NOTES ON FOURIER ANALYSIS (XXXVI): ON CERTAIN APPLICATIONS OF WIENER'S TAUBERIAN THEOREMS*)

By

GEN-ICHIRÔ SUNOUCHI

In this note the author gives two applications of Wiener's theorem. The general convergence theorem and its converse are discussed in § 1. Partial solution of the problem of Cheng [4] is also given. In § 2 the Cesàro summability problem of multiple Fourier series is discussed, to which the quasi-Tauberian theorem is applied. § 1 and § 2 are closely related but may be read independently.

1. A general convergence theorem and its converse.

We shall begin to state one of the fundamental theorem of Wiener.

THEOREM 1. *Let $\varphi(\lambda)$ be a function of bounded total variation over every interval $(\varepsilon, 1/\varepsilon)$ where $0<\varepsilon<1$, and let*

$$(1) \qquad \int_u^{2u} \lambda^{-1}|d\varphi(\lambda)| \leqq C \quad (0 < u < \infty).$$

Let $\lambda N_1(\lambda)$ be a continuous function for which

$$(2) \qquad \sum_{k=-\infty}^{\infty} \max_{2^k \leqq \lambda \leqq 2^{k+1}} \lambda |N_1(\lambda)| < \infty,$$

and let

$$(3) \qquad \lim_{\lambda\to\infty(0)} \frac{1}{\lambda}\int_0^\infty N_1\left(\frac{\mu}{\lambda}\right) d\varphi(\mu) = A\int_0^\infty N_1(\mu)\,d\mu.$$

Let for any real u

$$(4) \qquad \int_0^\infty N_1(\lambda)\lambda^{iu}d\lambda \neq 0.$$

Then if $\lambda N_2(\lambda)$ is any continuous function for which

$$(5) \qquad \sum_{k=-\infty}^{\infty} \max_{2^k \leqq \lambda \leqq 2^{k+1}} \lambda |N_2(\lambda)| < \infty,$$

we have

$$(6) \qquad \lim_{\lambda\to\infty(0)} \frac{1}{\lambda}\int_0^\infty N_2\left(\frac{\mu}{\lambda}\right) d\varphi(\mu) = A\int_0^\infty N_2(\mu)\,d\mu.$$

This theorem is a transformation of Theorem IX of Wiener [10] (p. 26). To see this it is sufficient to put

*) Received March 26, 1950.

$$\xi = \log \lambda, \qquad f(\xi) = \int \lambda^{-1} d\varphi(\lambda), \quad K_{1,2}(\xi) = \lambda N_{1,2}(\lambda)$$

or
$$\xi = -\log \lambda, \quad f(\xi) = \int \lambda^{-1} d\varphi(\lambda), \quad K_{1,2}(\xi) = \lambda N_{1,2}(\lambda).$$

THEOREM 1.1. *Let $f(t)$ be integrable $(0, B)$ and let*

$$\frac{1}{T^q}\int_0^T |f(t)|dt \leqq C, \text{ for all } T > 0, \quad (q > 0). \tag{7}$$

Let $t^q K(t)$ be a cintinuous function for which

$$\sum_{k=-\infty}^{\infty} \max_{2^k \leqq t \leqq 2^{k+1}} t^q |K(t)| < \infty. \tag{8}$$

Then

$$\lim_{T\to\infty(0)} \frac{1}{T^q}\int_0^T f(t)\left(1 - \frac{t}{T}\right)^\alpha dt = l\Gamma(\alpha+1), \text{ for some } \alpha \geqq 0, \tag{9}$$

implies

$$\lim_{R\to\infty(0)} \frac{1}{R^q}\int_0^\infty f(t)K\left(\frac{t}{R}\right)dt = l\frac{\Gamma(q+\alpha)}{\Gamma(q)}\int_0^\infty K(t)t^{q-1}dt. \tag{10}$$

PROOF. Put

$$\int_{\lambda_0}^{\lambda} f(t)/t^{q-1}dt = \varphi(\lambda), \tag{11}$$

then

$$\int_u^{2u} \lambda^{-1}|d\varphi(\lambda)| = \int_u^{2u} |f(t)|/t^q \leqq \frac{1}{u^q}\int_0^{2u} |f(t)|dt \leqq 2^q C \tag{12}$$

by (7). Thus the condition (1) of Theorem 1 is valid.

