NOTES ON FOURIER ANALYSIS (XXXIX): THEOREMS CONCERNING CESARO SUMMABILITY*)

By

SHIN-ICHI IZUMI AND GEN-ICHIRÔ SUNOUCHI

In this paper it is proved that, if

$$(1) \qquad \int_0^t \varphi_x(u)du = o\left(t/\log\frac{1}{t}\right), \quad \text{as} \quad t \to 0,$$

then the Fourier series of $f(t)$ is summable $(C, 1)$ at $t=x$, and if $0<\alpha<1$ and

$$(2) \qquad \int_0^t \varphi_x(u)du = o(t^{1/\alpha}), \quad \text{as} \quad t \to 0,$$

then the Fourier series of $f(t)$ is summable (C, α) at $t=x$. These theorems are known (Wang [7], [8]), but we give two kinds of proof. Each method is generalized to prove more general theorem. We prove that o in (1) and (2) cannot be replacd by O in these theorems.

§ **1.** THEOREM 1. *If*

$$(1) \qquad \int_0^t \varphi(u)du = o\left(t/\log\frac{1}{t}\right), \quad as \quad t \to 0,$$

where

$$\varphi(u) = \varphi_x(u) = \{f(x+u) + f(x-u) - 2f(x)\}/2,$$

then the Fourier series of $f(t)$ is summable $(C, 1)$ at $t=x$.

We prove this theorem in two ways, one using Young's function and the other using the Fejér kernel, respectively.

THE FIRST PROOF OF THEOREM 1. For $\alpha>0$, Young's function is defined by (Hobson [2] and Bosanquet [1])

$$\gamma_{1+\alpha}(u) = \int_0^1 (1-t)^\alpha \cos tu \, dt.$$

Then, as is well known, $\gamma_{1+\alpha}(u)$ and its derivative $\gamma'_{1+\alpha}(u)$ are bounded for $n \geqq 0$ and

$$(3) \quad \gamma_{1+\alpha}(u) \sim \frac{\Gamma(1+\alpha)}{u^{1+\alpha}} \cos\left(u - \frac{\alpha+1}{2}\pi\right) + O\left(\frac{1}{u^{\alpha+2}}\right) + O\left(\frac{1}{u^2}\right) \quad (u \to \infty)$$

and $\gamma'_{1+\alpha}(u)$ has the behaviour of the derivative of the right hand side of (3) as $u \to \infty$. Especially, for $0<\alpha\leqq 1$,

$$(4) \qquad \gamma_{1+\alpha}(u) = O(1/u^{1+\alpha}) \quad (u \to \infty).$$

The necessary and sufficient condition that the Fourier series of $f(t)$ is summable $(C, 1)$ at $t=x$, is that

---

*) Received Apr. 3rd., 1950.

$$(5) \qquad \sigma_\omega \equiv \frac{2\omega}{\pi}\int_0^\infty \gamma_2(\omega u)\varphi(u)du = o(1) \quad (\omega\to\infty),$$

where

$$\gamma_2(u) = O(1/u^2) \quad (u\to\infty), \ \gamma_2(u) = O(1) \quad (u\to 0)$$

by (4).

Letting $0<r<1/2$, we divide the integral (5) into two parts such as

$$\frac{\pi}{2}\sigma_\omega = \omega\int_0^\infty \gamma_2(\omega u)\varphi(u)du = \omega\int_0^{1/\omega^r} + \omega\int_{1/\omega^r}^\infty \equiv I_1 + I_2,$$

say. Then we have

$$I_2 = \omega\int_{1/\omega^r}^\infty \gamma_2(\omega u)\varphi(u)du = O\Big(\frac{1}{\omega}\int_{1/\omega^r}^\infty \frac{|\varphi(u)|}{u^2}du\Big)$$

$$= O\Big(\frac{1}{\omega}\Big[\omega^{2r} + \sum_{k=1}^\infty \frac{1}{k^2}\Big]\int_0^{2\pi}|\varphi(u)|du\Big) = O(\omega^{-1+2r}),$$

and

$$I_1 = \omega\int_0^{1/\omega^r}\gamma_2(\omega u)\varphi(u)du$$

$$= [\omega\gamma_2(\omega u)\varphi_1(u)]_0^{1/\omega^r} - \omega^2\int_0^{1/\omega^r}\gamma_2'(\omega u)\varphi_1(u)du$$

$$\equiv J_1 - J_2,$$

say, where $\varphi_1(u) = \int_0^u \varphi(t)dt$. We have

$$J_1 = O(\omega^{-1+2r}|\varphi_1(1/\omega^r)|) = O(\omega^{-1+2r})$$

and

$$J_2 = \omega^2\int_0^{1/\omega} + \omega^2\int_{1/\omega}^{1/\omega^r}\gamma_2'(\omega u)\varphi_1(u)du \equiv K_1 + K_2,$$

say, where

$$K_1 = o\Big(\omega^2\int_0^{1/\omega} O\Big(u/\log\frac{1}{u}\Big)du\Big) = o(1/\log\omega) = o(1),$$

$$K_2 = o\Big(\int_{1/\omega}^{1/\omega^r}\frac{du}{u\log 1/u}\Big) = o\Big(\log\frac{1}{r}\Big) = o(1).$$

Taking $0<r<1/2, I_2=o(1)$ and $I_1=J_1+o(1)=o(1)$. Thus we get (5), which is the required.

By this method of proof, we get the following generalization.

THEOREM 2. *If* $\alpha>0$ *and*

$$\varphi_\alpha(t) \equiv \frac{1}{\Gamma(\alpha)t}\int_0^t\Big(1-\frac{u}{t}\Big)^{\alpha-1}\varphi(u)du = o\Big(1/\log\frac{1}{t}\Big),$$

then the Fourier series of $f(t)$ *is summable* (C,α) *at* $t=x$.

