ARITHMETIC MEANS OF SUBSEQUENCES

BY

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Introduction. Let \( \{s_n\} \) be a sequence of real numbers which is summable \((C,1)\) to \( s : (s_1 + s_2 + \cdots + s_n)/n \to s \) as \( n \to \infty \). Let \( \{r_n(x)\} \) be the Rademacher system. If the limit of

\[
(1) \quad \varphi_n(x) = \left( \frac{\sum_{k=1}^{n} 1 + r_k(x)}{2} \right) / \left( \frac{\sum_{k=1}^{n} 1 + r_k(x)}{2} \right)
\]

for \( n \to \infty \), exists for almost all \( x \), we shall say that almost all the subsequences of \( \{s_n\} \) are summable \((C,1)\); if the limit of \( (1) \) does not exist for almost all \( x \), we say that almost all the subsequences of \( \{s_n\} \) are not summable \((C,1)\) (cf. [2]). These two cases are the all which may occur, since the existence set of the limit of \( (1) \) is homogeneous. If the limit of \( (1) \) exists only for \( x \) belonging to a set of the first category, it is called that nearly all the subsequences of \( \{s_n\} \) are not summable \((C,1)\).


Theorem. If \( \{s_n\} \) is summable \((C,1)\) to \( s \), then in order that almost all the subsequences of \( \{s_n\} \) are summable \((C,1)\), it is sufficient that

\[
(2) \quad \sum_{k=1}^{\infty} \frac{s_k^2}{k^2} < \infty,
\]

and it is necessary that

\[
(3) \quad \sum_{k=1}^{n} s_k^2 = o(n^2) \quad \text{as} \quad n \to \infty.
\]

In § 1 of this paper we shall give another sufficient condition, and in § 2 we shall construct an example which shows not only that this condition is the best possible one in a sense but also give a negative answer for the Buck-Pollard problem [2] whether the condition \( (3) \) is a sufficient one. In the last § we shall concern ourselves the summability \((C,1)\) of nearly all the subsequences.

*) Received May 20, 1950.
§ 1. By easy consideration, we may see that the existence almost everywhere of the limit of (1) is equivalent to:

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} s_k(x) = 0 \]

almost everywhere, provided that \( \{s_n\} \) is summable \((C,1)\) (See [2]).

**Theorem 1.** If \( \{s_n\} \) is summable \((C,1)\) to \(s\), and if

\[ \sum_{k=1}^{n} s_k^2 = o(n^{3/\log \log n}) \quad \text{as } n \to \infty, \]

then almost all the subsequences of \( \{s_n\} \) are summable \((C,1)\) to \(s\).

**Proof.** Let us put

\[ B_n = \sum_{k=1}^{n} s_k^2, \quad S_n(x) = \sum_{k=1}^{n} s_k(x), \quad S^*_n(x) = \max_{1 \leq k \leq n} |S_k(x)|, \quad \text{for } \(k = 1, 2, \ldots\). \]

For \( \delta > 0 \), we denote, by \( E_k \) \((k = 1, 2, \ldots)\) the set of all \( x \) such that \( |S_n(x)| > n\delta \) for at least one value of \( n \), \( 2^k - 1 < n \leq 2^k \). If we put

\[ G_k = \left[ x : S^*_n(x) > 2^{k-1}\delta \right] \quad \text{for } \(k = 1, 2, \ldots\), \]

we have evidently \( F_k \subset G_k \) \((k = 1, 2, \ldots)\). Hence if the inequality

\[ \sum_{k=1}^{\infty} |G_k| < \infty \]

holds for every \( \delta > 0 \) we can deduce that \( |S_n(x)|/n \to 0 \) as \( n \to \infty \) almost everywhere, and by the remark at the beginning of this § we may complete the proof. To prove (6), we use the Marcinkiewicz-Zygmund inequality ([4]; [5] Remark 1 § 3)

\[ \int_{\delta}^{1} \exp(aS^*_n(x)) \, dx \leq 32 \exp\left( \frac{1}{2} a^2 B_n \right), \quad a = a_n > 0. \]

From this we have

\[ |G_k| \exp(a2^{k-1}\delta) \leq \int_{\delta}^{1} \exp(aS^*_n(x)) \, dx \leq 32 \exp\left( \frac{1}{2} a^2 B_{2^k} \right), \]

and if we take \( a = 2^{k-1}\delta/B_{2^k} \), we have
On the other hand, from (5) it follows that
\[ B_{2h} (2^k)^2 \leq \delta (16 \log \log 2) \]
for large \( k (> k_0 \text{ say}) \). Consequently we have from (8)
\[ |G_k| \leq 32 \exp \left(- \frac{5}{2} \frac{2^k B_{2h}}{B_{2h}} \right) = 32 \exp \left(- \frac{5}{8} \frac{(2^k)^2}{B_{2h}} \right) \]
which is a term of a convergent series, and (6) is proved, q.e.d.