Put

$$N_1(\lambda) = \begin{cases} \lambda^{q-1}(1-\lambda)^\alpha/\Gamma(\alpha+1), & \text{if } 0 \leqq \lambda \leqq 1, \quad (\alpha > 0) \\ 0 & \text{if } 1 < \lambda, \end{cases} \tag{13}$$

then we have

$$\sum_{k=-\infty}^{\infty} \max_{2^k \leqq \lambda \leqq 2^{k+1}} \lambda N_1(\lambda) = C\sum_{k=-\infty}^{\infty} \max_{2^k \leqq \lambda \leqq 2^{k+1}} \lambda^q (1-\lambda)^\alpha < \infty, \tag{14}$$

$$\begin{aligned} &\lim_{\lambda\to\infty(0)} \frac{1}{\lambda\Gamma(\alpha+1)}\int_0^\infty N_2\left(\frac{\mu}{\lambda}\right)d\varphi(\mu) \\ &= \lim_{\lambda\to\infty(0)} \frac{1}{\lambda\Gamma(\alpha+1)}\int_0^\infty \frac{f(\mu)}{\mu^{q-1}}\left(\frac{\mu}{\lambda}\right)^{q-1}\left(1-\frac{\mu}{\lambda}\right)^\alpha d\mu \\ &= \lim_{\lambda\to\infty(0)} \frac{1}{\lambda^q\Gamma(\alpha+1)}\int_0^\infty f(\mu)\left(1-\frac{\mu}{\lambda}\right)^\alpha d\mu = l, \quad (\text{by } (9)) \end{aligned} \tag{15}$$

and

$$\frac{1}{\Gamma(\alpha+1)}\int_0^1 \lambda^{q-1}(1-\lambda)^\alpha \lambda^{iu} d\lambda = \frac{\Gamma(q+iu)}{\Gamma(q+\alpha+iu+1)} \neq 0. \tag{16}$$

Thus the conditions (2), (3) and (4) of Theorem 1 are valid. Putting

$$N_2(\lambda) = \lambda^{q-1}K(\lambda),$$

we get the theorem.

THEOREM 1.2. *Let n be a positive integer, $\mu > 0$ and $0 < q < n(\mu + 1/2)$.*

Let, in addition to (7),

(17)
$$\lim_{T\to\infty(0)} \frac{1}{T^q}\int_0^T f(t)\,dt = l,$$

then we have

(18)
$$\lim_{R\to 0(\infty)} R^q\int_0^\infty f(t)\left\{\frac{J_\mu(Rt)}{(Rt)^\mu}\right\}^n dt = lq\int_0^\infty \left\{\frac{J_\mu(t)}{t^\mu}\right\}^n t^{q-1}dt.$$

PROOF. In Theorem 1.1, put $\alpha=0$, and

(19)
$$K(t) = \{J_\mu(t)/t^\mu\}^n.$$

Since,

$$J_\mu(t) = \begin{cases} O(t^\mu), & \text{as } \mu\to 0,\\ O(t^{-1/2}), & \text{as } \mu\to\infty,\end{cases}$$

we have

(20)
$$\sum_{k=-\infty}^{\infty} \max_{2^k\leq t\leq 2^{k+1}} t^q|K(t)|<\infty,$$

if $0<q<n(\mu+1/2)$.

THEOREM 1.3. *Let* $\mu>0$ *and* $0<q<\mu+1/2$. *Let in addition to* (7),

(21)
$$\lim_{R\to 0(\infty)} R^q\int_0^\infty f(t)\frac{J_\mu(Rt)}{(Rt)^\mu}dt = l\frac{\Gamma(q+\alpha)}{\Gamma(q)}\int_0^\infty\left\{\frac{J_\mu(t)}{t^\mu}\right\}t^{q-1}dt$$

then we have

(22)
$$\lim_{T\to\infty(0)}\frac{1}{T^q}\int_0^T f(t)\left(1-\frac{t}{T}\right)^\alpha dt = l\Gamma(\alpha+1),\quad (\alpha>0).$$

