ON NORMAL COORDINATES OF A RIEMANN SPACE, WHOSE HOLONOMY GROU FIXES A POINT.*)

By

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§ 1. Consider a Riemann space V_n whose distance ds between two infinitely nearby points is given by

$$ds^2 = g_{\lambda\mu} dx^\lambda dx^\mu, \quad (\lambda, \mu, \nu, \cdots\cdots = 1, 2 \cdots\cdots n),$$

where the right hand member is a positive definite quadratic form.

Any normal coordinate system $(\bar{x}^\lambda)$ of V_n with a point O as origin is characterized by the condition that the following equations

$$\overline{\left\{ {\lambda \atop \mu\nu} \right\}} \bar{x}^\mu \bar{x}^\nu = 0 \tag{1}$$

are satisfied at every point in a neighbourhood of O $(\bar{x}^\lambda = 0)$ of V_n. Let $\bar{g}_{\lambda\mu}$ be the metric tensor of this coodinate system. According as $\bar{g}_{\lambda\mu}$'s have definite values or not at the origin, we call the normal coordinate system in consideration *ordinary* or *singular* respectively.

Consider an arbitrary coordinate syatem(x^λ). Then if a point is designated as the origin, one and only one normal coordinate system is determined so that both metric tensors have same values at this point and the transformations of normal coordinate systems with the same origin constitute a linear representation of the original coordinate transformations.

Now, if there is any point such that $|g_{\lambda\mu}| = 0$ or some of $g_{\lambda\mu}$'s have indefinite values with respect to some coordinate systems, we say that they are singular points of V_n. If P is not a singular point, there exists at least one coodinate system such that $g_{\lambda\mu}$'s with $|g_{\lambda\mu}| \neq 0$ have definite values at P. Hence the normal coordinate system coresponding to such coordinate system and having P as origin is ordinary. Accordingly, every singular point is characterized by the condition that any normal coordinate system with this point as its origin is necessarily a singular one.

§ 2. Now consider a V_n, whose holonomy group fixes a point O. Consider a normal coordinate system $(\bar{x}^\lambda)$ with O as its origin, then every geodesic issuing from O is expressible by the equations $\bar{x}^\lambda = \xi^\lambda s$ where S is the arc

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length and ξ^λ is the parameters of direction at O. Let M $(\bar{x}^\lambda)$ be any point on a geodesic $\bar{x}^\lambda=\xi^\lambda s$ and $e_\lambda(s)$ be the natural repère at that point. When we develop the tangent spaces $T_n(M)$ along this geodesic on which M lies we have

$$\frac{d}{ds}[M-x^\lambda e_\lambda]=\left(\frac{dx^\lambda}{ds}-\frac{d\bar{s}^\lambda}{ds}-\bar{x}^\alpha\left\{\begin{array}{c}\overline{\lambda}\\ \alpha\mu\end{array}\right\}\frac{dx^n}{ds}\right)e_\lambda$$

$$=\left\{\begin{array}{c}\overline{\lambda}\\ \alpha\mu\end{array}\right\}\bar{x}^\alpha\xi^\mu e_\lambda=0.$$

If M approaches O along the geodesic on which M lies, i. e. $M\to O$, $\bar{x}^\lambda=\xi^\lambda s\to 0$, then $[M-\bar{x}^\lambda e_\lambda]\to O$. Therefore when we develop tangent spaces $T_n(M)$ in $T_n(O)$ along this geodesic, the points $[M-\bar{x}^\lambda e_\lambda]$ which we consider at every point of this geodesic, have the same image overlaping with the origin O. This assertion is true for every geodesic issuing from O.

On the other hand, as O is invariant under the holonomy group, the following equations hold good along any curve $\bar{x}^\lambda=\bar{x}^\lambda(t)$ of V_n:

$$\frac{d}{dt}[M-x^\lambda e_\lambda]=-\bar{x}^\alpha\left\{\begin{array}{c}\overline{\lambda}\\ \alpha\mu\end{array}\right\}\frac{d\bar{x}^\mu}{dt}e_\lambda=0.$$

Hence we get

$$\bar{x}^\alpha\left\{\begin{array}{c}\overline{\lambda}\\ \alpha\mu\end{array}\right\}=0. \tag{2}$$

Accordingly we obtain the following

Theorem. *The necessary and sufficient condition that the holonomy group of a Reimann space V_n fixes a point is that there exists a coordinate system such that equations*

$$\left\{\begin{array}{c}\overline{\lambda}\\ \alpha\mu\end{array}\right\}\bar{x}^\alpha=0$$

are satisfied at every point of V_n.

From (2) we obtain

$$\frac{\partial\bar{g}_{\lambda\mu}}{\partial\bar{x}^\nu}\bar{x}^\nu=0, \tag{3}$$

so $g_{\lambda\mu}$ is homogeneous functions of degree 0 with respect to $\bar{x}^\lambda$.

From (1) and (3), we obtain

$$\frac{\partial\bar{g}_{\lambda\mu}}{\partial\bar{x}^\nu}\bar{x}^\lambda\bar{x}^\mu=0. \tag{4}$$

Equation (3) shows that the following theorem holds good:

Theorem. *If there exists a normal coordinate system with some point of V_n as its origin, such that the components of its metric tensor $g_{\lambda\mu}$ have, except the origin, constant values along each geodesics issuing from the origin respectively, the holonomy group of our space fixes a point. The* *convers is also true.*

Now we assume that our space is not euclidean, then we must consider that $g_{\lambda\mu}$'s have indefinite values at the origin O. Hence our normal coordinate system is a singular one, consequently O is a singular point of V_n.

