NOTES ON FOURIER ANALYSIS (XVII): THE INTEGRATED LIPSCHITZ CONDITION OF A FUNCTION AND FEJER MEAN OF FOURIER SERIES.*)

By
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I. Let $f(x)$ be a function of period 2π satisfying the intgrated Lipschitz condition Lip (α, p) $(0<\alpha\leqq 1, p\geqq 1)$, that is

$$\left(\int_0^{2\pi}|f(x+t)-f(x)|^p dx\right)^{1/p}=O(t^\alpha), \tag{1.1}$$

and let its Fouries seies be

$$f(x)\sim a_0/2+\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx). \tag{1.2}$$

If $f\varepsilon$ Lip(α, p) $(0<\alpha<1, p\geqq 1)$ and $\sigma_n(x, f)=\sigma_n(x)$ denotes the Fejér mean of (1. 2), then it will be easily seen that[1)]

$$\left(\int_0^{2\pi}|f(x)-\sigma_n(x)|^p dx\right)^{1/p}=O(n^{-\alpha}). \tag{1.3}$$

This does not hold generally for $\alpha=1$, $p=1$, but we have

$$\int_0^{2\pi}|f(x)-\sigma_n(x)|dx=O(n^{-1}\log n). \tag{1.4}$$

This will be seen by the following example. If we put

$$f(x)=\sum_{n=1}^{\infty}\frac{\sin nx}{n}=\frac{\pi-x}{2} \qquad (0\leqq x<2\pi), \tag{1.5}$$

then $f(x)$ belongs to Lip(1. 1)[2)]. Now

$$f(x)-\sigma_n(x)=\frac{1}{n}\sum_{k=1}^{n}\left\{f(x)-s_k(x)\right\}$$

$$=\frac{1}{n}\sum_{k=1}^{n}\sin kx-\frac{1}{n}\sum_{k=1}^{n}\frac{\sin kx}{k}+\sum_{k=n+1}^{\infty}\frac{\sin kx}{k}$$

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1) G. H. Hardy and J.E. Littlewood, A convergence criteria for Fourier series, Math. Zeitschr. 28, (1928).

2) It is well known that if $f(x)$ is of bounded variation then $f(x)$ belongs to Lip(1.1). cf. loc. cit. Lemma 9.

$$=\frac{1}{n}\bar{D}_n(x)-\frac{1}{n}\sum_{k=1}^{n}\frac{\sin kx}{k}-\frac{\bar{D}_n(x)}{n+1}+\sum_{k=n+1}^{\infty}\frac{\bar{D}_k(x)}{k(k+1)}$$

$$=\sum_{k=n+1}^{\infty}\frac{\bar{D}_k^*(x)}{k(k+1)}-\frac{1}{2}\sum_{n+1}^{\infty}\frac{\sin kx}{k(k+1)}+\frac{\bar{D}_n(x)}{n(n+1)}+O(1/n),$$

where

$$\bar{D}_n(x)=\sum_{k=1}^{n}\sin kx,\ \bar{D}_n^*(x)=\bar{D}_n(x)-\frac{1}{2}\sin nx=\frac{1-\cos nx}{2\operatorname{tg}\frac{1}{2}x}\geqq 0$$

$$(0<x<\pi).$$

Since[3]

$$\int_0^{2\pi}|\bar{D}_n(x)|\,dx\sim\log n,\ \int_0^{\pi}\bar{D}_n^*(x)\,dx\sim\log n,$$

we have

$$\int_0^{2\pi}|f(x)-\sigma_n(x)|\,dx\geqq A\sum_{k=n+1}^{\infty}\frac{\log n}{n^2}-\frac{B\log n}{n(n+1)}+O(1/n)$$

$$>C\log n/n,$$

where A, B and C are the positive constants.