PROOF. Put, in Theorem 1,

$$\varphi(\lambda) = \int_0^\lambda f(t)/t^{q-1}dt,$$

$$N_1(\lambda) = \lambda^{q-1}J_\mu(\lambda)/\lambda^\mu = J_\mu(\lambda)\lambda^{-\mu+q-1},$$

and

$$N_2(\lambda) = \begin{cases}\lambda^{q-1}(1-\lambda)^\alpha/\Gamma(\alpha+1), & 0\leqq\lambda\leqq 1 \quad (\alpha>0)\\ 0, & \text{otherwise.}\end{cases}$$

Since

(23)
$$\int_0^\infty N_1(\lambda)\lambda^{iu}d\lambda = \int_0^\infty\frac{J_\mu(\lambda)}{\lambda^\mu}\lambda^{q+iu-1}d\lambda = \frac{\Gamma\{(q+iu)/2\}}{2^{\mu-(q+iu)}\Gamma\{\mu-(q+iu)/2+1\}}\neq 0,$$

$(0<q<\mu+1/2)$. (Watson [9], p. 391).

Other conditions of Theorem 1 are evident.

Theorem 1.1 is one of the so-called general convergence theorem and many writers have discussed conditions of validity. The condition of our theorem is the best possible one in a sense. Theorem 1.2 is proved by Cheng [4] by direct calculation. The special case of $n=2$, $\mu=1/2$ is Jacob's generalization of Wiener's formula. Littauer [7] has applied the Tauberian method in this case, but his proof is elliptical, since his $R(\xi)$ is not integrable over $(-\infty,\infty)$ for $\alpha=0$. In the case $\alpha=0$ of Theorem 1.3, more stringent condition is required.

THEOREM 2. *Let, in addition to the conditions of Theorem 1,*

$$\varphi(\lambda) = \int_{\lambda_0}^{\lambda} g(\mu)d\mu \tag{24}$$

and

$$g(\lambda) \geqq 0, \quad \textit{for all} \quad \lambda > 0, \tag{25}$$

then

$$\lim_{\lambda \to 0(\infty)} \frac{1}{\lambda} \int_0^{\lambda} g(\mu)d\mu = A.$$

Proof is the repetition of Wiener's argument ([10], p. 31).

THEOREM 2.1. *Let $f(t)$ be integrable $(0, B)$ and let, in addition to (7) and (8) where $t^q K(t)$ is continuous,*

$$f(t) \geqq 0, \quad \textit{for all} \quad t > 0$$

and

$$\int_0^{\infty} K(t)t^{q-1}t^{iu}dt \neq 0, \quad \textit{for all real } u. \tag{26}$$

Then

$$\lim_{R \to \infty(0)} \frac{1}{R^q} \int_0^{\infty} f(t) K\left(\frac{t}{R}\right)dt = lq \int_0^{\infty} K(t)t^{q-1}dt \tag{10'}$$

implies

$$\lim_{T \to \infty(0)} \frac{1}{T^q} \int_0^{T} f(t)dt = l. \tag{9'}$$

PROOF. Since

$$\frac{1}{R^q} \int_0^{\infty} f(t) K\left(\frac{t}{R}\right)dt = \frac{1}{R} \int_0^{\infty} \frac{f(t)}{t^{q-1}} \left(\frac{t}{R}\right)^{q-1} K\left(\frac{t}{R}\right)dt,$$

we put

$$g(t) = f(t)/t^{q-1} \geqq 0,$$

and

$$N_1(t) = t^{q-1}K(t),$$

then (7) implies (1) and we get the theorem.

Since in the problem of Cheng [4] $K(t) = \{J_\mu(t)/t^\mu\}^n$, if $\mu > 0$ and $0 < q < n(\mu + 1/2)$, then (8) is valid, for

$$J_\mu(t) = \begin{cases} O(t^\mu) & \text{as} \quad t \to 0 \\ O(t^{-1/2}) & \text{as} \quad t \to \infty. \end{cases}$$