For, putting $\beta\geqq\alpha>0$, the Cesàro mean of the Fourier series of $f(t)$ of order β is equivalent to (Bosanquet [1])

$$\sigma_\omega^\beta \equiv \omega\int_0^\eta \varphi_\alpha(t)J_\beta^\alpha(\omega t)dt,$$

where

$$J^{\alpha}_{\beta}(t) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1)}\frac{1}{t^{1-\alpha+\beta}}\cos\left(t-\frac{\pi}{2}(1+\alpha+\beta)\right)+O\left(\frac{1}{t^2}\right)$$

as $t\to\infty$. Thus we can prove $\sigma^{\beta}_{\omega}=o(1)$ as the proof of Theorem 1.

THE SECOND PROOF OF THEOREM 1. The Cesàro mean of the Fourier series of $f(t)$ of the first order is, using the Fejér kernel,

$$\begin{aligned}\sigma_n &= \frac{1}{2\pi(n+1)}\int_0^{\pi}\varphi(t)\frac{\sin^2(n+1)t/2}{\sin^2 t/2}dt \\ &= \frac{2}{\pi 2}\int_0^{\pi}\varphi(t)\frac{\sin^2 nt/2}{t^2}dt + o(1) \\ &= \frac{2}{\pi n}\left(\int_0^{1/n^r}+\int_{1/n^r}^{\pi}\right)\varphi(t)\frac{\sin^2 nt/2}{t^2}dt + o(1) \\ &\equiv I_1 + I_2 + o(1),\end{aligned}$$

say, where $0<r<1/2$. Then we have

$$I_2 = O\left(\frac{1}{n^{1-2r}}\int_0^{\pi}|\varphi(t)|dt\right) = o(1),$$

and, by the integration by parts,

$$\begin{aligned}I_1 &= \frac{2}{\pi}\left(\int_0^{1/n}+\int_{1/n}^{1/n^r}\right)\left[\Phi_1(t)\frac{\sin nt}{t}+\frac{1}{n}\Phi_1(t)\frac{\sin^2 nt/2}{t^2}\right]dt \\ &\quad + \left[\frac{2}{\pi n}\Phi_1(t)\frac{\sin^2 nt}{t^2}\right]_0^{1/n^r} \equiv J_1 + J_2 + J_3,\end{aligned}$$

say, where

$$\Phi_1(t) = \varphi_1(t)/t = \frac{1}{t}\int_0^t \varphi(u)du.$$

By the hypothesis $\Phi_1(t)=o\left(1/\log\frac{1}{t}\right)$, whence

$$J_1 + J_3 = o(1),$$

$$J_2 = o\left(\int_{1/n}^{1/n^r}\frac{dt}{t\log\frac{1}{t}}\right) = o\left(\left[\log\log\frac{1}{t}\right]_{1/n}^{1/n^r}\right) = o(1).$$

Thus the theorem is proved.

§ 2. THEOREM 3. *In Theorem 1, o in (1) cannot be replaced by O.*

PROOF. It is sufficient to construct a function $f(t)$ such that the Fourier series is not summable $(C, 1)$ at $t=x$ and

$$\int_0^t \varphi(u)du = O\left(t/\log\frac{1}{t}\right). \tag{6}$$

Let (c_k) be a sequence of positive numbers and (M_k), (m_k), (n_k) be increasing sequences of integers, which will be determined later. Let us take a sequence of intervals

$$I_k \equiv \left(\frac{\pi}{n_k}, \frac{\pi}{n_k}+\frac{\pi}{m_k}\right) \quad (k=1,2,\ldots) \tag{7}$$

which are disjoint mutually. Let $f(t)$ be an even periodic function such that

(8) $$f(t) = (-1)^k c_k\left[t\cos M_k t + \frac{1}{M_k}\sin M_k t\right]$$

in $I_k(k=1, 2, \dots)$ and $f(t)=0$ in $(0,\pi)-\vee I_k$. Supposing $x=0, \varphi(u)=\varphi_0(u)=f(t)$ and

$$\int_{I_k}|f(t)|dt = c_k\int_{I_k}\left|t\cos M_k t + \frac{1}{M_k}\sin M_k t\right|dt$$
$$\leq \frac{c_k}{n_k m_k} + \frac{c_k}{n_k M_k}.$$

If we suppose that

(9) $$m_k|M_k,\quad n_k|M_k\quad (k=1, 2, \dots),$$

in order that f is integrable, it is sufficient that

(10) $$\sum_{k=1}^{\infty}\frac{c_k}{n_k m_k} < \infty.$$

We have also

$$\int_{I_k} f(t)\,dt = \left[\frac{c_k}{M_k}t\sin M_k t\right]_{t=\pi/n_k}^{\pi/n_k+\pi/m_k} = 0,$$

and then

$$\int_0^t f(u)du = \int_{\pi/n_k}^t f(u)du = \frac{c_k}{M_k}t\sin M_k t$$

for t in I_k. Taking

(11) $$m_k/n_k \to 0\quad (k\to\infty),$$

(6) is satisfied when

(12) $$c_k\log m_k/M_k \to a \neq 0.$$

Let us now consider the Fourier series of $f(t)$ and σ_n be its Cesàro mean of the first order. Then,

$$\sigma_n = \int_0^\pi f(t)\frac{\sin^2 nt/2}{nt^2}dt + o(1)$$
$$= \int_0^\pi \Phi_1(t)\frac{\sin nt}{t}dt - \frac{1}{n\pi}\int_0^\pi \Phi_1(t)\frac{\sin^2 nt/2}{t^2}dt + 0(1)$$
$$\equiv J_1 + J_2 + o(1) = J_1 + o(1),$$

say, where

$$\Phi_1(t) \equiv \frac{\varphi_1(t)}{t} \equiv \frac{1}{t}\int_0^t f(u)du.$$

$$\frac{\pi}{8}J_1 = \int_0^\pi \Phi_1(t)\frac{\sin nt}{t}dt = \sum_{i=1}^{\infty}\int_{I_i}\Phi_n(t)\frac{\sin nt}{t}dt.$$

