SOME REMARKS CONCERNING PRINCIPAL IDEAL THEOREM*)

By

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1.

In this paper I present a detailed account of the results previously announced in the Proceedings of the Academy of Tokyo, and which constitute the arithmetic complement to Terada's paper in this same volume. Concerning the historical note we refer to the above previous note and the preface in Terada's paper.

In the paragraph 6 of this paper I give a proof of original principal ideal theorem, which is in substance that of Iyanaga's paper [1], but which does not depend on the concept of "order ideal". § 7 contains further remark in this direction.

2.

Mr. Terada has proved recently the following generalization of Furtwängler's principal ideal theorem:

THEOREM 1. *Let K be the absolute class field over k, and Ω a cyclic intermediate field of K/k, then all the ambigous ideal classes of Ω will become principal in K.*

This Theorem was suggested by the special case $K=\Omega$, where no essential difficulty occurs. We can prove that special case for instance by the principal genus theorem, which asserts that $N_{K/k}\mathfrak{a}\sim 1$ implies $\mathfrak{a}\sim\mathfrak{b}^{1-s}$ for some ideal $\mathfrak{b}$, s being a generator of the Galois group $G(K/k)$. We see namely that in our case the correspondence $\mathfrak{a}\to\mathfrak{a}^{1-s}$ leads to an isomorphism of the ideal classes, and the required proof is at hand.

I will show in the following line in what manner Iyanaga's principal ideal theorem [2] for "ray class fields" (Strahlklassenkörper) can be generalized, and that this amounts to

THEOREM 2. *Let K be the ray class field mod $\mathfrak{f}(K/k)$ over k, and Ω a cyclic intermediate field of K/k. Let also $\mathfrak{m}$ denote the ideal Max $\{\mathfrak{f}(K/\Omega), \mathfrak{F}(\Omega/k)\}$ in Ω. If $\mathfrak{a}$ is an ideal in ambigous class modulo $\mathfrak{m}$, then $\mathfrak{a}$ lies in the ray modulo $\mathfrak{F}(K/k)$, when considered as an ideal in K. Thereby*

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$\mathfrak{F}$ *means the "Geschlechtermodul", whose construction is given in the below.*

3.

Let K/k be an arbitrary normal field with the Galois group $G=G(K/k)$, and Ω a normal subgroup which corresponds to the subgroup g of G. Let $\mathfrak{P}$ be a prime ideal in K, which has $G_i (i=1, 2, \dots)$ as Hilbert's ramification subgroups of order $(G_i)=N_i$ respectively, that is, G_i may consists of all the Galois substitutions σ with

$$A^\sigma \equiv A \mod \mathfrak{P}^i \quad (A \text{ in } K).$$

Corresponding subgroups for K/Ω are $g_i=G_i \cap g$, with the order n_i (sometimes also $n(i)$ when it is convenient in writing) respectively, and the ramification subgroups γ_i for Ω/k were determined by J. Herbrand in the following form.

If the subgroups $\gamma'_i=(g, G_i)/g$ satisfy the relations

$$\begin{aligned}\gamma'(1) &= \gamma'(2) = \dots = \gamma'(i'_1)\\ &> \gamma'(i'_1+1) = \gamma'(i'_1 + 2) = \dots = \gamma'(i'_2)\\ &> \cdots\end{aligned}$$

and if

$$\begin{aligned}\gamma(1) &= \gamma(2) = \dots = \gamma(i_1)\\ &> \gamma(i_1 + 1) = \gamma(i_2 + 2) = \dots = \gamma(i_2)\\ &> \cdots\end{aligned}$$

then we have the equalities (putting $e=n(1)$),

$$\begin{aligned}ei_1 &= n(1) + n(2) + \dots + n(i'_1),\\ ei_2 &= n(1) + n(2) + \dots + n(i'_2),\\ &\dots\dots\dots\dots\end{aligned}$$

and the isomorphisms

$$\gamma(i_j) \cong \gamma'(i'_j) \qquad (j=1, 2, \dots),$$

by which $\gamma(i)$ are completely determined.