Consequently the validity of the conjecture depends only on non-vanishing of

$$\int_0^{\infty} \left\{\frac{J_\mu(x)}{x^\mu}\right\}^n x^{q-1} x^{iu} dx$$

for all real u. Especially the cases $n=1$ and $n=2$ are evident. For, in the case $n=1$ by (23), and in the case $n=2$, by

$$\int_0^{\infty} \frac{J_\mu^2(x)}{x^{2\mu-(k+q+iu)+1}} dx \tag{27}$$

$$= \frac{2^{(k+q+iu)-2\mu}\Gamma\{2\mu-(k+q+iu)+1\}\Gamma\{(k+q+iu)/2\}}{2\Gamma\{[2\mu-(k+q+iu)+1]/2+1/2\}^2\Gamma\{\mu+[2\mu-(k+q+iu)+1]/2+1/2\}},$$

$(2\mu+1>2\mu-(k+q)+1>0)$. (Watson [9] p. 397).

Thus we get

THEOREM 2.2. *Let* $0<q<(\mu+1/2)$. *Let*

(28) $$\frac{1}{T^q}\int_0^T|f(t)|dt \leqq M, \quad \textit{for all } T>0$$

and

(29) $$f(t) \geqq 0, \quad \textit{for all } t.$$

Then

(30) $$\lim_{R\to 0(\infty)} R^{q-\mu}\int_0^\infty f(t)\left\{\frac{J_\mu(Rt)}{t^\mu}\right\}dt = lq\int_0^\infty \frac{J_\mu(t)}{t^\mu}t^{q-1}dt$$

implies

(31) $$\lim_{T\to\infty(0)}\frac{1}{T^q}\int_0^T f(t)dt = l.$$

THEOREM 2.3. *Let* $0<q<2(\mu+1/2)$ *and let* (28), *and* (29). *Then*

(32) $$\lim_{R\to 0(\infty)} R^{q-2\mu}\int_0^\infty f(t)\left\{\frac{J_\mu(Rt)}{t^\mu}\right\}^2 dt = lq\int_0^\infty \left\{\frac{J_\mu(t)}{t^\mu}\right\}^2 t^{q-1}dt$$

implies (31).

Put $k=0$ in (27), then we get the theorem.

THEOREM 2.4. *Let* $1-2\mu<q<2$ *and suppose* (28) *and* (29). *Then*

(33) $$\lim_{R\to 0(\infty)} R^{q-1}\int_0^\infty \frac{\{J_\mu(Rt)\}^2}{t}dt = lq\int_0^\infty \{J_\mu(t)\}^2 t^{q-2}dt$$

implies (31).

For the proof it is sufficient to put $k=2\mu-1$ in (27).

REMARK. In Theorem 2.1 (consequently in its corollaries), if $K(t)\geqq 0$ for all $t\geqq 0$ and $K(t)\to M\neq 0$ as $t\to 0$, then we can dispense with

$$\frac{1}{T^q}\int_0^T|f(t)|dt \leqq C, \quad \text{for all } T>0, \quad (q>0).$$

PROOF. We prove the case $K(t)=\{J_\mu(x)/x^\mu\}^{2n}$ for the sake of simplicity.

Put

(34) $g(t)=\begin{cases} f(t), & \text{if } t<N \\ 0, & \text{if } t\geqq N\end{cases}$ for some fixed N then we have

(35) $$\lim_{T\to\infty}\frac{1}{T^q}\int_0^T g(t)dt = 0,$$

and

(36) $$\lim_{R\to 0} R^q\int_0^\infty g(t)\left\{\frac{J_\mu(Rt)}{(Rt)^\mu}\right\}^{2n}dt = \lim_{R\to 0} R^q\int_0^N f(t)\left\{\frac{J_\mu(Rt)}{(Rt)^\mu}\right\}^{2n}dt$$

$$\leqq \lim_{R\to 0} MR^q\int_0^N |f(t)|dt = 0.$$

Consequently, if we put

(37) $$f(t) = g(t) + h(t),$$

then it is sufficient to prove the theorem for $h(t)$, which vanishes near to the origin. As the integrand is non-negative for any $0<R<\eta$ we have

(38) $$K > R^q \int_0^\infty h(t)\left\{\frac{J_\mu(Rt)}{(Rt)^\mu}\right\}^{2n} dt \geqq R^q \int_0^{C/R} h(t)\left\{\frac{J_\mu(Rt)}{(Rt)^\mu}\right\}^{2n} dt$$
$$\geqq MR^q \int_0^{C/R} h(t)dt,$$

for $$\lim_{x\to 0}\left\{\frac{J_\mu(x)}{x^\mu}\right\}^{2n} = M \neq 0,$$

that is, for $T>1/\eta$, we have
$$\frac{1}{T^q}\int_0^T h(t)dt \leqq C.$$

But, by (34) and (37), we have
$$\frac{1}{T^q}\int_0^T h(t)dt = 0, \quad \text{for } 0 < T \leqq N.$$

Thus
$$\frac{1}{T^q}\int_0^T h(t)dt \leqq \text{const.},$$

uniformly in T.