Putting $n\equiv M_k$ and dividing the above sum into three parts,

$$\frac{\pi}{8}J_1 = \sum_{i=1}^{k-1} + \int_{I_k} + \sum_{i=k+1}^{\infty} \equiv K_1 + K_2 + K_3,$$

say. We have

$$(-1)^k K_2 = \frac{c_k}{M_k}\int_{I_k}\frac{\sin^2 M_k t}{t}dt = \frac{c_k}{2M_k}\int_{I_k}\left(\frac{1}{t} - \frac{\cos 2M_k t}{t}\right)dt$$
$$= \frac{c_k}{2M_k}\log\left(1 + \frac{n_k}{m_k}\right) - \frac{c_k}{2M_k}\int_{2\pi M_k/n_k}^{2(\pi/n_k+\pi/m_k)M_k}\frac{\cos t}{t}dt$$
$$= \frac{c_k}{2M_k}\log\frac{n_k}{m_k} + O\left(\frac{c_k n_k}{M_k^2}\right) + o(1).$$

If we suppose that

(13) $$n_k = m_k^2 \quad (k = 1, 2, \ldots),$$

then $\log\dfrac{n_k}{m_k} = \log m_k$, whence $(-1)^k K_2 \to a/2$ by (12). Concerning K_1,

$$K_1 = \sum_{i=1}^{k-1}(-1)^i\int_{I_i}\frac{c_i}{M_i}\frac{\sin M_i t \sin M_k t}{t}dt$$
$$= \sum_{i=1}^{k-1}(-1)^i\frac{c_i}{2M_i}\int_{\pi/n_i}^{\pi/n_i+n/m_i}[\cos(M_k - M_i)t + \cos(M_k + M_i)t]\frac{dt}{t}$$
$$= \sum_{i=1}^{k-1}(-1)^i\frac{c_i}{2M_i}\left\{\int_{\pi(M_k-M_i)/n_i}^{(\pi/n_i+\pi/m_i)(M_k-M_i)}\frac{\cos t}{t}dt + \int_{\pi(M_k+M_i)/n_i}^{(\pi/n_i+\pi/m_i)(M_k+M_i)}\frac{\cos t}{t}dt\right\}$$
$$= O\left(\sum_{i=1}^{k-1}\frac{c_i}{M_i}\frac{n_i}{M_k - M_i}\right).$$

If we suppose that (M_k) is convex and

(14) $$\sum_{i=1}^{k-1}\frac{c_i n_i}{(k-i)M_i(M_{i+1} - M_i)} = o(1),$$

then $K_1 = o(1)$. Similarly $K_2 = o(1)$, when

(15) $$\sum_{i=k+1}^{\infty}\frac{c_i n_i}{M_i(M_i - M_k)} = o(1).$$

Thus σ_n does not converge when (M_k), (m_k) and (n_k) satisfy the conditions (9), (10), (11), (12), (13), (14) and (15).

Let us define the sequence (M_k), (m_k), (n_k) satisfying the required conditions. Firstly, let

$$M_1 = 2^5,\ m_1 = 2^2,\ n_1 = 2^4,\ c_1 = 2^5/(2\log 2).$$

Taking μ_2 such as $\mu_2^2 > 2n_1$

$$M_2 = \mu_2^5,\ m_2 = \mu_2^2,\ n_2 = \mu_2^4,\ c_2 = \mu_2^5/(2\log\mu_2).$$

Further, taking μ_3 such as $\mu_3^2 > 2n_2$, M_3, m_3, n_3, c_3 will be defined as above. In general, if M_{k-1}, m_{k-1}, n_{k-1} and c_{k-1} are defined, then we take μ_k such as $\mu_k^2 > 2n_{k-1}$ and put

$$M_k = \mu_k^5,\ m_k = \mu_k^2,\ n_k = \mu_k^4,\ c_k = \mu_k^5/(2\log\mu_k).$$

Thus (M_k), (m_k), (n_k) and (c_k) are completely defined and, as easily may be verified, satisfy the required conditions.

§ **3.** THEOREM 4. *If* $0 < \alpha < 1$ *and*

(2) $$\int_0^t \varphi(u)du = o(t^{1/\alpha}),$$

then the Fourier series of $f(t)$ is summable (C, α) at $t=x$.

We will also give two proofs.

THE FIRST PROOF OF THEOREM 4. The necessary and sufficient condition that the Fourier series of $f(t)$ is summable (C, α), is that

$$\sigma_\omega^\alpha \equiv \frac{2\omega}{\pi}\int_0^\infty \gamma_{1+\alpha}(\omega u)\varphi(u)du = o(1) \quad (\omega \to \infty). \tag{16}$$