Furthermore, even for the absolutely continuous function, the relation

$$\int_0^{2\pi}|f(x)-\sigma_n(x)|\,dx=O(n^{-1})$$

does not hold in general. For example, we take the function

$$f(x)=\sum_{k=2}^{\infty}\frac{\sin kx}{k\log k}, \tag{1.6}$$

which is absolutely continuous, but

$$\int_0^{2\pi}|f(x)-\sigma_n(x)|\,dx>D\log\log n/n, \tag{1.7}$$

where D is a positive constant, in fact,

$$f(x)-\sigma_n(x)=\frac{1}{n}\sum_{k=2}^{n}\frac{\sin kx}{\log k}-\frac{1}{n}\sum_{k=2}^{n}\frac{\sin kx}{k\log k}+\sum_{k=n+1}^{\infty}\frac{\sin kx}{k\log k}. \tag{1.8}$$

If we denote a_n-a_{n+1} by Δa_n, we have

$$\sum_{k=2}^{n}\frac{\sin kx}{\log k}=\sum_{k=2}^{n-1}\bar{D}_k(x)\Delta\frac{1}{\log k}+O(1)+\frac{\bar{D}_n(x)}{\log n}$$

$$=\sum_{k=2}^{n-1}\bar{D}_k^*(x)\Delta\frac{1}{\log k}-\frac{1}{2}\sum_{k=2}^{n-1}\sin kx\Delta\frac{1}{\log k}+\frac{\bar{D}_n(x)}{\log n}+O(1), \tag{1.9}$$

3) A. Zygmund. Theory of trigonometrical series, p. 28.

and then

$$\int_0^{2\pi} \Big| \sum_{k=2}^{n} \frac{\sin kx}{\log k} \Big| dx \geqq E \sum_{k=2}^{n-1} \log k \, \Delta \frac{1}{\log k} - O(1)$$
$$> F \log \log n.$$

Next, since

$$\sum_{k=n+1}^{\infty} \frac{\sin kx}{k \log k} = - \frac{\bar{D}_n(x)}{(n+1)\log(n+1)} + \sum_{k=n+1}^{\infty} \bar{D}_k(x) \Delta \frac{1}{k \log k},$$

we have

$$\int_0^{2\pi} \Big| \sum_{k=n+1}^{\infty} \frac{\sin kx}{k \log k} \Big| dx = O(1/n) + O\Big(\sum_{k=n+1}^{\infty} \frac{1}{k^2} \Big) = O(1/n). \tag{1.10}$$

Summing up the estimations (1.7), (1.8) (1.9) and (1.10), we get

$$\int_0^{2\pi} |f(x) - \sigma_n(x)| \, dx > G \log \log n/n.$$

II. Recently R. Salem and A. Zygmund[4] have proved that if $f(x)$ is integrable and if $\int_0^{2\pi} |f(x)-s_n(x)|\,dx = O(n^{-\alpha})$, then $\int_0^{2\pi} |\bar{f}(x)-\bar{s}_n(x)|\,dx = O(n^{-\alpha})$ for any $\alpha > 0$, where $s_n(x)$ is the n-th partial sum of the series (1. 2), and $\bar{f}(x)$ and $\bar{s}_n(x)$ denote the conjugate function of $f(x)$ and the partial sum of conjugate series of (1. 2), respectively.

For the Fejér means, however, the circumference is differement. If $0<\alpha<1$, and $p \geqq 1$, then under the condition

$$\Big(\int_0^{2\pi} |f(x) - \sigma_n(x)|^p dx \Big)^{1/p} = O(n^{-\alpha}) \tag{2.1}$$

we have

$$\Big(\int_0^{2\pi} |\bar{f}(x) - \bar{\sigma}_n(x)|^p dx \Big)^{1/p} = O(n^{-\alpha}), \tag{2.2}$$

where $\bar{\sigma}_n(x)$ is the Fejér means of the series conjugate to (1. 2). For, in the case $f(x)$ belongs to Lip (α, p), if (2. 2) holds for $0<\alpha<<1$, then, by Hardy and Littlewood's Theorem[5], $\bar{f}(x)$ belongs to Lip (α, p) and then (2. 2) holds good.