2. Cesàro summability of multiple Fourier series.

Quasi-Tauberian theorem of Wiener ([10], p. 77) reads as follows.

THEOREM 3. *Let $K_1(x)$ be bounded and continuous over every finite interval. Let $f(x)$ be of bounded variation over every finite interval. Let*

(39) $$\lim_{y\to\infty}\int_0^\infty K_1(y-x)df(x) = A\int_{-\infty}^\infty K_1(x)dx,$$

(40) $$\int_{-\infty}^\infty |d\{K_1(x)e^{-\lambda x}\}| \leqq C$$

and as $x\to-\infty$

(41) $$K_1(x)\sim A_1e^{\lambda x} \quad (\lambda > 0), \quad (A \neq 0)$$

holds. Let $k_1(u)$ and $k_2(u)$ be defined by

(42) $$k_1(u) = \int_{-\infty}^\infty K_1(x)e^{ux}dx$$

and

(43) $$k_2(u) = \int_{-\infty}^\infty K_2(x)e^{ux}dx.$$

Let $K_1(x)$ belong $L_2(-\infty,\infty)$ and let $k_2(u)/k_1(u)$ be analytic over $-\varepsilon\leqq\Re(u)\leqq\lambda+\varepsilon$, and let it belong to L_2 over every ordinate in that strip. Then we have

(44) $$\lim_{y\to\infty}\int_0^\infty K_2(y-x)df(x) = A\int_{-\infty}^\infty K_2(x)dx.$$

Let $f(x)=f(x_1, \dots, x_k)$ be a function of the Lebesgue class L, periodic in each of the k-variables, and having the period 2π, and put (cf. Bochner

[1] and Chandrasekharan [2])

(45) $\varphi(x, t) = \varphi(t) = \dfrac{\Gamma(k/2)}{2(\pi)^{k/2}} \displaystyle\int_{\sum_{i=1}^{k} \xi_i^2 = 1} f(x_1 + t\xi_1, \ldots, x_k + t\xi_k) d\sigma_\xi,$

where $d\sigma_\xi$ denotes $(k-1)$-dimenisonal volume element of the unit sphere. If $n > 0$, we define

(46) $$\varphi_n(x, t) = \frac{2}{B(n, k/2) t^{2n+k-2}} \int_0^t (t^2 - s^2)^{n-1} s^{k-1} \varphi(s) ds = \frac{c}{t} \int_0^t \left(1 - \frac{s^2}{t^2}\right)^{n-1} \left(\frac{s}{t}\right)^{k-1} \varphi(s) ds.$$

where

(47) $$c = 2/B(n, k/2).$$

$\varphi_n(x, t) \equiv \varphi_n(t)$ is called the spherical mean of order n of the function $f(x)$. Then

(48) $$\varphi_n(x, t)/t^r = \frac{c}{t} \int_0^t \left(1 - \frac{s^2}{t^2}\right)^{n-1} \left(\frac{s}{t}\right)^{k+r-1} \frac{\varphi(s)}{s^r} ds$$

and put $t = e^{-y}$, and $s = e^{-x}$, then (48) is

(49) $$c \int_y^\infty \exp\{(k + r)(y - x)\}\{1 - \exp(2(y - x))\}^{n-1} \varphi(e^{-x}) e^{rx} dx.$$

In Theorem 3, we put

(50) $$K^{(n)}(x) = \begin{cases} ce^{(k+r)x}(1 - e^{2x})^{n-1}, & x < 0, \quad (n \geqq 0) \\ 0, & x > 0, \end{cases}$$

then

(51) $$K^{(n)}(x) \sim ce^{(k+r)x} \quad \text{as} \quad x \to -\infty, \quad (k + r > 0).$$