Let $0<r<1$ and $\psi\equiv\psi(\omega)$ tend to ∞ sufficiently slowly. Dividing the integral (16) into two parts,

$$\frac{\pi}{2}\sigma_\omega^\alpha = \omega\int_0^\infty = \omega\int_0^{\psi/\omega^r} + \omega\int_{\psi/\omega^r}^\infty \equiv I_1 + I_2,$$

say, where

$$\begin{aligned} I_2 &= \omega\int_{\psi/\omega^r}^\infty (\omega u)^{-(1+\alpha)}|\varphi(u)|du \\ &= O\left\{\frac{1}{\omega^\alpha}\left(\omega^{(1+\alpha)r}\psi^{-(1+\alpha)} + \sum_{m=1}^\infty \frac{1}{m^{1+\alpha}}\right)\int_0^{2\pi}|\varphi|du\right\} \\ &= O(\omega^{-\alpha+(1+\alpha)r}\psi^{-(1+\alpha)}) = o(\omega^{-\alpha+(1+\alpha)r}) \end{aligned} \tag{17}$$

and

$$\begin{aligned} I_1 &= \omega\int_0^{\psi/\omega^r} \gamma_{1+\alpha}(\omega u)\varphi(u)du \\ &= \omega\Big[\gamma_{1+\alpha}(\omega u)\varphi_1(u)\Big]_0^{\psi/\omega^r} - \omega^2\int_0^{\psi/\omega^r}\gamma'_{1+\alpha}(\omega u)\varphi_1(u)du \\ &\equiv J_1 + J_2, \end{aligned}$$

say. Since $\varphi_1(u)=\int_0^u \varphi(t)dt=o(u^{1/\alpha})$, we have

$$\begin{aligned} J_1 &= \omega\omega^{-(1+\alpha)}\omega^{(1+\alpha)r}\varphi^{-(1+\alpha)}\varphi_1(\psi/\omega^r) \\ &= o(\omega^{-\alpha+(1+\alpha)r}(\psi^{1/\alpha-1-\alpha}/\omega^{r/\alpha})). \end{aligned}$$

We can suppose that

$$\psi^{1/\alpha-1-\alpha}/\omega^{r/\alpha} = o(1) \quad (\omega\to\infty).$$

Thus

$$J_1 = o(\omega^{-\alpha+(1+\alpha)r}). \tag{18}$$

Concerning J_2, we put

$$J_2 = \omega^2\int_0^{1/\omega} + \omega^2\int_{1/\omega}^{\psi/\omega^r} \equiv K_1 + K_2.$$

Then

$$K_1 = \omega^2\int_0^{1/\omega} o(u^{1/\alpha})du = o(\omega^{1-1/\alpha}) = o(1). \tag{19}$$

If we take $\chi=\chi(u)$ such as $\chi(u)$ tends to zero and

$$\varphi_1(u) = o(u^{1/\alpha}\chi),$$

then

$$K_2 = \omega^2\int_{1/\omega}^{\psi/\omega^r} \omega^{-(1+\alpha)}u^{-(1+\alpha)}o(u^{1/\alpha}\chi(u))du$$

$$= o\left(\omega^{1-\alpha}\chi(\omega^{-r/2})\int_{1/\omega}^{\psi/\omega^r} u^{-1-\alpha-1/\alpha}du\right)$$
$$= o(\omega^{1-\alpha-(1/\alpha-\alpha)r}\chi(\omega^{-r/2})\psi^{1/\alpha-\alpha}).$$

If we suppose
$$\chi(\omega^{-r/2})\psi(\omega)^{1/\alpha-\alpha} = O(1) \quad (\omega \to \infty),$$
which is always possible, then

(20) $$K_2 = o(\omega^{(1-\alpha)-(1/\alpha-\alpha)r}).$$

Let us take $r \equiv \alpha/(1+\alpha)$. Then, by (17), (18), (19) and (20),
$$\frac{\pi}{2}\sigma_\omega^\alpha = I_1 + I_2 = (J_1 + J_2) + I_2$$
$$= (J_1 + K_1 + K_2) + I_2 = o(1).$$
Thus the theorem is completely proved.

REMARK. Hsiang [3] has proved that if

(*) $$\lim_{t\to 0}\int_t^\pi \varphi(u)/u^{1/\alpha}du$$

exists, then the Fourier series is summable $(C, 1)$, but not summable (C, β) for $0<\beta<\alpha$. Since
$$\int_0^t \varphi(u)du = t^{1/\alpha}\int_t^\pi \frac{\varphi(u)}{u^{1/\alpha}}du - \frac{1}{\alpha}\int_u^t u^{\frac{1-\alpha}{\alpha}}du\int_u^\pi \frac{\varphi(v)}{v^{1/\alpha}}dv,$$
(*) implies (2). Hence Theorem 4 shows that if (*) holds, then the Fourier series is summable (C, α), Theorem 4 has early proved by Wang [7]. But by the method used here, we can generalize the Wang theorem in the following form.

THEOREM 5. *If* $\gamma>\beta\geq 1$, *and*
$$\Phi_\beta(t) \equiv \frac{1}{\Gamma(\beta)}\int_0^t (t-u)^{\beta-1}\varphi(u)du = o(t^\gamma),$$
then the Fourier series of $f(t)$ *is summable* $(C, \beta-(\gamma-\beta)/(\gamma-\beta+1))$.

PROOF. Put $\gamma-\beta=\eta>0$, then the theorem is equivalent to

(21) $$\varphi_\beta(t) \equiv \frac{1}{t}\int_0^t\left(1-\frac{u}{t}\right)_0^{\beta-1}\varphi(u)du = o(t^\eta),$$

implies $(C, \beta-\eta/(1+\eta))$-summability.

Using the formula in the proof of Theorem 2, we have
$$\sigma^\alpha(\omega) = \omega\int_0^1 \varphi_\delta(t)J_\alpha^\delta(\omega t)dt + o(1)$$
$$= I(\omega) + o(1)$$
and
$$J_\alpha^\delta(x) \sim \frac{\Gamma(\alpha+1)}{\Gamma(\delta+1)}\frac{\cos\left\{x-\dfrac{\pi}{2}(1+\alpha+\delta)\right\}}{x^{1+\alpha-\delta}} + O\left(\frac{1}{x^2}\right)$$
as $x \to \infty$.

