = 0 \text{ for } s \varepsilon F_i',$$

then z_i belongs to R and

$$y_i(s) = z_i(s) x(s)$$

converges strongly to y, namely, y belongs to $[x]$. This proves the lemma.

Lemma 2. If S is completely normal and compact, then every ideal I is principal.

Proof: Let F be the closed set on which every element $y(s)$ of I vanishes. By the compactness and a theorem due to I. Gelfand and A. Kolmogoroff[1], this set F is not void. Hence, by virtue of Lemma 1, it is sufficient to show the existence of $x(s)$ in I satisfying

$$F = \{ s \mid x(s) = 0 \}.$$

Let F_i and G_i be the sets defined in Lemma 1. Since F_i is disjoint with F, by the complete normality and the compactness of S, there exists a non-negative function $x_i(s)$ in I with the norm unity, strictly positive on F_i and vanishing on F. Hence, if we put

$$x=\sum_{i=1}^{\infty}\frac{1}{2^i}x_i,$$

then the series converges and x belongs to I, for x_i belongs to I. Naturally, $x(s)$ vanishes exactly on F, which proves the lemma.

Since the above lemmas give the sufficiency, it remains to prove the necessity. For this purpose it suffices to show the existence of $x(s)$ in R satisfying

$$F=\left\{ s \mid x(s)=0\right\},$$

where F is any closed set in S. Let P be the set of ali elements of R vanishing on F. Then

$$I=\bigvee_{y\in P}[y]$$

exists, where the join operation is taken in the lattice of the ideals of R, and by the hypothesis there exists an element x in R such as $[x]=I$. If we put

$$F'=\left\{ s \mid x(s)=0\right\},$$

then evidently F' contains F. Let us suppose that F' does not coincide with F and s' be the element of their difference. Then by the complete regularity of S there is an element y in R vanishing on. F and strictly positive at s'. Therefore, $[y]$ contains $[x]$ as a proper subideal, which is a contradiction, for I is maximal ideal whose element vanishing on the set F. Thus it completes the proof of the theorem.

References

1. I. Gelfand and A. Kolmogoroff, On rings of continuous functions on topological spaces, C.R. (Doklady) de l'URSS, 22 (1939), 11-15.

2. A. Komatu, Isôkûkan-ron (in Japanese), Tokyo 1947.

3. G. Silov, Ideals and subrings of the ring of continuous functions. C.R.(Doklady)de l'URSS, 22(1939), 7-10.

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