ON THE STRUCTURE OF SPACES WITH NORMAL PROJECTIVE CONNEXIONS WHOSE GROUPS OF HOLONOMY FIX A HYPERQUADRIC OR A QUADRIC OF (N-2)-DIMENSION.[*]

By

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Several years ago, we have studied the spaces with normal conformal connexions whose groups of holonomy fix a point or a hypersphere.[1] The most fundamental theorem that we have found is the following: If the group of holonomy of a space C_n with a normal conformal connexion is a subgroup of the Möbius' group which fixes a point (or a hypersphere), the C_n is a space with a normal conformal connexion corresponding to the class of Riemann spaces conformal to each other including an Einstein space with a vanishing (or non vanishing) scalar curvature. The converse is also true. Making use of the fact that a subgroup of the Möbius' group which fixes a hypersphere is in a close relation with the Poincaré's representation of non-Euclidean geometry, we could further generalize the Poincaré's representation of non-Euclidean geometry to Einstein spaces.

In the present paper, we shall apply that idea to spaces with normal projective connexions. In Klein's representation of non-Euclidean geometry the fundamental group of the space is the subgroup of all projective transformations which fix a hyperquadric. Hence we are led to consider those spaces with normal projective connexions whose groups of holonomy fix a hyperquadric. In connection with this, we also consider those spaces with normal projective connexions whose groups of holonomy fix an $(n-2)$ dimensional quadric in a hyperplane.

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*) Received March 1st, (1948).

1) S.Sasaki, On the spaces with normal conformal connexions whose groups of holonomy fix a point or a hypersphere, I, II, III, Jap. J. of Math., 34 (1942) pp. 615-622, pp. 623-633, 35 (1943) pp. 791-795.
K. Yano, Conformal and concircular geometries in Einstein spaces, Proc. Imp. Acad. Japan, 19 (1943) pp. 444-453.

§ 1. The structure of spaces with normal projective connexions whose groups of holonomy fix a hyperquadric.

Let there be given a space with a projective connexion P_n. If we take repères semi-naturel $[R_0, R_i]$, the projective connexion of the space is given by the following formulae:

$$(1.1)\qquad \begin{cases} dR_0 = P_i dx^i R_0 + dx^i R_i, \\ dR_j = \gamma^0{}_{jk} dx^k R_0 + \gamma^i{}_{jk} dx^k R_i. \end{cases}$$

We shall call

$$(1.2)\qquad \Gamma^0_{jk} = \gamma^0_{jk}, \qquad \Gamma^i_{jk} = \gamma^i_{jk} - \delta^i_j\, p_k,$$

the parameters of the projective connexion. If P_n is normal, Γ^0_{jk} is determined, by means of Γ^i_{jk}, as follows:

$$(1.3)\qquad \Gamma^0_{jk} = -\frac{1}{n^2-1}(nR_{jk} + R_{kj}),$$

where we have put

$$R^j{}_k = R^i{}_{jki},$$

$$R^i{}_{jkl} = \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \frac{\partial \Gamma^i_{jl}}{\partial x^k} + \Gamma^i_{hl}\, \Gamma^h_{jk} - \Gamma^i_{hk}\, \Gamma^h_{jl}.$$

If we apply to the repère a transformation of the hyperplane at infinity

$$(1.4)\qquad \overline{R}_0 = R_0, \qquad \overline{R}_j = R_j + \phi_j R_0,$$

the parameters of the projective connexion will change in the following way:

$$(1.5)\qquad \begin{cases} \overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \phi_k + \delta^i_k \phi_j, \\ \overline{\Gamma}^0_{jk} = \Gamma^0_{jk} + \dfrac{\partial \phi_j}{\partial x^k} - \Gamma^i_{jk}\phi_i - \phi_j \phi_k, \end{cases}$$

$(1.5)_1$ is the so-called projective change of affine connexions.

Now, the covariant differential of a projective contravariant vetor X^λ ($\lambda, \mu, \nu = 0, 1, 2, \ldots\ldots, n$) is given by

$$(1.6)\qquad \begin{cases} DX^0 = dX^0 + \gamma^0_{jk}\, X^j\, dx^k + X^0\, P_k dx^k, \\ DX^i = dX^i + \gamma^i_{jk}\, X^j\, dx^k + X^0\, dx^i. \end{cases}$$

If the group of holonomy H of the given space fixes a hyperquadric

$$Q_{n-1}: \qquad a_{\lambda\mu}\, X^\lambda\, X^\mu = 0$$

of the tangent projective space P^0_n, $d(a_{\lambda\mu}\, X^\lambda\, X^\mu)$ must vanish for every point of Q_{n-1} (that is $d\,(a_{\lambda\mu} X^\lambda X^\mu)$ must be proportional to $(a_{\lambda\mu} X^\lambda X^\mu)$ in virtue of the relation $DX^\lambda = 0$. The converse is also true.