If we put $\beta=\delta+1$, then (21) is equivalent to

(22) $$\varphi_{\delta+1}(u) = o(t^\eta).$$

So $\varphi_\delta(u)$ is integrable in the sense of Cauchy-Lebesgue. Put

$$\Phi(t) \equiv \int_0^t \varphi_\delta(u)du,$$

then by (22), we get

$$\Phi(t) = o(t^{1+\eta}).$$

Then

$$I(\omega) \equiv \omega \int_0^1 \varphi_\delta(t) J_\alpha^\delta(\omega t)dt = \omega \int_0^{\omega^{-r}\psi} + \omega \int_{\omega^{-r}\psi}^1$$
$$= I_1 + I_2$$

say, Firstly

$$I_2 = \omega \int_{\omega^{-r}\psi}^1 \varphi_\delta(t) J_\alpha^\delta(\omega t)dt$$
$$= \Big[\omega\Phi(t)\, J_\alpha^\delta(\omega t)\Big]_{\omega^{-r}\psi}^1 - \omega^2 \int_{\omega^{-r}\psi}^1 \Phi(t) J_\alpha'^\delta(\omega t)dt.$$

If we assume

$$\varepsilon = \alpha - \delta > 0 \quad \text{and} \quad r = \varepsilon/(1+\varepsilon),$$

then by applying the second mean value theorem, we have

$$I_2 = o(\omega^{-\varepsilon+r(1+\varepsilon)}) + \omega^2 \int_{\omega^{-r}\psi}^1 \Phi(t) \frac{\sin\left\{\omega t - \frac{\pi}{2}(1+\alpha+\delta)\right\}}{(\omega t)^{1+\alpha-\delta}} dt + o(1)$$
$$= o(1) + \omega^{1-\varepsilon}\omega^{r(1+\varepsilon)}\psi^{-(1+\varepsilon)} \int_{\omega^{-r}\theta}^1 \left|\sin\left\{\omega t - \frac{\pi}{2}(1+\alpha+\delta)\right\}\right| dt$$
$$= o(1) + O(\omega^{-\varepsilon}\omega^{\varepsilon}\psi^{-(1+\varepsilon)}) = o(1).$$

Next we get

$$I_1 = \omega \int_0^{\omega^{-r}\psi} \varphi_\delta(t) J_\alpha^\delta(\omega t)dt$$
$$= o(\omega^{1-\varepsilon+(1+\eta-\varepsilon)r}) = o(1),$$

by the analogous method to the proof of Theorem 4, for the kernel is same order. The order of summabllity α is determind by

$$\alpha = \delta + 1/(1+\eta) = \beta - 1 + 1/(1+\eta) = \beta - \eta/(1+\eta)$$

where

$$\varepsilon = \alpha - \delta = 1/(1+\eta).$$

REMARK. The order of summability by Wang's theorem is

$$\beta - \eta(\beta + 1 - n)/(n+\eta),$$

where $n \geqq \gamma > n-1$ and $n \geqq 2$. Since

$$\frac{1}{1+\eta} > \frac{1+\beta-n}{n+\eta}, \quad (\text{for } n-\beta > 0,\ n \geqq 2)$$

our theorem is better than Wang's.

THE SECOND PROOF OF THEOREM 4. Let the Cesàro mean of Fourier series of $f(t)$ of the α-th order σ_n^α. Then

$$\sigma_n^\alpha = \frac{1}{\pi}\int_0^\pi \varphi(t) K_n^\alpha(t)\,dt,$$

$K_n^\alpha(t)$ being Fejér kernel. It is known that

$$(23)\qquad \begin{cases} |K_n^\alpha(t)| \leqq C/n^\alpha t^{1+\alpha} & (nt \geqq 1), \\ |[K_n^\alpha(t)]'| \leqq n^2, & \\ |[K_n^\alpha(t)]'| \leqq Cn^{1-\alpha}/t^{1+\alpha} & (nt \geqq 1), \end{cases}$$

C being an absolute constant. This is proved for $1<\alpha<2$ by Zygmund [10, p. 48 and p. 56] using Abel's transformation twice, but in our case it is sufficient to use it once. Now

$$\begin{aligned} \sigma_n^\alpha = &-\left[\int_0^{1/n} + \int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}}\right][K_n^\alpha(t)]'\varphi_1(t)\,dt \\ &+ \Big[\varphi_1(t) K_n^\alpha(t)\Big]_0^{\psi/n^{\alpha/(1+\alpha)}} + \int_{\psi/n^{\alpha/(1+\alpha)}}^\pi \varphi(t) K_n^\alpha(t)\,dt \\ = &\; I_1 + I_2 + I_3 + I_4, \end{aligned}$$

say, where $\psi \equiv \psi(n)$ increases indefinitely and sufficiently slowly. Using (23),

$$I_1 = o\left(n^2 \int_0^{1/n} t^{1/\alpha}dt\right) = o(n^2/n^{1+1/\alpha}) = o(1)$$

$$I_2 = o\left(n^{-1-\alpha}\int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}} t^{1/\alpha-\alpha-1}dt\right) = o(1)$$

$$I_3 = o\left(\Big[t^{1/\alpha}K_n^\alpha(t)\Big]_0^{\psi/n^{\alpha/(1+\alpha)}}\right) = o\left(\frac{\psi^{1/\alpha-\alpha-1}}{n^{1/(1+\alpha)}}\right)$$

$$I_4 = O\left(\frac{1}{n^\alpha}\int_{\psi/n^{\alpha/(1+\alpha)}}^\pi \frac{|\varphi|}{t^{1+\alpha}}\right) = O\left(\frac{1}{\psi}\int_0^\pi |\varphi|\,dt\right) = O\left(\frac{1}{\psi}\right).$$

Thus we get $\sigma_n^\alpha = o(1)$.

