NOTES ON BANACH SPACE (XII): A REMARK ON A THEOREM OF GELFAND AND NEUMARK*)

By

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Following C.E.Rickart [7], a Banach algebra R over complex numbers having a principal unit 1 is called a *star-algebra* if it has an operation x^* satisfying

$$
\begin{aligned}
&1^\circ \quad (\lambda x + \mu y)^* = \lambda^* x^* + \mu^* y^*,\\
&2^\circ \quad (xy)^* = y^* x^*,\\
&3^\circ \quad x^{**} = x,\\
&4^\circ \quad |xx^*| = |x|^2.
\end{aligned}
$$

It is reported by the American literatures, although the original paper of I. Gelfand and M. Neumark [3] is not yet available in this country, that they have proved in 1943 that a commutative star-algebra is isometrically isomorphic with the algebra of all complex-valued functions on a compact Hausdorff space. Recently, R. Arens [1] simplified and clarified the proof of this theorem. In the general case, it is also reported, they have proved that the algebra is isometrically ismorphic with a ring of operators on a certain Hilbert space under an additional assumption, which states that $1 + xx^*$ has an inverse in R for any x.

The purpose of the present note is to show, that the later theorem is also t ue when the above condition is replaced by another one, and that the theorem is proved in a similar manner as in that of I. E. Sagal [8]. In the below, to save the spice, it is assumed that the readers are familiar with the stir-algebras, and so we will only describe the outline of the proofs when they are already known.

1. In this section, we may prove some geneneral properties of star algebras. The essential materials are taken from I. E. Segal [8].

Definition 1. A linear functional f on R is said to be *positive*, symbolically $f \geqq 0$, if $f(xx^*) \geqq 0$ for any x of R. A positive linear functional is

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called a *state* if $f(1)=1$ holds, and a state is called *pure* if it can not be expressed by a convex combination of other two states.

LEMMA 1. *The set Z of all states forms a convex weakly* compact set in the conjugate space of* R, *and the set P of all pure states is the set of all extreme points of Z. If Z is total then P is total.*

PROOF: The convexity of Z is obvious. It is somewhat easily proved that a state has the norm at most unity, whence it is a weakly* totally bounded set in the conjugate space by a corollary of the Tychonoff theorem (Conerning on this point, cf. for example, Kakutani [4]). Suppose that the f_α converge weakly* to f. Then $f_\alpha(xx^*) \geqq 0$ and $f_\alpha(1)=1$ imply $f(xx^*) \geqq 0$ and $f(1)=1$, that is, f is a state and Z is closed in the weak* topology. This proves the first half of the lemma since P is the set of all extreme points of Z by Definition 1. Now, let us suppose that Z is total. Then by a theorem of Krein-Milman [5] (cf.also Yosida-Fukamiya [9]) P is non-void and the weak closure of its convex hull is coincides with Z. Hence the totality of Z implies that of P. This completes the proof.

LEMMA 2. *If f is a state, then the set M of all elements x with $f(xx^*)=0$ forms a closed right-ideal of* R. *Moreover, if we put $f(xy^*)=(x,y)$ for $\bar{x}$ and $\bar{y}$ of* R$/M$ *which contains x and y respecively, then it gives an inner product of* R$/M$, *whence there exists a Hilbert space H containing* R$/M$ *as a dense subset. Consequently,* R *is represented continuously and homomorphically as a ring of operators on H.*

PROOF: Let A be the star-subalgebra of R generated[1] by 1 and xx^*. Then A is commutative, hence by the Gelfand-Neumark-Arens Theorem it is isometrically isomorphic with a certain $C(S)$. Hence $|x|^2 - xx^*$ corresponds to a positive function on S, and so there exists an element Z in A such as $|x|^2 - xx^* = zz^*$. And consequently, it holds $f(yzz^*y^*) = |x|^2 f(yy^*) - f(yxx^*y^*) \geqq 0$ for any y in R. This inequality and the positive linearity imply the first and the second statements of the lemma as usual calculations. Moreover, the inequality implies that R forms an operator-domain of the continuous endomorphisms of H if we define the multiplication by $\bar{x}a = \overline{xa}$. The norm $\|a\|$ as operators does not exceeds the norm $|a|$ of the algebra, since $(\bar{y}x, \bar{y}x) \leqq |x|^2(\bar{y}, \bar{y})$ holds for any x and y. By this definition the conjugate element

1) A subalgebra A of a star algebra R is called "generated by a set S of elements of R" if A is the smallest self-adjoint, metrically closed subalgebra containing S.

a^* of a corresponds to the adjoint operation of a, since we have $(\bar{x}a, \bar{y}) = f(xay^*) = f(x(ya^*)^*) = (\bar{x}, \bar{y}a^*)$. This proves the remainder of the lemma.