The last condition is easily reduced to the following relations:

$$(1.7)\qquad \begin{cases} \dfrac{\partial a_{00}}{\partial x^k}-2a_{0k}=a_{00}(\tau_k+2p_k), \\ \dfrac{\partial a_{0j}}{\partial x^k}-\Gamma^i_{jk}\,a_{0i}-a_{00}\Gamma^0_{jk}-a_{jk}=a_{0j}\ (\tau_k+2p_k), \\ \dfrac{\partial a_{ij}}{\partial x^k}-\Gamma^h_{ik}\,a_{hj}-\Gamma^h_{jk}\,a_{ih}-a_{0i}\,\Gamma^0_{jk}-a_{0j}\,\Gamma^0_{ik}=a_{ij}\ (\tau_k+2p_k). \end{cases}$$

In this paper we shall confine ourselves only to the domain where a_{00} does not vanish. Hence we can put

$$(1.8)\qquad a_{00}=\varepsilon \qquad (\varepsilon=\pm 1).$$

Then $(1.7)_1$ shows us

$$\tau_k+2p_k=-2\varepsilon a_{0k}.$$

For the sake of simplicity, let us put $a_{0k}\equiv a_k$, and denote by a comma the covariant differentiation with respect to Γ^i_{jk}. Then $(1.7)_{2,3}$ reduce to the following relations:

$$(1.9)\qquad \begin{cases} a_{j,k}-\varepsilon\Gamma^0_{jk}-a_{jk}=-2\varepsilon a_j a_k, \\ a_{ij,k}-a_i\,\Gamma^0_{jk}-a_j\,\Gamma^0_{ik}=-2\varepsilon a_k a_{ij}. \end{cases}$$

Now, if we define

$$(1.10)\qquad a_{ij}\equiv g_{ij}+\varepsilon a_i a_j,$$

(1.9) reduces to

$$(1.11)\qquad \begin{cases} a_{j,k}-\varepsilon\Gamma^0_{jk}-g_{jk}=-\varepsilon a_j a_k \\ g_{ij,k}+\varepsilon a_{i,k}a_j+\varepsilon a_i a_{j,k}-a_i\,\Gamma^0_{jk}-a_j\,\Gamma^0_{ik}=-2\varepsilon a_k\ (g_{ij}+\varepsilon a_i a_j). \end{cases}$$

Solving $(1.11)_1$ with respect to $a_{j,k}$ and putting into $(1.11)_2$, we get

$$g_{ij,k}+\varepsilon g_{ik}\,a_j+\varepsilon g_{jk}\,a_i+2\varepsilon g_{ij}\,a_k=0.$$

If we put

$$(1.12)\qquad \bar{\Gamma}^i_{jk}=\Gamma^i_{jk}-\varepsilon\delta^i_{jk}a-\varepsilon\delta^i_k u_j,$$

the last relation becomes

$$(1.13)\qquad \frac{\partial g_{ij}}{\partial x^k}-\bar{\Gamma}^h_{ik}\,g_{hj}-\bar{\Gamma}^h_{jk}\,g_{ih}=0.$$

On the other hand, we get from (1.10)

$$\det|g_{ij}|=|a_{ij}-\varepsilon a_i a_j|=|a_{ij}|-\varepsilon\Sigma a_i a_j A_{ij}.$$

(where A_{ij} means the confactor of the element a_{ij} in the det $|a_{ij}|$) and

$$\begin{vmatrix} \varepsilon & a_1 & a_2 \cdots\cdots a_n \\ a_1 & a_{11} & a_{12} \cdots\cdots a_{1n} \\ \vdots & \vdots & \vdots \qquad\quad \vdots \\ a_n & a_{n1} & a_{n2} \cdots\cdots a_{nn} \end{vmatrix} = \varepsilon|a_{ij}|-\sum_{i,j}a_i\,a_j\,A_{ij}.$$

Therefore, if the hyperquadric Q_{n-1} is non-degenerate, i. e. if det $|a_{\lambda\mu}|\neq 0$, then det $|g_{ij}|\neq 0$. Hereafter we assume that the hyperquadric Q_{n-1} is non-degenerate. Then we see from (1.13) that $\bar{\Gamma}_{jk}$'s are the Christoffel's symbols constructed from g_{jk}.

