ON BOREL'S DIRECTIONS OF MEROMORPHIC FUNCTIONS OF FINITE ORDER*)

By

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Contents

1. Introduction.

Let $w(z)$ be meromorphic for $|z| < \infty$ and

$$T(r) = \int_0^r \frac{S(r)}{r}\, dr,$$

where

$$S(r) = \frac{1}{\pi} \int_0^r \int_0^{2\pi} \left(\frac{|w'(te^{i\theta})|}{1 + |w(te^{i\theta})|^2} \right)^2 t\, dt\, d\theta \tag{1}$$

be its Nevanlinna's characteristic function and

$$\overline{\lim_{r\to\infty}} \log T(r)/\log r = \rho \tag{2}$$

be its order. If $\rho < \infty$, then by Borel's theorem, for any $\varepsilon > 0$,

$$\sum_{\nu} 1/|z_\nu(a)|^{\rho+\varepsilon} < \infty$$

for any a and if $0 < \rho < \infty$,

$$\sum_{\nu} 1/|z_\nu(a)|^{\rho-\varepsilon} = \infty$$

for any a, with two possible exceptions, where $z_\nu(a)$ are zero points of $w(z) - a$.

Varilon[1] proved that there exists a direction J, which is called a Borel's direction, such that

$$\sum_{\nu} 1/|z_\nu(a, \Delta)|^{\rho-\varepsilon} = \infty,$$

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*) Received October 1, 1949.

1) G. Valiron: Recherches sur le théorème de M. Borel dans la théorie des fonctions méromorphes. Acta Math. **52** (1928).

for any a, with two possible exceptions, where Δ is any angular domain, which contains J and $z_\nu(a, \Delta)$ are zero points of $w(z) - a$ in Δ.

In §3, we will prove this Valiron's theorem simply by means of Theorem 2 of §2. In §5, we consider meromorphic functions in a half-plane $\Re z \geqq 0$ and establish theorems, which are analogous to Nevanlinna's fundamental theorems for meromorphic functions for $|z| < R$ ($\leqq \infty$) and by means of which we prove theorems of Valiron and Nevanlinna in § 6.

2. Main theorems.

THEOREM 1. *Let $w = w(z)$ be meromorphic in $|z| < 1$ and the number of zero points of $(w(z) - a_1)\ (w(z) - a_2)\ (w(z) - a_3)$ in $|z| < 1$ be $\leqq n$, where multiple zeros are counted only once, then*

$$S(r) \leqq n + A/(1 - r), \quad (0 \leqq r < 1),$$

where A is a constant, which depends on a_1, a_2, a_3 only.

PROOF. Let $z_1, \cdots, z_\nu$ $(\nu \leqq n)$ be zero points of $\prod_{i=1}^{3}(w(z) - a_i)$ in $|z| < 1$ and ff these points from $|z| < 1$ and D_0 be the remaining domain and the part of D_0, which lies in $|z| \leqq r$ (< 1). Let F_r be the Riemann read upon the w-sphere, which is generated by $w = w(z)$, when z $D_0(r)$, then F_r is a covering surface of the basic domain F_0, which is from the w-sphere by taking off three points a_1, a_2, a_3. Let $\rho(r)$ be s characteristic of F_r, then by Ahlfors' fundamental theorem on surfaces,[2]

$$\overset{+}{\rho}(r) \geqq S(r) - hL(r), \qquad \overset{+}{\rho}(r) = \text{Max.}\ (\rho(r), 0), \tag{1}$$

$$L(r) = \int_0^{2\pi} \frac{|w'(re^{i\theta})|}{1 + |w(re^{i\theta})|^2} r\, d\theta \tag{2}$$

ι constant, which depends on a_1, a_2, a_3 only.
hwarz's inequality, we have

$$[L(r)]^2 \leqq 2\pi^2 r \frac{dS(r)}{dr}. \tag{3}$$

the hypothesis, $\overset{+}{\rho}(r) \leqq n$, we have by (1),

$$S(r) - n \leqq hL(r), \qquad (0 \leqq r < 1). \tag{4}$$

nce if $S(r') - n > 0$ for all r' $(r \leqq r' < 1)$, then by (3), (4), $1 - r < \int_r^1 \frac{1}{r'} dr'$ $2\pi^2 h^2 \int_r^1 \frac{1}{(S(r') - n)^2} dS(r') \leqq 2\pi^2 h^2 / (S(r) - n)$, or

) L. Ahlfors: Zur Theorie der Überlagelungsflächen, Acta Math., **65** (1935).

$$S(r) \leqq n + 2\pi^2 h^2 / (1-r). \tag{5}$$

If $S(r') - n \leqq 0$ for some $r'(r \leqq r' < 1)$, then $S(r) \leqq S(r') \leqq n$, so that (5) holds. Hence (5) holds for $0 \leqq r < 1$, which proves the theorem.