THEOREM 6. *In Theorem 4, o in (2) connot be replaced by O.*

Proof runs similarly as Theorem 3. (cf. the succeeding paper, Izumi [5]).

REMARK. Theorem 6 is better than the second part of Hsiang's theorem [3]. For, if $\varphi_1(u) = O(u^{1/\beta})\,(0<\beta<\alpha)$, then

$$\int_\eta^t \frac{\varphi(u)}{u^{1/\alpha}}du = \left[\frac{\varphi_1(u)}{u^{1/\alpha}}\right]_\eta^t + \frac{1}{\alpha}\int_\eta^t \frac{\varphi_1(u)}{u^{1/\alpha+1}}du$$

exists as $\eta \to 0$.

§ **4.** We can now generalize Theorem 5.

THEOREM 7. *If* $0<\beta<\gamma$ *and*

$$\Phi_\beta(t) \equiv \frac{1}{\Gamma(\beta)}\int_0^t (t-u)^{\beta-1}\varphi(u)\,du = o(t^\gamma),$$

then the Fourier series of $f(t)$ *is* (C, α) *summable at* $t = x$, *where*

$$\alpha > \beta/(\gamma - \beta + 1).$$

PROOF. It is known that

$$s_n = O(n^{\gamma/(\beta+1)})$$

(F. T. Wang [9]) and

$$\sigma^{(\gamma+\varepsilon)} = (n^{\beta-\gamma}) \quad (\varepsilon > 0)$$

(F. T. Wang [7] and Hyslop [4]). Thus by Riesz's convexity theorem [6], we get the theorem.

THEOREM 8. *If* $0<\beta<\gamma, \beta\leqq 1+(\gamma-\beta)$ *and*

$$\Phi_\beta(t) = o(t^\gamma),$$

then the Fourier series of $f(t)$ *is summable* $(C, \beta/(\gamma-\beta+1))$ *at* $t=x$.

REMARK. It is conjectured that the condition $\beta\leqq 1+(\gamma-\beta)$ is superfluous, that is, may be taken such as

$$\alpha = \beta/(\gamma-\beta+1)$$

in Theorem 7. We could prove this for integreal β.

PROOF. We will begin by the case $0<\beta<1$. This case is contained in the next case, but the proof of this case suggests that of the general case. Let us consider the Cesàro mean of Fourier series of $f(t)$ order α, which we denote by σ_n^α. We have, putting $\alpha\equiv\beta/(\gamma-\beta+1)$,

$$\begin{aligned}
\sigma_n^\alpha &= \frac{1}{\pi}\int_0^\pi \varphi(t) K_n^\alpha(t)\,dt \\
&= \frac{1}{\pi}\int_0^{\psi/n^{\alpha/(1+\alpha)}} \varphi(t) K_n^\alpha(t)\,dt + \frac{1}{\pi}\int_{\psi/n^{\alpha/(1+\alpha)}}^\pi \varphi(t) K_n^\alpha(t)\,dt \\
&= -\frac{1}{\pi}\int_0^{\pi/n^{\alpha/(1+\alpha)}} \Phi_1(t)[K_n^\alpha(t)]'dt + o(1),
\end{aligned}$$

as in the proof of Theorem 4. If we denote the last integral by I, then

$$\begin{aligned}
I &= \int_0^{\psi/n^{\alpha/(1+\alpha)}} [K_n^\alpha(t)]'dt \int_0^t \Phi_\beta(u)(t-u)^{-\beta}du \\
&= \int_0^{1/n} dt \int_0^t du + \int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}} dt \int_0 du \equiv I_1 + I_2,
\end{aligned}$$

say. By (22)

$$\begin{aligned}
I_1 &= O\Big(n^2 \int_0^{1/n} dt \int_0^t u^\gamma (t-u)^{-\beta} du\Big) \\
&= O\Big(n^2 \int_0^{1/n} t^{\gamma-\beta+1} dt\Big) = O(n^2/n^{\gamma-\beta+2}) = O(1/n^{\gamma-\beta}) \\
&= o(1).
\end{aligned}$$

Concerning I_2,

$$\begin{aligned}
I_2 &= \int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}} [K_n^\alpha(t)]'dt \int_0^t \Phi_\beta(u)(t-u)^{-\beta}du \\
&= \int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}} du \int_u^{u+1/n} dt + \int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}-1/n} du \int_{u+1/n}^{\psi/n^{\alpha/(1+\alpha)}} dt \\
&\quad + \int_0^{1/n} du \int_{1/n}^{u+1/n} dt - \int_{\psi/n^{\alpha/(1+\alpha)}-1/n}^{\psi/n^{\alpha/(1+\alpha)}} du \int_{\psi/n^{\alpha/(1+\alpha)}}^{u+1/n} dt
\end{aligned}$$

$$\equiv J_1 + J_2 + J_3 - J_4,$$

say. By (23)

$$J_1 = o\left(\int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}} u^\gamma du \int_u^{u+1/n} n^{1-\alpha} t^{-(1+\alpha)} (t-u)^{-\beta} dt\right)$$

$$= o\left(n^{1-\alpha} \int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}} u^{\gamma-\alpha-1} du \int_u^{u+1/n} (t-u)^{-\beta} dt\right)$$

$$= o\left(\frac{n^{1-\alpha}}{n^{1-\beta}} \int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}} u^{\gamma-\alpha-1} du\right) = 0\left(n^{\beta-\alpha-\frac{\alpha}{1+\alpha}(\gamma-\alpha)}\right)$$

$$= o(1),$$

for $\gamma > \alpha$ and $(1+\alpha)(\beta-\alpha)-\alpha(\gamma-\alpha)=0$. Now

$$J_2 = \int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}-1/n} \Phi_\beta(u) du \int_{u+1/n}^{\psi/n^{\alpha/(1+\alpha)}} [K_n^\alpha(t)]'(t-u)^{-\beta} dt$$

$$= \int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}-1/n} \Phi_\beta(n) du \left\{\left[K_n^\alpha(t)(t-u)^{-\beta}\right]_{u+1/n}^{\psi/n^{\alpha/(1+\alpha)}} - \int_{u+1/n}^{\psi/n^{\alpha/(1+\alpha)}} K_n^\alpha(t)(t-u)^{-\beta-1} dt\right\}.$$