Now, comparing the projective change of affine connexions (1.12) with (1.5) we get

$$\bar{\Gamma}^0_{jk}=\Gamma^0_{jk}-\varepsilon a_{jk}-a_j a_k.$$

We see from (1.11) that the following relation holds good:

$$(1.14) \qquad \bar{\Gamma}^0_{jk}=-\varepsilon g_{jk}.$$

As $\bar{\Gamma}_{jk}$'s are Christoffel's symbols, $\bar{R}_{jk}$ is the Ricci's tensor of the Riemann space $\bar{g}_{jk}$, and hence they are symmetric with respect to j and k.

Therefore, we get from (1. 3)

$$(1.15) \qquad \bar{\Gamma}^0_{jk}=\frac{-1}{n-1}\bar{R}_{jk}.$$

Comparing this with the last equation we get finally

$$(1.16) \qquad \bar{R}_{jk}=(n-1)\,\varepsilon\, g_{jk}.$$

Accordingly we obtain the following theorem.

Theorem 1. If the group of holonomy of a space with a normal projective connexion P_n is a subgroup of the group of all projective transformations in P_n which fix a non-degenerate hyperquadric Q_{n-1}, the P_n is a space with a normal projective connexion corresponding to the class of affinely connected spaces with corresponding paths including an Einstein space with non-vanishing scalar curvature, in other word, the P_n is projective to an Einstein space with non-vanishing scalar curvature. The converse is also true.

Hereafter we shall denote the space with a normal projective connexion having the same system of paths with a given affinely connected space A_n by $P_n(A_n)$.

Let E_n be an Einstein space with non vanishing scalar curvature R. If we perform the trivial conformal transformation

$$(1.17) \qquad \bar{g}_{ij}=c^2 g_{ij}, \qquad c^2=\frac{R}{\varepsilon\,(n-1)} \qquad \varepsilon R>0,$$

then the Riemann space $\bar{E}_n(\bar{g}_{ij})$ is an Einstein space with scalar curvature $(n-1)$. Both Einstein spaces have the same system of paths. We can easily see from (1.11) that the hyperquadric ......

$$\varepsilon\,(X^0)^2+g_{ij}X^iX^j=0$$

is invariant under the transformations of the holonomy group of the space with normal projective connexion $P_n(E_n)$. ......

§ 2. Relations between the Klein's representation of Non-Euclidean geometry and the metrics of Einstein spaces with non vanishing scalar

curvature.

Let E_n be an Einstein space with positive definite fundamental tensor and of non vanishing scalar curvature.

By Theorem 1, the group of holonomy of the space with the normal projective connexion P_n (E_n) fixes a hyperquadric Q_{n-1}. We shall study the relation between the Non-Euclidean geometry with Q_{n-1} as the absolute figure and the metric of the Einstein space E_n.

The Case where $R<0$. In this case,applying the given space E_n an appropriate trivial conformal transformation, we can obtain Einstein space $\bar{E}_n$ with scalar curvature$-(n-1)$. Both Einstein spaces have the same system of geodesics. Consider geodesics in $\bar{E}_n$. As we consider only development of tangent spaces along a curve, we can assume without any loss of generality that $p_i=0$. Hence the connexion of the space with the normal projective connexion P_n $(\bar{E}_n)$ is expressible by the following equation:

$$(2.1)\qquad \begin{aligned} dR_0 &= dx^i R_i, \\ dR_j &= g_{jk}\, dx^k R_0 + \{^{\,i}_{jk}\}\, dx^k R_i. \end{aligned}$$

Denoting by s the arc length of a geodesic g in $\bar{E}_n$, we develop the geodesic in the tangent space at the point $s=0$. Then we get

$$(2.2)\qquad R_0(s) = R_0(0) + R_0'(0)\,s + R_0''(0)\frac{s^2}{2} + \cdots\cdots$$

While, geodesics are characteriged by the differential equations

$$(2.3)\qquad x''^i + \{^{\,i}_{jk}\} x'^j x'^k = 0,$$

hence we get

$$\begin{aligned} R_0' &= x'^i R_i, \\ R_0'' &= x''^i R_i + x'^i\,(g_{jk}\,x'^k R_0 + \{^{\,i}_{jk}\}\,x'^k R_i) \\ &= R_0, \\ R_0''' &= x'^i R_i, \\ R_0^{(4)} &= R_0. \end{aligned}$$