As for the infinite spots, we have entirely same results. Readers may compare with a paper due to Mr. Iyanaga [3]. In the following consideration we will confine ourselves exclusively to the case of finite spots.

4.

By the Hasse's formula for the conductor of the abelian field K/k we have

$$\mathfrak{f}(K/k) = \prod \mathfrak{P}^\nu, \quad \nu = \sum_{i=1}^{r} N_i \quad (N_r > N_{r+1} = 1), \tag{4.1}$$

and which we write for the sake of convenience

$$[\mathfrak{f}(\mathfrak{P}\,; K/k)] = \nu = \sum (G_i) \quad (G_i \neq 1).$$

Conversely if we define the conductor of the arbitrary normal field

K/k by this formula, formal properties obviously remain true.

From this formula, applied to the field Ω/k, and the Herbrand's in the preceding section we have easily (cf. [6])

(4.2) $$[\mathfrak{f}(\mathfrak{P};\Omega/k)]=\sum(G_i)\quad(G_i\not\subset g).$$

We have also an analogous formula for the Geschlectermodul $\mathfrak{F}(\Omega/k)$, which is defined by the formula

(4.3) $$[\mathfrak{F}(\mathfrak{q}\,;\,\Omega/k)]=\sum 1\quad(\text{summed over }\gamma_i\neq 1)$$

where $\mathfrak{P}/\mathfrak{q}$, $\mathfrak{q}$ in Ω, and the corresponding formula to (4.2) is

(4.4) $$[\mathfrak{F}(\mathfrak{P};\Omega/k)]=\sum(g_i)\quad(G_i\not\subset g).$$

Next we define an ideal $\mathfrak{f}(K,\Omega,k)$ in Ω, which unifies the concepts of "conductor" and "Geschlechtermodul", and which is given by the formula

(4.5) $$[\mathfrak{f}(\mathfrak{P};K,\Omega,k)]=\sum(G_i\cap g)\quad(G_i\neq 1).$$

We see immediately, that it follows

(4.6) $$\mathfrak{f}(K,K,k)=\mathfrak{F}(K/k),\quad \mathfrak{f}(K,k,k)=\mathfrak{f}(K/k)$$

respectively. More generally we can prove the

THEOREM 3. $\mathfrak{f}(K,\Omega,k)=\mathrm{Max}\{\mathfrak{f}(K/\Omega),\mathfrak{F}(\Omega/k)\}$.

PROOF. We distinguish two cases according to the behavior of ramification groups.

CASE I. There is a group $G_i\neq 1$ with $G_i\subset g$. In this case the condition $G_i\neq 1$ coincides with the condition $g_i=G_i\cap g\neq 1$, so that we have

$$\begin{aligned}[\mathfrak{f}(\mathfrak{P};K,\Omega,k)]&=\sum(g_i)\quad(g_i\neq 1)\\&=[\mathfrak{f}(\mathfrak{P};K/\Omega)]\geqq[\mathfrak{F}(\mathfrak{P};\Omega/k)]\quad(=\sum(g_i)\ (G_i\not\subset g)).\end{aligned}$$

CASE II. There is no group $G_i\neq 1$ with $G_i\subset g$. In this case the right-hand side of the equality (4.5) coincides with that of the equality (4.4), we have accordingly

$$\begin{aligned}[\mathfrak{f}(\mathfrak{P};\,K,\Omega,k)]&=[\mathfrak{F}(\mathfrak{P};\,\Omega/k)]\\&\geqq[\mathfrak{f}(\mathfrak{P};\,K/\Omega)]\quad(=\sum(g_i)\quad(g_i\neq 1)).\end{aligned}$$

From these both results we conclude the required equality, and especially that $\mathfrak{f}(K,\Omega,k)$ is an ideal in Ω.