In the case $\alpha = 1$, the above fact fails to be true. This may be seen by the following example.

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4) R. Salem and A. Zygmund, The approximation by partial sums of Fourier series. Trans. American Math. Soc., 59 (1946).

5) Loc. cit.[1]

(2.3) $$f(x)=\sum_{n=2}^{\infty}\frac{\cos nx}{n\log n},$$

(2.4) $$f(x)=-\sum_{n=2}^{\infty}\frac{\sin nx}{n\log n}.$$

For, we have

$$f(x)-\sigma_n(x)=\frac{1}{n}\sum_{k=2}^{n}(k-1)\frac{\cos kx}{k\log k}+\sum_{k=n+1}^{\infty}\frac{\cos kx}{k\log k},$$

and then

$$\int_0^{\pi}|f(x)-\sigma_n(x)|\,dx=O\Big(\frac{1}{n}\Big).$$

But we have seen in § 1 that

$$\int_0^{2\pi}|\bar{f}(x)-\bar{\sigma}_n(x)|\,dx\geqq D\log\log n/n.$$

III. We can, however, prove the following theorem.

Theorem 1. If

(3.1) $$\int_0^{2\pi}|f(x)-\sigma_n(x)|\,dx=O(1/n)$$

then

(3.2) $$\int_0^{2\pi}|\bar{f}(x)-\bar{\sigma}_n(x)|\,dx=O(\log n/n).$$

Proof will be done by the analogous way as Kawata's Theorem[6].

IV. We shall now prove the following theorem.

Theorem 2. Let $f(x)$ belong to Lip(1.1), then the necessary and sufficient condition that

$$\int_0^{2\pi}|f(x)-\sigma_n(x)|\,dx=O(1/n),$$

is that $f(x)$ belongs to Lip(1. 1).

The proof of Theorem 2 is based on the following two lemmas.

Lemma 1. Let $f(x)$ belong to Lip (1.1), then the necessary and sufficient condition that the conjugate function $\bar{f}(x)$ belongs to Lip(1. 1) is that the condition

(4.2) $$\int_0^{2\pi}\Big|\int_h^{\pi}\frac{\varphi(x,t)}{t^2}\,dt\Big|\,dx=O(1)$$

holds, where

6) T. Kawata. The Lipschitz condition of a function and Fejér means of Fourier series. Under the press.

$$\varphi(x,\ t)=f(x+t)+f(x-t)-2f(x).$$

Proof. We can write

$$f(x)=-\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(x+t)-f(x)}{2\ \mathrm{tg}\frac{1}{2}t}dt,$$

so we get

$$f(x+h)=-\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(x+t)-f(x+h)}{2\mathrm{tg}\frac{1}{2}(t-h)}dt.$$

$$f(x+h)-f(x)=-\frac{1}{2\pi}\Big(\int_{-\pi}^{-2h}+\int_{2h}^{\pi}\Big)[f(x+t)-f(x)][\mathrm{ctg}\tfrac{1}{2}(t-h)-\mathrm{ctg}\tfrac{1}{2}t]dt$$

$$+\frac{1}{2\pi}[f(x+h)-f(x)]\int_{2h}^{\pi}[\mathrm{ctg}\tfrac{1}{2}(t-h)-\mathrm{ctg}\tfrac{1}{2}(t+h)]dt$$

$$+\frac{1}{2\pi}\int_{-2h}^{2h}\frac{f(x+t)-f(x)}{\mathrm{tg}\ \frac{1}{2}t}dt-\frac{1}{2\pi}\int_{-2h}^{2h}\frac{f(x+t)-f(x+h)}{\mathrm{tg}\frac{1}{2}(t-h)}$$

$$\equiv\ J_1+J_2+J_3+J_4,$$

say.