Accordingly, we obtain

$$(2.4)\qquad R_0(s) = \cosh s\; R(0) + \sinh s\; R'(0).$$

Now, the hyperquadric Q_{n-1} invariant under the group of holonomy of the space P_n (E_n) is given by

$$(2.5)\qquad -(x^0)^2 + g_{jk} X^j X^k = 0.$$

We can easily see that the points of intersection Y, Z of this hyperquadric Q_{n-1} and the straight line g^* which is the image of the geodesic are given by $\lambda R_0(0) + \mu R'_0(0)$ where $\lambda^2 = \mu^2$. Hence, the value of the double ratio

$$d=(R_0(0)\ R_0(s),\ Y\ Z)$$

is immediately calculated, giving

$$d=e^{2s}.$$

Accordingly, we get the relation

$$s=\frac{1}{2}\log d.$$

In the general case where $R<0$ and $\neq -(n-1)$, we get also

$$(2.6)\qquad s=\sqrt{\frac{-(n-1)}{4R}}\log d.$$

The case where $R>0$. In this case we can transform the given Einstein space E_n to an Einstein space $\bar{E}_n$, with scalar curvature $(n-1)$ by a trivial conformal transformation. E_n and $\bar{E}_n$ have the same system of geodesics. As we develop the tangent spaces of E_n only along curves, we can assume without any loss of generality that the connexion of $\bar{E}_n$ is given by the following equation

$$(2.7)\qquad \begin{cases} dR_0=dx^iR_i, \\ dR_j=-g_{jk}dx^kR_0+\{^{\,i}_{jk}\}dx^kE_i. \end{cases}$$

Hence, along a geodesic of $\bar{E}_n$ we can easily see that

$$R_0'=x'^i R_i, \qquad R''_0=-R_0.$$

Accordingly, the straight line g^* which is the image of the geodesic g in the tangent space at a point $s=0$ is given by

$$(2.8)\qquad R_0(s)=\cos s.\ R_0(0)+\sin s\ R'_0(0).$$

The points of intersection Y, Z of this straight line and the invariant hyperquadric Q_{n-1} of the group of holonomy

$$(2.9)\qquad (X^0)^2+g_{jk}X^jX^k=0$$

are given by $\lambda R_0(0)+\mu R'(0)$, where $\lambda^2+\mu^2=0$. Therefore the value of the double ratio $d=(R_0(0)R_0(s),YZ)$ is e^{2is}.
Accordingly we get

$$s=\frac{1}{2i}\log d.$$

In the general case where $R>0$, we get

$$(2.10)\qquad s=\frac{1}{i}\sqrt{\frac{n-1}{4R}}\log d.$$

Hence we obtain the following

Theorem 2. The group of holonomy of the space with a normal projective connexion P_n corresponding to Einstein space E_n with positive definite fundamental tensor and of non vanishing scalar curvature fixes an real (oval) or imaginary (nullteilig) hyperquadric according as the scalar curvature R is negative or positive respectively. The arc length of a geodesic segment PQ in E_n is expressible by (2.6) or (2.10) making use of the double ratio of four points P, Q and the points of intersection of the straight line (image of the geodesic PQ) and the invariant hyperquadric.

§ 3. The structure of spaces with normal projective connexions whose groups of holonomy fix an $(n-2)$ dimensional quadric in a hyperplane.

In §1 we have studied the structure of spaces with normal projective connexions whose groups of holonomy fix a hyperquadric. They are spaces with normal projective connexions corresponding to the classes of affinely connected spaces characterized by the property that they iuclude at least an Einstein space with non vanishing scalar curvature. The converse is also true. At that time, there did not appear Einstein spaces with vanishing scalar curvature. The fact that the group of holomomy fixes a hyperquadric is non-Euclidean type, hence, if we consider invariant figures of Euclidean type that is a hyperplane and an $(n-2)$ dimensional quadric in it there will appear Einstein spaces with vanishing scalar curvature. Being led by such conjecture, we shall study on the structure of spaces with normal projective connexions whose groups of holonomy fix an $(n-2)$ dimensional quadric in a hyperplane.