Let

(52) $$V_\mu(x) = J_\mu(x)/x^\mu,$$

then

(53) $$V_\mu(x) = \begin{cases} O(1), & \text{as} \quad x \to 0, \\ x^{-(\mu+1/2)}, & \text{as} \quad x \to \infty. \end{cases}$$

If we denote by $\sigma^{(m)}(R, x)$ the m-th spherical Riesz mean of the Fourier series of $f(x)$, then

(54) $$\sigma^{(m)}(R, x) = \sigma^{(m)}(R) = 2^m \Gamma(m + 1) R^k \int_0^\infty t^{k-1} \varphi(x, t) V_{m+k/2}(tR) dt = dR^k \int_0^\infty t^{k-1} \varphi(t) V_{m+k/2}(tR) dt,$$

where

$$d = 2^m \Gamma(m + 1).$$

Since

(55) $$\lim_{R \to \infty} R^k \int_1^\infty t^{k-1} \varphi(t) V_{m+k/2}(tR) dt = O(R^{-m+(k+1)/2}),$$

we can neglect this term in the following lines. Put

(56) $$R^r \sigma^{(m)}(R) \equiv dR^{r+k} \int_0^1 t^{k-1} \varphi(t) V_{m+k/2}(tR) dt$$

$$= dR\int_0^1 \varphi(t)t^{-r}(tR)^{k+r-1}V_{m+k/2}(tR)dt,$$

and let $R=e^y, t=e^{-x}$, respectively, then (56) becomes

(57) $$d\int_0^\infty \exp\{(k+r)(y-x)\}V_{m+k/2}(e^{y-x})\varphi(e^{-x})e^{rx}dx.$$

Comparing with Theorem 3, we put

(58) $${}^{(m)}K(x) = de^{(k+r)x}V_{m+k/2}(e^x),$$

then

(59) $${}^{(m)}K(x) \sim O(e^{(k+r)x}) \neq 0, \quad \text{as} \quad x\to -\infty (k+r>0)$$

by (53).

$K^{(n)}(x)\in L_2(-\infty,\infty)$, but ${}^{(m)}K(x)\in L_2(-\infty,\infty)$, if and only if

(60) $$m > r + (k-1)/2,$$

by (53). The Mellin transform of $K^{(n)}(x)$ is

(61) $$k_n(u) = c\int_{-\infty}^0 e^{(k+r)x}(1-e^{2x})^n e^{kx}dx$$

$$= \frac{c}{2}\int_0^1 t^{(k+r+u-2)/2}(1-t)^n dt = \frac{c\Gamma\{(k+r+u)/2\}\Gamma(n+1)}{2\Gamma\{(k+r+u)/2+n+1)}$$

and the transform of ${}^{(m)}K(x)$ is

(62) $$l_m(u) = d\int_0^\infty e^{(k+r+u)x}V_{m+u/2}(e^x)dx$$

$$= d\int_0^\infty t^{(r+k+u-1)}V_{m+u/2}(t)dt$$

$$= d\int_0^\infty J_{m+k/2}(t)t^{(r+k+u-1-m-k/2)}dt.$$

$$= \frac{d\Gamma\{(r+k+u)/2\}}{2^{m-k/2-r-u+1}\Gamma\{m+1-(r+u)/2\}} \quad \text{(Watson [9] p. 391).}$$

In this case u is imaginary, the condition of validity of (62) is

$$0 < r+k < m + k/2 + 3/2$$

and this is contained in (60). Then we have

(63) $$\frac{l_m(u)}{K_n(u)} = \text{const}\cdot\frac{\Gamma\{k+r+2n+2+u)/2\}}{\Gamma\{(2m-r-u+2)/2\}}$$

$$\sim \text{const.}|\mathfrak{J}(u)|^{\mathfrak{R}(u)+r+n-m+k/2},$$

as $|\mathfrak{J}(u)|\to\infty$. From Theorem 3, if

(64) $$m > n + (k+1)/2 + r,$$

then

(65) $$\lim_{y\to\infty}\int_0^\infty K^{(n)}(y-x)\varphi(e^{-x})e^{rx}dx = A\int_{-\infty}^\infty K^{(n)}(x)dx$$

implies

(66) $$\lim_{y\to\infty}\int_0^\infty {}^{(m)}K(y-x)\varphi(e^{-x})e^{rx}dx = A\int_{-\infty}^\infty {}^{(m)}K(x)dx,$$

and if

(67) $$n > m - r - (k-1)/2,$$

then (66) implies (65). In the latter case, the condition of analyticity of $k_n(u)/l_m(u)$ is contained in (60). Thus we get the follwing theorem.