THEOREM 4. *If K/k is a normal field with Galois group $\mathfrak{G}$, and Ω, k' two intermediate fields which correspond to the subgroups g and G of $\mathfrak{G}$ respectively, then we have*

(4.7) $$[\mathfrak{f}(\mathfrak{P};\Omega,k',k)]=\sum(G_i),\quad \mathfrak{G}_i\not\subset g.$$

Thereby $\mathfrak{P}$ is a prime ideal in K, and G_i, $\mathfrak{G}_i$ are corresponding ramification groups for K/k' and K/k respectively.

PROOF. We denote the quotient groups $\mathfrak{G}/g$, G/g by δ and γ respectively, and δ_i, γ_i corresponding Hilbert's subgroups.

CASE I. There is a group $\delta_i\neq 1$ with $\delta_i\subset\gamma$.

We have then by the proof of preceding theorem

$$[\mathfrak{f}(\mathfrak{q}\,;\,\Omega, k', k)] = \mathfrak{f}(\mathfrak{q}\,;\,\Omega/k')], \ (\mathfrak{P}/\mathfrak{q}, \mathfrak{q} \text{ in } \Omega)$$

and then using the formula (4.2) applied to the fields K, Ω and k' instead of K, Ω and K respectively,

$$[\mathfrak{f}(\mathfrak{P}\,;\Omega, k', k)] = \sum(G_i), G_i \not\subset g,$$

but as we see easily by our assumption $G_i \not\subset g \rightleftarrows \mathfrak{G}_i \not\subset g$, this reduces to the equality (4.7).

CASE II. There is no group $\delta_i \neq 1$ with $\delta_i \subset \gamma$. By the proof of preceding theorem and the formula (4.4), again applied to the fields K, k' and k,

$$[\mathfrak{f}(\mathfrak{q}\,;\,\Omega, k', k)] = [\mathfrak{F}(\mathfrak{q}\,;\,k', k)],$$

$$[\mathfrak{F}(\mathfrak{P}\,;\,k', k)] = \sum(\mathfrak{G}_i), \mathfrak{G}_i \not\subset G.$$

But from the assumption follows $\mathfrak{G}_i \not\subset g \rightleftarrows \mathfrak{G}_i \not\subset G$, as we have $\delta_i = \mathfrak{G}_i/g$, $\gamma = G/g$, and the theorem is proved.

5.

We denote generally by $S(\mathfrak{m})$ the "ray" (Strahl) modulo $\mathfrak{m}$. Our main theorem 2 is an immediate consequence of the following theorem, if we assume Terada's result of group theoretical part of our theorem. Apart from this application theorem 5 may be of some interest in itself.

THEOREM 5. *Let Ω be the ray class field modulo $\mathfrak{f}(\Omega/k)$ over k,* and *k' an arbitrary (not always cyclic) intermediate field. Let further $\bar{\Omega}$ be the ray class field modulo $\mathfrak{F}(\Omega/k)$ over Ω and K maximal abelian extension over k' contained in $\bar{\Omega}$. Then K is the ray class field modulo some ideal $\mathfrak{m}$ over k'. We may assume*

$$\mathfrak{m} = \mathfrak{f}(\Omega, k', k) = \mathrm{Max}\{\mathfrak{f}(\Omega/k'), \mathfrak{F}(k'/k)\}.$$

NOTE. By the application to the theorem 2 we restrict k' to the cyclic field, and write K and Ω instead of Ω and k' respectively.

PROOF. We shall prove this by showing the following two partial results.