By (21), we have

$$J_2 = o\left[\int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}} u^\gamma du \left\{\frac{n^{\beta-\alpha}}{(u+1/n)^{\alpha+1}} + \frac{1}{\psi^{\alpha+1}}\left(\frac{\psi}{n^{\alpha/(1+\alpha)}} - u\right)^{-\beta} + \frac{1}{n^\alpha}\int_{u+1/n}^{\psi/n^{\alpha/(1+\alpha)}} \frac{dt}{t^{\alpha+1}(t-u)^{\beta+1}}\right\}\right]$$

$$= o\left(n^{\beta-\alpha} \int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}} u^{\gamma-\alpha-1} du + \frac{1}{\psi^{\alpha+1}} \int_0^{\psi/n^{\alpha/(1+\alpha)}} u^\gamma \left(\frac{\psi}{n^{\alpha/(1+\alpha)}} - u\right)^{-\beta} du + \frac{1}{n^\alpha} \int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}} u^{\gamma-\alpha-\beta-1} du\right)$$

$$= o\left(n^{\beta-\alpha-\frac{\alpha}{1+\alpha}(\gamma-\alpha)} + \frac{1}{\psi^{\alpha+1}} + \frac{1}{n^{\gamma-\beta}} + \frac{1}{n^{\alpha+\frac{\alpha}{1+\alpha}(\gamma-\alpha-\beta)}}\right)$$

$$= o(1).$$

Since $J_3 + J_4 = o(1)$, we get $I_2 = o(1)$, and then $I = o(1)$. Thus the theorem is proved for the case $0 < \beta < 1$.

Let us now turn to the case $0 < \alpha = \beta/(\gamma-\beta+1) \leqq 1$. There is an integer $k > 1$ such that $k-1 \leqq \beta < k$. We suppose that $k-1 < \beta < k$, for the case $\beta = k$ can be easily deduced by the following argument. As we have already seen,

$$\sigma_n^\alpha = \frac{1}{\pi} \int_0^{\psi/n^{\alpha/(1+\alpha)}} \varphi(t) K_n^\alpha(t) dt + o(1).$$

By k time application of integration by parts, the last integral, which we denote by I', becomes

$$I' = (-1)^k \int_0^{\psi/n^{\alpha/(1+\alpha)}} \Phi_k(t)[K_n^\alpha(t)]^{(k)} dt + \sum_{h=0}^{k-1} \left[\Phi_{h+1}(t)[K_n^\alpha(t)]^{(h)}\right]_{t=0}^{\psi/n^{\alpha/(1+\alpha)}}$$

$$\equiv (-1)^k I_1' + I_2',$$

say. Now, since

$$\Phi_1(t) = o(1), \quad \Phi_\beta(t) = o(t^\gamma),$$

we have

$$\Phi_{h+1}(t) = o(t^{\gamma h/(\beta-1)})$$

by the M. Riesz theorem [6]. On the other hand, by Zygmund [10, p. 259], we have

$$[K_n^\alpha(t)]^{(h)} = O\Big(\frac{n^{h-\alpha}}{t^{\alpha+1}} + \frac{n^{h-s}}{t^{s+1}} + \sum_{j=1}^{s} \frac{1}{n^j t^{j+h+1}}\Big) \tag{24}$$

for $nt \geqq 1$, s being sufficiently large integer and

$$[K_n^\alpha(t)]^{(h)} = O(n^{h+1}) \tag{25}$$

for all t. Since $0 < \alpha \leqq 1$, we have, for $h \geqq 0$,

$$\frac{n^{h-\alpha}}{t^{\alpha+1}} \geqq \sum_{j=1}^{s} \frac{1}{n^j t^{j+h+1}}.$$

Hence we have

$$[K_n^\alpha(t)]^{(h)} = O(n^{h-\alpha}/t^{\alpha+1}). \tag{26}$$

Thus

$$\begin{aligned}[\Phi_{h+1}(t)\{K_n^\alpha(t)\}^{(h)}]_{t=\psi/n^{\alpha/(1+\alpha)}} &= o\big([t^{\frac{\gamma}{\beta-1}h-\alpha-1} n^{h-\alpha}]_{t=\psi/n^{\alpha/(1+\alpha)}}\big) \\ &= o\,\big(\psi^{\frac{\gamma}{\beta-1}h-\alpha-1}/n^{(\frac{\gamma}{\beta-1}\frac{\alpha}{1+\alpha}-1)h}\,\big) = o(1)\end{aligned}$$

for $h \geqq 0$. Therefore $I_2' = o(1)$.