LEMMA 3. *If* R *has a total set of states such that for any x with the norm unity there exists a state f with $f(xx^*) = 1$, then* R *is isometrically isomorphic with a ring of operators of a Hilbert space which is the direct sum of H_f of Lemma* 2 *on the set P of all pure states.*

PROOF: Let H be the direct sum of H_f on P, that is, H is the space of all functions $\mathfrak{z}(f)$ on P assuming the values in H_f such that $\Sigma_f(\mathfrak{z}(f), \mathfrak{z}(f))$ is bounded. Since Z is total, some $\mathfrak{z}_1 = \bar{x}(f)$ does not vanish, where $\bar{x}(f)$ means the residue class of R/M_f containing x. Let $\mathfrak{z}_0$ be the residue class of R/M_f containing 1. Then evidently x transforms $\mathfrak{z}_0$ into $\mathfrak{z}_1$ in H_f, whence the mapping from R to a ring of operators R$'$ is one-to-one, that is, R is algebraically isomorphic with R$'$. Hence it remains to show that the mapping is norm-preserving. Since the weak* closure of the convex hull of P is Z, there exists a convex combination $\sum_{i=1}^{n} \alpha_i f_i$ of pure states such that

$$|f(xx^*) - \sum_{i=1}^{n} \alpha_i f_i(xx^*)| < \varepsilon$$

for any positive number ε. Let $\mathfrak{z}$ be an element of H such that $\mathfrak{z}(f_i) = \alpha_i^{1/2} \bar{x}_i$ $(i = 1, 2, \cdots, n)$ and $\mathfrak{z}(f) = 0$ for $f \neq f_i$ $(i = 1, 2, \cdots, n)$ where $\bar{x}_i = \bar{x}(f_i)$. Then by the above inequality we have

$$|\mathfrak{z}|^2 = \sum_i \alpha_i (\bar{x}_i, \bar{x}_i) = \sum_i \alpha_i f_i(xx^*) > 1 - \varepsilon$$

for x and f such that $|x| = 1$ and $f(xx^*) = 1$. Let furthermore $\mathfrak{z}_0$ be an element in H such that $\mathfrak{z}_0(f_i) = \alpha_i^{1/2} \bar{\mathfrak{z}}_0$ and $\mathfrak{z}_0(f) = 0$ otherwise where $\mathfrak{z}_0$ is the residue class of R$/M_f$ containing 1. Then evidently the norm of $\mathfrak{z}_0$ is unity and x transforms $\mathfrak{z}_0$ into $\mathfrak{z}$, whence the norm $\|x\|$ is greater than $1 - \varepsilon$. Since ε is arbitrary, we have $\|x\| = 1$, which is required.

2. Before proving that R satisfies the hypothesis of Lemma 3 under an additional condition, we make some remarks on the real linear space E of all hermitean elements of the alegbra. Firstly, it is not difficult to prove, that R is the direct sum $E + iE$ as real Banach space and each real linear functional of E is extensible to complex linear functional of R without increasing the norm. Nextly, we can define the *spectra* of an element x of R as the set of all λ's such that $\lambda - x$ has no inverse in R, as usually. Since it is proved by C. E. Rickart [7] that a star-subalgebra of R always contains the inverses of its elements if they exist, the spectra of an hermitean element x is determined uniquely by the commutative star-subalgabra $\mathcal{A}$ generated by 1 and x.

It is a direct consequence of the Gelfand-Neumark-Arens Theorem, that the spectra of an hermitean element are real. Hence it is possible to introduce an *order* in E such that $x \geqq 0$ if and only if the spectra of x contains no negative number. It is also a consequence of the above theorem, that each hermitean element of R is expressible as a difference of two non-negative elements and the order satisfies the archimedean postulate, that is, $-|x| \leqq x \leqq |x|$ for any x in E. Hence to prove the theorem, it is sufficient to show that F has sufficiently many states having the required property. After these considerations, now we introduce a definition as follows:

DEFINITION 2. A star-algebra R is called *archimedean* if $\sum_{i=1}^{n} x_i x_i^* = 0$ implies $x_i = 0$.