FIRST: If $K \subset \bar{\Omega}$ and K/k' is abelian then we have $\mathfrak{f}(K/k') \leqq \mathfrak{f}(\Omega, k', k)$, or

$$\mathfrak{f}(K/\Omega) \leqq \mathfrak{F}(\Omega/k) \to \mathfrak{f}(K/k') \leqq \mathfrak{f}(\Omega, k', k). \tag{5.1}$$

If we use the same notations as in theorem 4, the left side inequality will become (restricting to $\mathfrak{P}$-contribution),

$$\sum(g_i) \quad (g_i \neq 1) \leqq \sum(g_i) \quad (\mathfrak{G}_i \not\subset g), \tag{5.2}$$

or subtracting the same terms (g_i) with $(g_i \neq 1,\ \mathfrak{G}_i \not\subset g)$ from both sides

$$\sum(g_i)(g_i \neq 1, \mathfrak{G}_i \subset g) \leqq \sum(g_i)(g_i = 1, \mathfrak{G}_i \not\subset g). \tag{5.2$'$}$$

We can transform likewise the right member of the logical relation

(5. 1), successively into the form

(5.3) $$\sum(G_i)(G_i \neq 1) \leqq \sum(G_i)(\mathfrak{G}_i \not\subset g),$$

or

(5.3)′ $$\sum(G_i)(G_i \neq 1, \mathfrak{G}_i \subset g) \leqq \sum(G_i)(G_i = 1, \mathfrak{G} \not\subset g).$$

If the assertion (5.1) is not true, the left side of (5.3)′ does not vanish (at least for a prime spot $\mathfrak{P}$), so that there exists a group $\mathfrak{G}_i \subset g$ with $G_j = \mathfrak{G}_j \cap G \neq 1$. We see immediately that the left side of (5.2)′ does not vanish, while the right side is zero, in contradiction to our assumption.

SECOND: If K is the class field for the ideal group $S(\mathfrak{f}(\Omega, k', k))$ in k', then $K \subset \overline{\Omega}$; or what comes to the same thing

(5.4) $$\mathfrak{f}(K/k') \leqq \mathfrak{f}(\Omega, k', k) \to \mathfrak{f}(K/\Omega) \leqq \mathfrak{F}(\Omega/k),$$

or by the above-mentioned reductions

(5.4)′ $$(5.3)' \to (5.2)'.$$

If then (5.2)′ is not true, we have for some prime spot $\mathfrak{P}$

$$\sum(g_i)(g_i \neq 1, \mathfrak{G}_i \subset g) > \sum(g_i)(g_i = 1, \mathfrak{G}_i \not\subset g),$$

and there is a group $\mathfrak{G}_j \subset g$ with $g_j \neq 1$. This implies that the right side of (5.3)′ is zero, while the left side of the same equality is equal to that of (5.2)′, which is >0 by the assumption.

6.

In this paragraph I will give a little remark about the Iyanaga's proof of original principal ideal theorem. We can namely by a slight modification avoid the concept of "order ideal" in his proof, though it is very important in itself.

The theorem to be treated here is the well known

"*Let G be a metabelian group and $S_\sigma(S_1=1)$ represent a system of representatives of the commutator factor $G/G'=\Gamma$, then factor set $D_{\sigma,\tau}= S_\sigma S_\tau S_{\sigma\tau}^{-1}$ satisfies the relation $\prod_\sigma D_{\sigma,\tau}=1$ for all τ*".

We construct as in Iyanaga's paper, Artin's splitting group $\mathfrak{U}$ generated by G' and the symbols $A_\sigma(A_1=1)$, and with Γ as operator system by the rules

$$U^\sigma = S_\sigma U S_\sigma^{-1} \quad (U \epsilon G'),$$

$$A_\tau^\sigma = A_\sigma^{-1} A_{\sigma\tau} D_{\sigma,\tau}^{-1}.$$

One of the important technics lies in the principle

(6.1) $$A_\tau^C,\ c=\sum c_\sigma \sigma \text{ (for all } \tau) \to \sum c_\sigma \sigma = c_1 \Gamma.$$

Let now σ_1, σ_2, ... and σ_k be a basis of Γ with the orders e_1, e_2, ... and $e_k (e_1 e_2 \dots e_k = n)$ respectively. As we have

$$A_{\sigma\tau} \equiv A_\sigma A_\tau^\sigma \quad (G')$$

$\mathfrak{U}$ is symbolically generated (that is to say generated by all symbolic powers) by A_i and G', where A_i denotes A_σ with $\sigma=\sigma_i$.