$$\begin{aligned} I_1' &= \int_0^{\psi/n^{\alpha/(1+\alpha)}} \Phi_k(t)[K_n^\alpha(t)]^{(k)} dt \\ &= \int_0^{\psi/n^{\alpha/(1+\alpha)}} [K_n^\alpha(t)]^{(k)} dt \int_0^t \Phi_\beta(u)(t-u)^{k-\beta-1} du \\ &= \int_0^{\psi/n^{\alpha/(1+\alpha)}} \Phi_\beta(u) du \int_u^{\psi/n^{\alpha/(1+\alpha)}} [K_n^\alpha(t)]^{(k)} (t-u)^{k-\beta-1} dt \\ &= \int_0^{\psi/n^{\alpha/(1+\alpha)}} du \int_u^{u+1/n} dt + \int_0^{\psi/n^{\alpha/(1+\alpha)}-1/n} du \int_{u+1/n}^{\psi/n^{\alpha/(1+\alpha)}} dt \\ &\qquad - \int_{\psi/n^{(1+\alpha)}-1/n}^{\psi/n^{\alpha/(1+\alpha)}} du \int_{\psi/n^{\alpha/(1+\alpha)}}^{u+1/n} dt \\ &\equiv J_1' + J_2' - J_3', \end{aligned}$$

say, and

$$J_1' = \int_0^{1/n} du \int_u^{u+1/n} dt + \int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}} du \int_u^{u+1/n} dt \equiv K_1' + K_2',$$

say. Then we have, by (25),

$$K_1' = o\Big(n^{k+1} \int_0^{1/n} u^\gamma du \frac{1}{n^{k-\beta}}\Big) = o\Big(\frac{1}{n^{\gamma-\beta}}\Big) = o(1),$$

and, by (26)

$$K_2' = o\Big(\int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}} u^\gamma du \int_u^{u+1/n} \frac{n^{k-\alpha}}{t^{1+\alpha}} (t-u)^{k-\beta-1} dt\Big)$$

$$= o\left(n^{k-\alpha}\int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}} u^{\gamma-\alpha-1}du\int_u^{u+1/n}(t-u)^{k-\beta-1}dt\right)$$
$$= o\left(\frac{n^{k-\alpha}}{n^{k-\beta}}\int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}} u^{\gamma-\alpha-1}du\right)$$
$$= o(n^{\beta-\alpha-\frac{\alpha}{1+\alpha}(\gamma-\alpha)}) = o(1).$$

Hence $J_1'=o(1)$.

Concerning J_2', if we use integration by parts k times in the inner integral, then we have

$$J_2' = \int_0^{\psi/n^{\alpha/(1+\alpha)}-1/n}\Phi_\beta(u)du\int_{u+1/n}^{\psi/n^{\alpha/(1+\alpha)}}[K_n^\alpha(t)]^{(k)}(t-u)^{k-\beta-1}dt$$
$$= \int_0^{\psi/n^{\alpha/(1+\alpha)}-1/n}\Phi_\beta(u)du\left\{(-1)^k C\int_{u+1/n}^{\psi/n^{\alpha/(1+\alpha)}} K_n^\alpha(t)(t-u)^{-\beta-1}du\right.$$
$$\left. + \sum_{h=0}^{k-1}C_h\left[[K_n^\alpha(t)]^{(h)}(t-u)^{h-\beta}\right]_{t=u+1/n}^{\psi/n^{\alpha/(1+\alpha)}}\right\}$$
$$\equiv L' + \sum_{h=0}^{k-1}L_h',$$

say, where C and C_h are constants arising by differentiation.

By (21)

$$L' = o\left(\int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}} u^\gamma du\int_{u+1/n}^{\psi/n^{\alpha/(1+\alpha)}}\frac{dt}{n^\alpha t^{1+\alpha}(t-u)^{\beta+1}}\right)$$
$$= o\left(\frac{1}{n^\alpha}\int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}} u^{\gamma-\alpha-\beta-1}du\right)$$
$$= o\left(\frac{1}{n^{\gamma-\beta}} + \frac{1}{n^{\alpha(\gamma-\beta+1)/(1+\alpha)}}\right) + o(1) = o(1),$$

and by (26)

$$L_h' = o\left(\frac{n^{h-\alpha}}{n^{h-\beta}}\int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}} u^{\gamma-\alpha-1}du + \frac{n^h}{\psi^{\alpha+1}}\int_{1/n}^{\psi/n^{\alpha/(1+\alpha)}-1/n} u^\gamma\left(\frac{\psi}{n^{\alpha/(1+\alpha)}}-u\right)^{h-\beta}du\right)$$
$$= o(n^{\beta-\alpha-\frac{\alpha}{1+\alpha}(\gamma-\alpha)} + n^{\beta-\frac{\alpha}{1+\alpha}(\gamma+1)}) = o(1).$$

Thus $J_2'=o(1)$. Since we have easily $J_3'=o(1)$, we get $I'=o(1)$, which is the required.

Literature.

[1] Bosanquet, L. S., On the Cesàro summation of Fourier series and allied series, Proc. London Math. Soc., 37 (1934), 17-32.

[2] Hobson, E. W., The theory of functions of a real variable, Vol. II, Cambridge (1925).

[3] Hsiang, F. C., The summability $(C, 1-\varepsilon)$ of Fourier series, Duke Math. Journ., 13 (1946), 43-50.

[4] Hyslop, J. M., Note on a group of theorems in the theory of Fourier series, Journ.

London Math. Soc., 24 (1949) 91–100.
[5] IZUMI, S., Notes on Fourier Analysis (XXXIII): Negative examples in the theory of Fourier series, in the press.
[6] RIESZ, M., Sur un théorème de la moyenne et ses applications, Acta Szeged, 1 (1922), 114–126.
[7] WANG, F.T., A note on Cesàro summability of Fourier series, Annals of Math., 44 (1943), 397–400.
[8] WANG, F.T., A remark on (C) summability of Fourier series, Journ. London Math. Soc., 22 (1947), 40–47.
[9] WANG, F.T., On the convergence factor of Fourier series at a point, II, Science Rep. Tôhoku Imp. Univ., 24 (1935), 665–696.
[10] ZYGMUND, A., Trigonometrical series, Warsawa (1935).

Mathematical Institute,
Tôhoku University, Sendai.