$$\int_{-\pi}^{\pi}|J_3|\,dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}dx\ \Big|\int_{-2h}^{2h}\frac{f(x+t)-f(x)}{\mathrm{tg}\frac{1}{2}t}dt\Big|\leqq\frac{1}{2\pi}\int_{-2h}^{2h}\frac{dt}{t}\int_{-\pi}^{\pi}|f(x+t)-f(x)|\,dx$$

$$=O\Big(\int_{-2h}^{2h}dt\Big)=O(h).$$

Similarly $\int_{-\pi}^{\pi}|J_4|\,dx=O(h)$. And

$$J_2=[f(x+h)-f(x)]\int_{2h}^{\pi}O\Big(\frac{h}{(t-h)(t+h)}\Big)dt,$$

so we get

$$\int_{-\pi}^{\pi}|J_2|\,dx=O(h).$$

Finally, we estimate for J_1,

$$J_1=-\frac{1}{2\pi}\Big(\int_{-\pi}^{-2h}+\int_{2h}^{\pi}\Big)[f(x+t)-f(x)][\mathrm{ctg}\frac{t-h}{2}-\mathrm{ctg}\frac{t}{2}]dt$$

$$=-\frac{1}{2\pi}\int_{2h}^{\pi}[f(x+t)+f(x-t)-2f(x)][\mathrm{ctg}\frac{t-h}{2}-\mathrm{ctg}\frac{t}{2}]dt$$

$$+\frac{1}{2\pi}\int_{2h}^{\pi}[f(x-t)-f(x)][\mathrm{ctg}\frac{t-h}{2}+\mathrm{ctg}\frac{t+h}{2}-2\mathrm{ctg}\frac{t}{2}]dt$$

$$\equiv\ J_1'+J_2'',$$

$$\int_{-\pi}^{\pi}|J_1''|\,dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\int_{2h}^{\pi}[f(x-t)-f(x)][\mathrm{ctg}\frac{t-h}{2}+\mathrm{ctg}\frac{t+h}{2}-2\mathrm{ctg}\frac{t}{2}]dt\right|dx$$

$$=O\Big(h^2\int_{2h}^{\pi}\frac{dt}{(t+h)(t-h)}\Big)=O(h).$$

And now

$$J_1'=-h\int_{2h}^{\pi}\frac{\varphi(x,t)}{t^2}dt+O\Big(\int_{2h}^{\pi}\varphi(x,t)\Big[\frac{h}{t^2}-\frac{1}{t-h}+\frac{1}{t}\Big]dt$$

$$=-h\int_{2\eta}^{\pi}\frac{\varphi(x,t)}{t^2}dt+O\Big(h^2\int_{2h}^{\pi}\frac{\varphi(x,t)}{t^2(t-h)}dt\Big),$$

so we get

$$\int_{-\pi}^{\pi}|J_1'|\,dx=h\int_{-\pi}^{\pi}\Big|\int_{2h}^{\pi}\frac{\varphi(x,t)}{t^2}dt\Big|dx+O\Big(h^2\int_{2h}^{\pi}\frac{dt}{t(t-h)}\Big)$$

$$=h\int_{-\pi}^{\pi}\Big|\int_{2h}^{\pi}\frac{\varphi(x,t)}{t^2}dt\Big|\,dx+O(h).$$

Summing up the estimations for J_1, J_2, J_3 and J_4, we complete the proof of Lemma 1.

Lemma 2. Let $f(x)$ belong to Lip(1.1). Then in order that the relation (4.1) holds, it is necessary and sufficient that the condition (4.2) holds.