Any element of G has the form $U_\sigma S_\sigma (U_\sigma \epsilon G')$, and the commutator of two such elements is

$$U_\sigma S_\sigma U_\tau S_\tau (U_\sigma S_\sigma)^{-1} (U_\tau S_\tau)^{-1} = U_\sigma^{1-\tau} U_\tau^{\sigma-1} A_\sigma^{\tau-1} A_\tau^{1-\sigma}.$$

Thus the elements $U_1, U_2, \ldots, U_l$ of G' have the form

$$U_j = \prod_{i=1}^{k} A_i^{-f(j,i)} \prod_{s=1}^{l} U_s^{-g(j,s)} \quad (j=1,2,\ldots,l) \tag{6.2}$$

where $f(j,i)=f_{ji}$ and $g(j,s)=g_{j,s}$ have respectively the form

$$\sum c_{\sigma,\tau}(\sigma-\tau) \quad \text{and} \quad \sum d_{\sigma,\tau}(\sigma-\tau). \tag{6.3}$$

On the other hand we have

$$A_{\sigma^2} \equiv A_\sigma^{1+\sigma} \quad (G'),$$
$$A_{\sigma^3} \equiv A_\sigma A_{\sigma^2}^{\sigma} \equiv A_\sigma^{1+\sigma+\sigma^2} \quad (G'),$$

so that, if we put

$$N_i = 1 + \sigma_i + \sigma_i^2 + \ldots + \sigma_i^{e(i)-1}$$

and recalling the relation $A_1=1$,

$$A_i^{N_i} \equiv 1 \quad (G'),$$
$$A_i^{N_i} = U_j \quad (j=j(i)). \tag{6.4}$$

Solving the equations (6.2) and (6.3) in U and A, by Cramér's method, we have

$$A_i^\Delta = 1, \quad U_j^\Delta = 1 \tag{6.5}$$

where

$$\Delta = \left|\begin{array}{ccc|ccc} N_1 & & & -1 & & \\ & \ddots & & & & -1 \\ & & N_k & -1 & & \\ \hline f_{11} & \cdots & f_{1k} & 1+g_{11} & \cdots & g_{1l} \\ & \cdots\cdots & & & \cdots\cdots & \\ f_{l1} & \cdots & f_{lk} & g_{l1} & \cdots & 1+g_{ll} \end{array}\right|. \tag{6.6}$$

From the principle (6.1) follows immediately $\Delta=c\Gamma$. But this gives by the ring homomorphism

$$\sum c_\sigma \sigma \rightarrow \sum c_\sigma$$

the relation $n=cn$ or $c=1$, as is easily seen by (6.6) and (6.3).

As $1=A_\tau^\Gamma=\prod_\sigma (A_\sigma^{-1} A_{\sigma\tau} D_{\sigma,\tau}^{-1})$ holds, our proof is now completed.

q. e. d.

Of course this proof is in substance quite the same as that of Iyanaga's. I intended only to extract the essential point of his proof.

It seems to me, that the most important point (assuming Artin's splitting group to be known) lies in the method of showing the equality $\Delta=\Gamma$ by means of the principle (6.1). Concerning this I will show in the next paragraph, the equality $\Delta=\Gamma$ is in fact an " identity ", in some sense.

7.

The object of this paragraph is the following general theorem concerning the determinants, whose elements lie in a commutative ring R.

THEOREM 6. *Let N_i, Δ_i and $A^{(i)}_{r,s}(i, r, s=1, 2, \ldots, k)$ be the elements in an abstract commutative ring R, which satisfy the relations*

(7.1) $$N_i\Delta_i = 0,$$

(7.2) $$A^{(i)}_{r,s} = - A^{(i)}_{s,r}, \quad A^{(i)}_{r,r} = 0.$$

Then the determinant $|\alpha_{ij}|$ is equal to $N_1N_2 \ldots N_k$, where

$$\alpha_{ij} = N_i\delta_{ij} + \sum_s A^{(i)}_{js}\Delta_s, \quad (\delta_{ii} = 1, \ \delta_{ij} = 0, \ i \neq j).$$

This can be proved, if the following theorem is established.