Let $w(z)$ be meromorphic in an angular domain $\Delta: |\arg z| \leqq \alpha$ and put

$$S(r; \Delta) = \frac{1}{\pi} \int_1^r \int_{-\alpha}^{\alpha} \left(\frac{|w'(te^{i\theta})|}{1+|w(te^{i\theta})|^2} \right)^2 t dt d\theta,$$

$$T(r; \Delta) = \int_1^r \frac{S(t; \Delta)}{t} dt,$$

$$N(r, a; \Delta) = \int_1^r \frac{n(t, a; \Delta)}{t} dt, \tag{1}$$

where $n(r, a; \Delta)$ is the number of zero points of $w(z) - a$ in a sector: $|\arg z| \leqq \alpha$, $0 \leqq |z| \leqq r$, where multiple zeros are counted only once.

THEOREM 2. *Let $w(z)$ be meromorphic in an angular domain $\Delta_0: |\arg z| \leqq \alpha_0$ and $\Delta: |\arg z| \leqq \alpha < \alpha_0$ be an angular domain contained in Δ_0. Then for any $\lambda > 1$*

$$T(r; \Delta) \leqq 3 \sum_{i=1}^{3} N(\lambda r, a \; ; \Delta_0) + A(\log^2,$$

where A is a constant, which depends on a_1, a_2, a_3, α, α_0, λ only.

PROOF. We put $k = \lambda^{1/3} > 1$ and for $r > 1$, let

$$N = [\log r / \log k], \qquad k^N \leqq r < k^{N+1}, \tag{2}$$

so that

$$k^{N+3} = \lambda k^N \leqq \lambda r. \tag{3}$$

Let Q_ν^0, Q_ν be curvilinear quadrilaterals:

$$Q_\nu^0: \ |\arg z| \leqq \alpha_0, \ k^{\nu-2} \leqq |z| \leqq k^{\nu+1},$$

$$Q_\nu: \ |\arg z| \leqq \alpha, \ k^{\nu-1} \leqq |z| \leqq k^\nu, \tag{4}$$

$$S_\nu = \frac{1}{\pi} \int_{Q_\nu} \int \left(\frac{w'(te^{i\theta})}{1+|w(te^{i\theta})|^2} \right)^2 t \, dt \, d\theta$$

and n_ν^0 be the number of zero points of $\prod_{i=1}^{3}(w(z) - a_i)$ in Q_ν^0. If we map Q_1^0 on $|\zeta| < 1$ conformally, such that the center of Q_1^0 becomes $\zeta = 0$, then Q_1 is mapped on a domain, which lies in $|\zeta| \leqq \rho < 1$.

Since Q_ν^0 is similar to Q_1^0, we have by Theorem 1,

$$S_\nu \leqq n_\nu^0 + A, \tag{5}$$

where A is a constant, which depends on $a_1, a_2, a_3, \alpha, \alpha_0, \lambda$ only.

In the following, we denote such constants by the same letter A.

We put

$$n(r; \Delta_0) = \sum_{i=1}^{3} n(r, a_i; \Delta_0). \tag{6}$$

Since Q_ν^0 overlap only twice,

$$\int_{k^{\nu-1}}^{k^\nu} \frac{S(t;\Delta)}{t} \leqq S(k^\nu; \Delta) \log k = (S_1 + \cdots + S_\nu) \log k$$
$$\leqq (n_0^1 + \cdots + n_\nu^0) \log k + A\nu \leqq 3n(k^{\nu+1}; \Delta_0) \log k + A\nu,$$

so that

$$\int_1^r \frac{S(t;\Delta)}{t} dt \leqq \int_1^{k^{N+1}} \frac{S(t;\Delta)}{t} dt = \sum_{\nu=1}^{N+1} \int_{k^{\nu-1}}^{k^\nu} \frac{S(t;\Delta)}{t} dt$$
$$\leqq 3\,(n(k^1; \Delta_0) + \cdots + n(k^{N+2}; \Delta_0)) \log k + AN^2. \tag{7}$$

Since

$$\int_{k^\nu}^{k^{\nu+1}} \frac{n(t;\Delta_0)}{t} dt \geqq n(k^\nu; \Delta_0) \log k,$$

we have from (7), (3),

$$T(r;\Delta) = \int_1^r \frac{S(t;\Delta)}{t} dt \leqq 3\int_1^{k^{N+3}} \frac{n(t;\Delta_0)}{t} dt + AN^2$$
$$\leqq 3\int_1^{\lambda r} \frac{n(t;\Delta_0)}{t} dt + A(\log r)^2 = 3\sum_{i=1}^{3} N(\lambda r, a_i; \Delta_0) + A(\log r)^2.$$

3. Existence of Borel's directions.

1. Now we will prove Valiron's theorem:

THEOREM 3. *Let $w(z)$ be a meromorphic function of finite order $\rho > 0$, then there exists a direction J:* $\arg z = \alpha$, *such that for any* $\varepsilon > 0$.