§ 2. Theorem 2. There exists a sequence \( \{s_n\} \) summable \( (C,1) \), which satisfies the condition
\[ \sum_{k=1}^{n} s_k^2 = O (n^2/\log \log n) \quad \text{as } n \to \infty, \]
and such that almost all the subsequences of this sequence are not summable \( (C,1) \).

This theorem gives us a negative answer for the Buck-Pollard problem, and comparing Theorem 1 and 2, we may say that the condition (5) is the best possible one of this form.

For the proof we will construct an example.

Let us put \( s_1 = 0 \) and \( s_n = (-1)^n \sqrt{n/\log \log n} \) \( (n = 1, 2, \ldots) \), then, as easily be seen, \( \{s_n\} \) is summable \( (C,1) \) to 0. We have
\[ B_n = \sum_{k=1}^{n} s_k^2 = \sum_{k=1}^{n} k/\log \log k \sim n^2/\log \log n \quad \text{as } n \to \infty, \]
and (9) is satisfied. Since \( B_n \to \infty \) and \( s_n = o (\sqrt{B_n/\log \log B_n}) \) as \( n \to \infty \), the conditions of the law of the iterated logarithm are fulfilled [3]. Hence
\[ \limsup_{n \to \infty} S_n(x)/\sqrt{2B_n \log \log B_n} = 1, \]
that is, \( \limsup_{n \to \infty} S_n(x)/n = \text{constant } = 0 \) almost everywhere. Thus the example was established.

§ 3. Theorem 3. If \( \{s_n\} \) is summable \( (C,1) \) but not convergent, then nearly all the subsequences of \( \{s_n\} \) are not summable \( (C,1) \).

Proof. If all the subsequences of \( \{s_n\} \) are summable \( (C,1) \), then \( \{s_n\} \) must be convergent (See e.g. [1]), hence from the assumption of the theorem there exists a subsequence \( \{s_{n_i}\} \) which is not summable \( (C,1) \). Let \( \{s_{n_i}\} = \left\{s_n \frac{1+o(x_n)}{2}\right\}, \)

1) \( P_n \sim Q_n \) means that \( P_n \) and \( Q_n \) are of the same order as \( n \to \infty \). \( P_n \sim Q_n \) means that \( P_n/Q_n \to 1 \) as \( n \to \infty \).
0 < x_0 < 1, where the terms with indices \( n \) such that \( \frac{1}{n} (1 + r_n(x_0)) = 0 \), are regarded to be omitted; evidently \( x_0 \) belongs to the set \( R \) of all dyadic irrationals.

Since \( \phi_n(x_0) \) is divergent, there exists a positive integer \( p_0 \) and a sequence of positive integers \( m_1 < m_2 < m_3 < \cdots \to \infty \), such that

\[
\left| \phi_{m_1}(x_0) - \phi_{n}(x_0) \right| > \frac{1}{p_0} \quad (i = 1, 2, \ldots).
\]

If we put \( E_{p,q} = R \cap [x; \left| \phi_m(x) - \phi_n(x) \right| \leq \frac{1}{p} (m, n > q)] \) \( (p, q = 1, 2, \ldots) \), then the set of \( x \in R \) for which the limit of (1) exists, may be represented as

\[
E = \bigcap_{p=1}^{\infty} \bigcup_{q=1}^{\infty} E_{p,q}.
\]

If we suppose that the set \( E \) is of the second category in \( R \), so is the set \( \bigcup_{p=1}^{\infty} E_{p,q_0} \) and then for some \( q_0 \), the set \( E_{p_0,q_0} \) is still of the second category in \( R \). The function \( \phi_n(x) \) being continuous in \( R \), the set \( E_{p_0,q_0} \) is closed in \( R \), and hence it contains an interval \( I \subset R \). Since there is a point \( x_1 \in I \) such that the difference \( |x_0 - x_1| \) is dyadically rational, we have

\[
\frac{1}{n} \sum_{k=1}^{n} r_k(x_0) - \frac{1}{n} \sum_{k=1}^{n} r_k(x_1), \quad \frac{1}{n} \sum_{k=1}^{n} s_k r_k(x_0) - \frac{1}{n} \sum_{k=1}^{n} s_k r_k(x_1)
\]

as \( n \to \infty \). Hence from (10) we have

\[
\left| \phi_{m_i}(x_1) - \phi_{n_i}(x_1) \right| > \frac{1}{2p_0}
\]

for large \( i \), which contradicts the fact \( x_1 \in I \subset E_{p_0,q_0} \).

Consequently the set \( E \) is of the first category in \( R \). The complement of \( R \), \( (0,1) - R \) being enumerable, the set \( E \) is of the second category in \( (0,1) \), q.e.d.

**References**


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