LEMMA 4. *By the above ordering the real linear space E of all hermitean elements of an archimdean algebra forms an archimedean ordered linear space, and the set N of all positive elements forms a convex cone having an inner point.*

PROOF: Firstly, we will prove, that $y = \sum_{i=1}^{n} x_i x_i^* \geqq 0$ for any x in R. Let us suppose the contrary and $y = v - w$ where v and w are positive hermitean elements of R (The existence of such v and w follows from the above remark). Since by the Gelfand-Neumark-Arens Theorem we can choose v and w such as $vw = wv = 0$, whence it holds $wyw = \sum_i wx_ix_i^*w^* = -w^3$. Since $w \geqq 0$ implies the existence of an hermitean element u whose square is w, it follows by the "archimedean" postulate $w = 0$. This is a contradiction.

From this it follows at once that ordering satisfies all the postulates of ordered linear spaces of G. Birkhoff [2; 105], and the set N of all non-negative elements forms a convex cone in E. The remainder of the lemma follows immediately from the Gelfand-Neumark-Arens Theorem, and we have that the norm of an hermitean element x is the infinimum of λ's such as $-\lambda \leqq x \leqq \lambda$. This proves the lemma.

LEMMA 5. *If* R *is archimedean, then the states are total and a state f exists with $f(xx^*) = 1$ for any x of the norm unity.*

PROOF: To prove that Z is total, it suffices to show that $f(x) = 0$ for all f in Z implies $x = 0$, where x is hermitean. Suppose that $|x| = 1$ (i. e. $x \leqq 1$) and $x \neq \lambda$. Then the half line starting from 1 and passing through x intersects at y with the frontier of N, the positive cone, since otherwise the interval (0, 1) of E (as an ordered space) contains the half line and a contradiction.

Suppose firstly $x \neq y$. Since N is convex body in E as proved in Lemma 4, by the well-known theorem of S. Mazur [6] there exists a contact hyperplane F of N pathing through y and not through x. Since N is a cone in E, F contains automatically the origin with y. Hence there exists a linear functional f' on E whose vanishing points coincide with F and $f'(x) \geqq 0$ for all positive x. Therefore, there exists a state f which is a scalar multiple of f', whence $f(y) = 0$ and so $f(x) \neq 0$.

Suppose nextly $x = y$, or more generally, suppose that x lies in the closure of N with the norm unity. Since $x \leqq 1$ by the assumption, $1 - x$ is non-negative and lies on the frontier of N, because otherwise there exists $\varepsilon > 0$ such that $1 - x$ situates at the center of a sphere with radius 2ε being contained in N and so $1 - x - \varepsilon$ is positive which contrary to the hypothesis that the norm of x is unity. Hence by the theorem of S. Mazur, there exists a contact hyperplane F of N passing through the origin and $1 - x$, and consequently a state f existe with $f(x) = 1$. This proves the remainders of the lemma, since x is of norm unity and so xx^* is positive and its norm is unity too.

THEOREM (Gelfand-Neumark). *An archimedean algebra is isometrically isomorphic with a ring of operators in a certain Hilbert space.*

REMARK. Firstly, lemma 4 shows, that the archimedean postulate implies Gelfand-Neumark's condition, since xx^* is positive and so has no negative spectrum. On the other hand, it is possible to deduce that Gelfand-Neumark's condition implies ours. Hence a star-algebra is archimedean if and only if it satisfies Gelfand-Neumark's condition. Therefore our method does not yield essential extention of the theorem. Secondly, it is also to be mentioned, that in the proof of the theorem Lemma 4 plays an essential rôle and neither the "archimedean" postulate nor Gelfand-Neumark's condition do appear explicitly. Hence, we can replace them by another suitable condition, which implies Lemma 4 to hold the theorem still true.

PROOF OF THE THEOREM: By Lemma 5 the existence of the set of state, whicn satisfies the hypothesis of Lemma 3, is proved, the theorem follows from Lemma 3 at once.

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Addendum (February 17, 1950): After this note is presented to the Editors, I. Kaplansky's "*Normed algebras*" (Duke Math. Journ., 16 (1949), 399-418) reached our university. In his paper, I. Kaplansky proved (among many other things) that two Banach star-algebras are isometrically isomorphic if and only if they are algebraically isomorphic. If we employ this result, instead of Lemma 5, we need only to prove the theorem that there exists a state f with $f(xx^*) \neq 0$ for any x, and Lemma 3 becomes redundant. Furthermore, he discussed the Gelfand-Neumark Theorem under the "archimedean postulate" (in much sharper form) with some algebraic restriction.

Also after this note is presented, the author has an oppotunity to read a manuscript of M. Fukamiya entitled as "*Normed ring with an involution*", which discusses the Gelfand-Neumark Theorem in different manner from more general point of view. M. Fukamiya's paper will appear soon in this Journal.