THEOREM 3.1. *Let* $r>-k$. (*a*) *If* $m>n+(k-1)/2+r$, $(n\geqq 1)$ *then* $\lim_{t\to 0}\varphi_n(t)/t^r=s$ *implies* $\lim_{R\to\infty}R^r\sigma^{(m)}(R)=ls$, *and* (*b*) *if* $n>m-r-(k-3)/2$ *and* $m>r+(k-1)/2$, *then* $\lim_{R\to\infty}R^r\sigma^{(m)}(R)=s$ *implies* $\lim_{t\to 0}\varphi_n(t)/t^r=s/l$, *where*

$$l = 2^{(k-2)/2-m-r}\Gamma\{(k+r)/2+n\}\{\Gamma(m-r/2+1)\Gamma(n)\}^{-1}.$$

The special case $r=0$ and $k=1$, is the well known theorem of Bosanquet-Paley-Wiener. The case (*b*) where $k=1$, and $s=0$ is solved by Hyslop [5] under some restrictions and the complete solution is due to Izumi [6]. Most general case (*a*) is given by Chandrasekharan [2] and the case (*b*) is new. This indicates that the order condition of the theorem is best possible in a sense.

THEOREM 4. *Under the hypothesis of Theorem 3,*

$$\int_{-\infty}^{\infty}\left|d_y\int_0^{\infty}K_1(y-x)df(x)\right|<\infty$$

implies

$$\int_{-\infty}^{\infty}\left|d_y\int_0^{\infty}K_2(y-x)df(x)\right|<\infty.$$

This is due to the author [8]. Corresponding to Theorem 3.1, we get

THEOREM 4.1. *Let* $r>-k$. (*a*) *If* $\varphi_n(t)/t^r$ *is of bounded variation in* $0<t<\infty$, *then* $R^r\sigma^{(m)}(R)$ *is of bounded variation in* $0<R<\infty$, *for* $m>n+(k-1)/2+r$, $(n\geqq 1)$, *and* (*b*) *if* $R^r\sigma^{(m)}(R)$ *is of bounded variation in* $0<R<\infty$, *then* $\varphi_n(t)/t^r$ *is of bounded variation in* $0<t<\infty$ *for*

$$n>m-r-(k-3)/2 \text{ and } m>r+(k-1)/2.$$

The case $r=0$ is given by Chandrasekharan [3] with direct calculation.

Literature

[1] BOCHNER, S., Summation of multiple Fourier seires by spherical means, Trans. Amer. Math. Soc., 40 (1936), 175–207.

[2] CHANDRASEKHARAN, On the summation of multiple Fourier series I, Proc. London Math. Soc., 50 (1948), 210–222.

[3] CHANDRASEKHARAN, On the summation of multiple Fourier series II, Proc. London Math. Soc., 50 (1948), 223–229.

[4] CHENG, M. T., Some Tauberian theorems with application to multiple Fourier series, Annals of Math., 50 (1949), 763–776.

[5] HYSLOP, J. M., Note on a group of theorems in the theory of Fourier series, Journ. London Math. Soc., 24 (1949), 91–100.

[6] IZUMI, S., On Cesàro summability of Fourier series, appear in Journ. London Math. Soc.

[7] LITTAUER, S. B., Note on a theorem of Jacob, Journ. London Math. Soc., 4 (1930), 226–231.
[8] SUNOUCHI, G., Quasi-Tauberian theorems, Tôhoku Math. Journ. 1 (1950), 167–185.
[9] WATSON, G. N., A treaties on the theory of Bessel functions, Cambridge, 1922.
[10] WIENER, N., Tauberian theorems, Annals of Math., 33 (1932), 1–100.

Mathematical Institute,
Tôhoku University, Sendai.