THEOREM 7. *Under the same assumptions as in the preceding theorem, we have $|\beta_{ij}| = 0$, where $\beta_{ij} = \sum_s A^{(i)}_{js}\Delta_s$.*

We first assume theorem 7 and deduce the theorem 6. Expanding the determinant $|\alpha_{ij}|$ in terms of N_i, we have

$$|\alpha_{ij}| = |\beta_{ij}| + \sum N_1D_1 + \sum N_1N_2D_{12} + \ldots + N_1 \ldots N_k.$$

We see then easily that all terms vanish except the last one, for instance

$$N_1D_1 = N_1\left|\sum_{s=1}^{k} A^{(i)}_{js}\Delta_s\right|_{i,j\geqq 2} = N_1\left|\sum_{s=2}^{k} A^{(i)}_{js}\Delta_s\right|_{i,j\geqq 2}$$

and the determinant in the right-hand side is 0, according to the theorem 7, applied to the case of $(k-1)$-th order.

PROOF OF THEOREM 7.

We show in the expansion

(7.3) $$|\beta_{ij}| = \sum_{a(1),\ldots,a(k)} \sum_{r(1),\ldots,r(k)} \begin{vmatrix} A^{(1)}_{1,r(1)} & \cdots & A^{(1)}_{k,r(k)} \\ & \cdots\cdots & \\ A^{(k)}_{1,r(1)} & & A^{(k)}_{k,r(k)} \end{vmatrix} \Delta_1^{a(1)} \ldots \Delta_k^{a(k)}$$

all the coefficients of $\Delta_1^{a(1)} \ldots \Delta_k^{a(k)}$ vanish identically in the variables $A^{(1)}_{rs}(r<s)$. Thereby $r_1, r_2, \ldots, r_k$ runs over all the possible combinations of the values

(7.4) $$C = (\underbrace{1, \ldots, 1}_{a_1} \ \underbrace{2 \ldots 2}_{a_2} \ldots\ldots \underbrace{k \ldots k}_{a_k}).$$

So we have to prove the identity

(7.5) $$\sum_{r(1),\ldots,r(k)} ((1r_1)(2r_2) \ldots (kr_k)) = 0, \ ((r(1), \ldots, r(k))\epsilon C)$$

where $((1r_1)(2r_2) \ldots (k, r_k))$ means the determinant in the summand of the equality (7.3).

The proof will be performed by induction in k.

We now distinguish two cases.

CASE I. At least one a_i is 0 (for instance $a_1=0$ and $a_2>0$).

In this case the coefficent of $A^{(1)}_{12}$ in our sum (7.3) is

(7.6) $$\sum \begin{vmatrix} A^{(2)}_{2,r(2)} \cdots A^{(2)}_{k,r(k)} \\ \cdots\cdots\cdots\cdots \\ A^{(k)}_{2,r(2)} \cdots A^{(k)}_{k,r(k)} \end{vmatrix}$$

where $r(2), \ldots, r(k)$ belongs to the class

$$C' = (\underbrace{2 \ldots 2}_{a_2-1}, \underbrace{3 \ldots 3}_{a_3}, \ldots, \underbrace{k \ldots k}_{a_k}),$$

and by the assumption of our induction (7.6) will then vanish.

As every term in (7.5) contains $A^{(i)}_{1,r(1)}$ as a factor, above mentioned proof can be considered to be "typical", though we restricted $i=1$ and $r(1)=2$.

To be noticed is, in this case, $A^{(1)}_{2,r(2)}$ give no contribution to the sum with $A^{(1)}_{1,2}$ as a factor, for $r(2)$ can not be 1, so that $A^{(1)}_{2,r(2)} \neq -A^{(1)}_{1,2}$.