§ 3. It has been known[1] that, under a suitably selected coordinate system, ds^2 of our space take the form

$$ds^2=(dx^n)^2+(x^n)^2 g_{ij}(x^k)dx^i dx^j \qquad (5)$$
$$(i, j, k, \cdots\cdots=1, 2, \cdots\cdots, n-1),$$

where $x^n=0$ is the image of the invariant point O of the holonomy group and x^n represents the distance from O to the point in consideration.

Let $\bar{x}^\lambda$ be normal coordinates with the invariant point O as its origin, then

$$(x^n)^2=g_{\lambda\mu}\bar{x}^\lambda\bar{x}^\mu. \qquad (6)$$

Hence x^n is a homogeneous function of degree 1 with respect to $\bar{x}^\lambda$.

The coordiniate system O btained from (5) by the following coordinate transformation

$$x^i=f^i(\bar{x}^\lambda), \quad x^n=f(\bar{x}^\lambda)$$

where f^i, f are homogeneous functions of degree 0 and 1 respectively with respect to $\bar{x}^\lambda$ such that the Jacobian $|\partial f/\partial x|$ except the origin does not vanish, is a singular one. One of the simplest example is given by

$$x^n=\delta^n_\lambda\bar{x}^\lambda, \quad x^i=\delta^i_\lambda\bar{x}^\lambda/\delta^n_\mu\bar{x}^\mu,$$

that is

$$\bar{x}^i=x^i x^n, \quad \bar{x}^n=x^n.$$

§ 4. Now if we assume that the Christoffel's symbols of the first kind are symmetric with respect to three indices, then from (3), (4) and (6) we get

$$\bar{g}_{\lambda\mu}=\frac{1}{2}\frac{\partial^2(x^n)^2}{\partial\bar{x}^\lambda\partial\bar{x}^\mu}. \qquad (7)$$

Accordingly, we obtain the following

Theorem. *If there exists a coordinate system such that the metric tensor of a Reimann space V_n is respresentable in the from (7), where x^n is a homogeneous function of degree* 1 *with respect to $\bar{x}^\lambda$, the holonomy group of V_n fixes a point.*

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1) S. Sasaki: On the structure of Riemann spaces, whose holonomy-group fixes a direction or a point, (in Japanese), Nippon. Sûgaku Butsuri Gakkaishi, (1941), pp. 193–200.

§ 5. Suppose that the holonomy group of V_n fixes $m(<n)$ linearly independent points. Then if we develope tangent spaces along any curve, every point which lies on the $(m-1)$ dimensional plane determined by the m linearly independent points is also pointwise invariant under the holonomy group. Hence they are singular.

We shall now show that if under a suitably selected coordinate system m independent equations

$$(x^{\alpha}-a^{\alpha}_{p})\left\{ {\lambda \atop \alpha\mu} \right\}=0, \quad (p=1,\ldots\ldots m), \tag{8}$$

where a^{α}_{p} are constants, hold good for the Riemann space V_n in consideration, then the holonomy group of V_n fixes m linearly independent points.

To prove this, we shall first consider a point $M+(a^{\alpha}{}_{p}-x^{\alpha})e_{\alpha}$ in each tangent space $T_n(M)$. If we develope the tangent spaces along a curve $x^{\lambda}=x^{\lambda}(t)$, we have

$$\frac{d}{dt}\left[M+(a^{\alpha}_{p}-x^{\alpha})e_{\alpha}\right]=(a^{\alpha}_{p}-x^{\alpha})\left\{ {\lambda \atop \alpha\mu} \right\}e_{\lambda}\frac{dx^{\mu}}{dt}.$$

Hence, the points $M+(a^{\alpha}_{p}-x^{\alpha})e_{\alpha}$ are invariant under the holonomy group. If $M=A_p(a^{\lambda}_{p})$, so $a^{\alpha}_{p}-x^{\alpha}=0$ and hence the fixed points are images of the points $A_p(a^{\alpha}_{p})$. When (8) are satisfied, the points $\lambda^{p}a^{\mu}_{p}\Big/\sum\limits_{p=1}^{m}\lambda^{p}$, where λ^{p} are the parameters, are also fixed by the holonomy group. From (8), we get also

$$\frac{\partial g_{\lambda\mu}}{\partial x^{\nu}}(x^{\nu}-a^{\nu}{}_{p})=0$$

Now let ξ^{λ}_{p} be the parameters of direction at the point (a^{ν}_{p}), then the curves defined by $x^{\lambda}=a^{\lambda}_{p}+\xi^{\lambda}_{p}s$ satisfy the differential equations of geodesics, accordingly they represent geodesics issuing from the point A_p. Along every geodesic of this system, $g_{\lambda\mu}$'s are constants respectively except the point A_p.

Therefore $g_{\lambda\mu}$ are constants along geodesics $x^{\lambda}=\dfrac{\lambda^{p}a^{\lambda}_{p}}{\sum\limits_{p}\lambda^{p}}+\xi^{\lambda}s$ respectively, where ξ^{λ} denote the parameters at the point $\lambda^{p}a^{\lambda}_{p}/\sum\limits_{p}\lambda^{p}$.

§ 6. Consider the set $\alpha^{(p)}_{n}$ of all line elements at a point $P(x^{\lambda})$ of a Finsler space with $ds=\int F(x,x')dt$, where F is a homogeneous function of degree

1 with respect to x', then we can consider $\alpha_n^{(p)}$ as an n-dimensional space where $\dot{x}'^{\lambda}$ play the role of coordinate system. When we introduce in $\alpha_n^{(p)}$ a Riemann metric

$$g_{\lambda\mu}(x,x')=\frac{1}{2}\ \frac{\partial^2 F^2}{\partial x'^{\lambda}\partial x'^{\mu}}\quad (x \text{ fixed}),$$

then it is evident that $\alpha_n^{(p)}$ is a Riemann space whose holonomy group fixes the center P.

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