Now, we suppose that the invariant $(n-2)$ dimensional quadric of the group of holonomy be given by the intersection of a hyperplane

$$\pi: \qquad a_\lambda X^\lambda = 0$$

and a hypercone

$$K: \qquad g_{ij} X^i X^j = 0.$$

We assume that K is non-degenerate, that is

$$\det |g_{ij}| \neq 0.$$

In order that the hyperplane π be invariant by transformations of the group of holonomy $d(a_\lambda X^\lambda)$ be proportional to $a_\lambda X^\lambda$ under the condition $DX^\lambda = 0$. This condition is reducible to

$$(3.1) \qquad \begin{cases} \dfrac{\partial x_0}{\partial a^k} - a_k = (\tau_k + p_k)\, a_0, \\ \dfrac{\partial a_j}{\partial x^k} - \Gamma^i_{jk} a_i - \Gamma^0_{jk} a_0 = (\tau_k + p_k)\, a_j, \end{cases}$$

where $\tau_k dx^k$ is the proportionality factor. We consider in this paper only the domain where $a_0 \neq 0$, hence there is no loss of generality even if we put $a_0 = 1$. If we put $a_0 = 1$, then $(3.1)_1$ tells us

$$\tau_k + p_k = -a_k$$

and hence $(3.1)_2$ becomes

(3.2) $$a_{j,k} - \Gamma^0_{jk} = -a_j a_k,$$

where, denotes the formal covariant derivative with respect to Γ^i_{jk}.

In the next place, as the intersection Q_{n-2} of the hyperplane π and the hypercone K is invariant under the transformations of the group of holonomy,

$$d\,(g_{ij} X^i X^j) = g_{ij,k}\, dx^k X^i X^j - 2g_{ij}\, dx^i X^j X^0$$

must be proportional to $g_{ij}X^iX^j$ when we put $X^0 = -a_i X^i$. If we denote the proportionality factor by $\psi_k dx^k$, the condition reduces to

(3.3) $$g_{ij,k} + g_{jk}a_i + g_{ik}a_j = \psi_k g_{ij}.$$

If we put

(3.4) $$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} - \delta^i_j\, a_k - \delta^i_k a_j,$$

(3.5) $$q_k = 2a_k + \psi_k,$$

then (3.3) can be written as

(3.6) $$g_{ij;k} = q_k\, g_{ij},$$

where ; denotes the covariant differentiation with respect to $\bar{\Gamma}^i_{jk}$. (3.2) and (3.6) are the necessary and sufficient condition that the $(n-2)$ dimensional quadric Q_{n-2} is invariant under the group of holonomy. As we suppose that $|g_{ij}| \neq 0$, equation (3.6) shows that the affinely connected space $\bar{\Gamma}^i{}_{jk}$ is a Weyl space with the fundamental tensor g_j and with a linear from $q_k dx^k$.

The equations (3.4) means geometrically that the transformations of the hyperplane at infinity, that is the plane where all R_i's lie, and usually called as the projective change of affine connexions. When the projective change of affine connexions (3.4) is performed, it is well known that Γ^0_{jk} of the projective connexion is transformed as follows:

(3.7) $$\bar{\Gamma}^0_{jk} = \Gamma^0_{jk} - a_{j,k} - a_j\, a_k.$$

Comparing the last equation with (3.2), we find

(3.8) $$\bar{\Gamma}^0_{jk} = 0.$$

Now, as the projective connexion in consideration is normal, $\bar{\Gamma}^0_{jk}$ is expressible by the contracted curvature tensor $\bar{R}_{jk}$ of the affine connexion $\bar{\Gamma}^i_{jk}$ as follows:

(3.9) $$\bar{\Gamma}^0_{jk} = -\frac{1}{n^2-1}\,(n\,\bar{R}_{jk} + \bar{R}_{kj}).$$

Accordingly, we see from (3. 8) that the following relation holds good:

(3.10) $$\bar{R}_{jk} = 0.$$

If $q_k \equiv 0$, it is evident that the Weyl space in consideration is no other than an Einstein space with vanishing scalar curvature. More generally, if q_k is a gradient i e. $q_k = \frac{\partial \log \sigma}{\partial x^k}$, then putting

$$g^*_{ij} = \sigma^{-1} g_{ij}, \tag{3.11}$$

we can easily see that (3. 6) becomes

$$g^*_{ij;k} = 0. \tag{3.12}$$

Hence, the affinely connected space $\overline{\Gamma}^i_{jk}$ is also an Einstein space with vanishing scalar curvature.

Consequently wn get the following theorem:

Theorem 3. If the group of holonomy of a space with a normal projective connexion P_n fixes an $(n-2)$ dimensional quadratic Q_{n-2} in an hyperplane π, there exists at least a Weyl space such that $R_{jk}=0$ (in particular, Einstein spaces with vanishing scalar curvature are remarkable example of them) in the class of affinely connected spaces having the same system of paths with P_n. The converse is also true.

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