Proof. We have

$$\sigma_{n-1}(x)-f(x)=\frac{1}{\pi}\int_0^{\pi}\varphi(x,t)\frac{\sin^2 nt/2}{n\sin^2 t/2}dt$$

$$=\frac{1}{\pi}\int_0^{\pi/n}+\frac{1}{\pi}\int_{\pi/n}^{\pi}\equiv I_1+I_2,$$

say, since $f(x)$ belongs to Lip(1.1), we have

(4.3) $$\int_0^{2\pi}|I_1|\,dx=\frac{1}{\pi}\int_0^{\pi/n}O(t)\frac{n^2t^2}{nt^2}dt=O\Big(\frac{1}{n}\Big).$$

Therefore,

$$\sigma_{n-1}(x)-f(x)-\frac{1}{\pi n}\int_{\pi/n}^{\pi}\frac{\varphi(x,t)}{\sin^2 t/2}dt$$

$$=I_n-\frac{1}{\pi n}\int_{\pi/n}^{\pi}\varphi(x,t)\frac{\cos nt}{\sin^2 t/2}dt\equiv P_n-Q_n,$$

say. If we put $R_n(t)\equiv 1/\pi n(\sin t/2)^2$, then, for $n\geqq 1$,

$$Q_n=\int_{\pi/n}^{\pi}\varphi(x,t)R_n(t)\cos nt\,dt$$

$$= -\int_{0}^{\pi(n-1)/n} \varphi(x, t+\pi/n) R_n(t+\pi/n) \cos nt\, dt,$$

$$2Q_n = \int_{\pi/n}^{\pi(n-1)/n} \varphi(x,t)[R_n(t) - R_n(t+\pi/n)] \cos nt\, dt$$

$$+ \int_{\pi/n}^{\pi(n-1)/n} [\varphi(x,t) - \varphi(x, t+\pi/n)] R_n(t+\pi/n) \cos nt\, dt$$

$$- \int_{0}^{\pi/n} \varphi(x, t+\pi/n) R_n(t+\pi/n) \cos nt\, dt$$

$$+ \int_{\pi(n-1)/n}^{\pi} \varphi(x,t) R_n(t) \cos nt\, dt \equiv I_n + J_n + K_n + L_n.$$

By the mean value theorem

$$|R_n(t) - R_n(t+\pi/n)| \leqq Cn^{-2}t^{-3},$$

so that

$$|I_n| \leqq Cn^{-2} \int_{\pi/n}^{\pi-\pi/n} |\varphi(x,t)|\, t^{-3} dt \leqq Cn^{-3} \int_{\pi/n}^{\pi} |\varphi(x,t)|\, t^{-3} dt.$$

Since $R_n(t+\pi/n) \leqq 1/nt^2$, and

$$\varphi(x,t) - \varphi(x, t+\pi/n) = f(x+t) - f(x+t-\pi/n) + f(x-t) - f(x-t-\pi/n),$$

we find

$$|J_n| \leqq Cn^{-1} \int_{\pi/n}^{\pi} |f(x+t) - f(x+t-\pi/n)|\, t^{-2}\, dt$$

$$+ Cn^{-1} \int_{\pi/n}^{\pi} |f(x-t) - f(x-t-\pi/n)|\, t^{-2}\, dt.$$

Moreover, since $R_n(t+\pi/n) < Cn$ for $0 \leqq t \leqq \pi/n$,

$$|K_n| \leqq Cn^{-1} \int_{0}^{\pi/n} |\varphi(x, t+\pi/n)|\, dt.$$

Finally

$$|L_n| \leqq Cn^{-1} \int_{\pi(n-1)/n}^{\pi} |\varphi(x,t)|\, dt.$$

Since $\int_0^{2\pi} |f(x+t) - f(x)|\, dx = O(t)$ by the assumption we can immediately deduce that $\int |P_n|\, dx$, $\int |I_n|\, dx$, $\int |J_n|\, dx$, $\int |K_n|\, dx$, $\int |L_n|\, dx$ are all $O(1/n)$. Thus the lemma is proved.

Proof of Theorem 2 is immediate from Lemma 1 and 2.

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