CASE II. All a_i are positive, that is $a_i=1$.

In this case

$$P = \begin{pmatrix} 1 & 2 & \ldots & k \\ r(1) & r(2) & \ldots & r(k) \end{pmatrix}$$

is a permutation of order k. We denote the corresponding determinant also by P, and decompose P in cycles $A_1 A_2 A_3 \ldots$, for instance

$$\begin{pmatrix} 1\,2\,3\,4\,5 \\ 4\,1\,5\,2\,3 \end{pmatrix} = \begin{pmatrix} 1\,4\,2 \\ 4\,2\,1 \end{pmatrix} \begin{pmatrix} 3\,5 \\ 5\,3 \end{pmatrix} = (1\,4\,2)(3\,5).$$

If P contains a transposition (ab), then the corresponding determinant is 0, because of the relation $A^{(i)}_{a,b} = -A^{(i)}_{b,a}$. We can therefore assume $A_1 \neq A_1^{-1}$, $A_2 \neq A_2^{-1}, \ldots$.

Now we write the sum (7.5) in the form

$$\sum P = \sum A_1 A_2 A_3 \ldots = \sum_A \sum_\varepsilon A_1^{\varepsilon(1)} A_2^{\varepsilon(2)} A_3^{\varepsilon(3)} \ldots \quad (\varepsilon = \pm 1).$$

We first assume P itself a cycle

$$P = (12 \ldots k),$$

then the sum $P + P^{-1}$ is 0, owing to the simple computation:

$$\begin{aligned} P + P^{-1} &= ((12)(23) \ldots (k1) + ((1k)(21)(32) \ldots \quad (k, k-1)) \\ &= ((12)(23) \ldots (k1)) + (-1)^k ((k1)(12)(23) \ldots \quad (k-1, k)) \\ &= ((12)(23) \ldots (k1)) - ((12)(23) \ldots \quad (k1)). \end{aligned}$$

General terms are also similarly treated as in the following schema,

$$\sum_\varepsilon A_1^{\varepsilon(1)} A_2^{\varepsilon(2)} A_3^{\varepsilon(3)} = \sum_P (A_1 P + A_1^{-1} P) = 0 \quad (P = A_2^{\varepsilon(2)} A_3^{\varepsilon(3)})$$

and the proof is completed. q. e. d.

We remarked at the end of the preceding paragraph, that the principal ideal theorem is an identity in some sense. This will be explained here. Under the identity we do not mean $\Delta = \Gamma$ (in § 6) itself. If we let correspond to each $U_j (\epsilon G')$, symbols $N_j (=1)$ and $A_j (=U_j)$, and if t and s denote the suffixes representing both $i (=1, 2, \ldots, k)$ and $j (=1, 2, \ldots, l)$, then we have by (6. 2), (6. 3), (6. 4) and (6. 5)

$$A_t^{N_t} = \prod_s A_s^{f(t,\, s)} \tag{7.6}$$

or written in additive notation

$$N_t c_t = \sum_s f_{t,\, s} c_s$$

where $f_{t,\, s}$ is of the form

$$\sum_r A_{rs}^{(t)} \Delta_r \ \ (A_{rs} = - A_{sr}, \ \ \Delta_r = 1 - \sigma_r).$$

To be noticed is, we have to put $\sigma_r = 1$ for $r = j$ (corresponding to an element in G'), so that

$$U_\sigma^{1-\tau} = A_j^{1-\tau} = A_j^{1-\tau} A_\tau^{\sigma(j)-1}.$$

As $N_t \Delta_t = 0$ holds, all the conditions of the theorem 6 are satisfied, so we have $N_1 \dots N_k c_i = 0$ or $A_i^\Gamma = 1$, "identically" in $A_{rs}^{(t)}$, and this was just our object.*)

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Concerning the principal ideal theorem compare also the papers

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*) After completing my investigation, I obtained also an alternative proof of Theorem 1 based on the remark in § 7. For the detail I refer to a forthcoming paper.