(i) $$\sum_\nu 1 / |z_\nu(a;\Delta)|^{\rho-\varepsilon} = \infty$$

for any a, with two possible exceptions, and if

$$\int^\infty \frac{T(r)}{r^{\rho+1}} dr = \infty$$

then

(ii) $$\sum_{\nu} 1/|z_\nu(a, \Delta)|^\rho = \infty,$$

for any a, with two possible exceptions, where Δ is any angular domain, which contains J, and $z_\nu(a, \Delta)$ are zero points of $w(z) - a$ in Δ, multiple zeros being counted only once.

PROOF. Suppose that for some $k > 0$,

$$\int^{\infty} \frac{T(r)}{r^{k+1}} dr = \infty. \tag{1}$$

Then dividing $(0, 2\pi)$ into 2^n equal parts, we see that there exists an angular domain Δ_n of magnitude $2\pi/2^n$, such that $\Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_n \cdots$,

$$\int^{\infty} \frac{T(r; \Delta_n)}{r^{k+1}} dr = \infty, \qquad (n=1, 2, \cdots). \tag{2}$$

Let Δ_n converge to a direction J: $\arg z = \alpha$, then for any angular domain Δ: $|\arg z - \alpha| \leqq \delta$, which contains J, $\Delta_n \subset \Delta$ for $n \geqq n_0$, so that

$$\int^{\infty} \frac{T(r; \Delta)}{r^{k+1}} dr = \infty. \tag{3}$$

Let Δ_0: $|\arg z - \alpha| \leqq 2\delta$, then by Theorem 2,

$$\int_1^r \frac{T(r; \Delta)}{r^{k+1}} dr \leqq 3 \sum_{i=1}^{3} \int_1^r \frac{N(\lambda r, a_i; \Delta)}{r^{k+1}} dr + O(1) \qquad (\lambda > 1),$$

so that from (3),

$$\int^{\infty} \frac{N(r, a; \Delta_0)}{r^{k+1}} dr = \infty, \quad \text{or} \quad \sum_{\nu} 1/|z_\nu(a, \Delta_0)|^k = \infty$$

for any a, with two possible exceptions.

Since $\int^{\infty} \frac{T(r)}{r^{\rho-\varepsilon+1}} dr = \infty$, if we take $k = \rho - \varepsilon$, then we have (i) and for $k = \rho$, we have (ii). q. e. d.

2. Theorem 3 can be extended as follows.

THEOREM 4. *Let C: $z = z(t)$ $(0 \leqq t < \infty)$ $(z(0) = 0,\ z(\infty) = \infty)$ be a simple curve, which connects $z = 0$ to $z = \infty$ and for any $\delta > 0$, let $\Delta(\delta)$ be the set of points, which is covered by all discs: $|z - z(t)| \leqq |z(t)|\,\delta$ $(0 \leqq t < \infty)$ and $\Delta_\theta(\delta)$ be the set obtained from $\Delta(\delta)$ by rotating an angle θ. Let $w = w(z)$ be a meromorphic function of finite order $\rho > 0$ for $|z| < \infty$. Then there exists a certain θ_0, such that for any $\delta > 0$, $\varepsilon > 0$,*

(i) $\sum_{\nu} 1/|z_\nu(a; \Delta_{\theta_0}(\delta))|^{\rho-\varepsilon} = \infty$, *for any a, with two possible exceptions and if $w(z)$ is of order ρ of divergence type, then*

(ii) $\sum_{\nu} 1/|z_\nu(a; \Delta_{\theta_0}(\delta))|^{\rho} = \infty$, *for any a, with two possible exceptions, where $z_\nu(a; \Delta_{\theta_0}(\delta))$ are zero points of $w(z) - a$ in $\Delta_{\theta_0}(\delta)$.*

First we prove a lemma.

LEMMA. *Let E be a closed set contained in $|z| \leqq 1$ and $0 < \rho < 1$. Then we can cover E by N circles C_i $(i = 1, 2, \cdots, N)$ of radius ρ with its center $(z_i \in E)$, such that $N \leqq 16\pi/(\sqrt{3}\,\rho^2)$ and circles C_i^0 $(i = 1, 2, \cdots, N)$ of radius 2ρ with center z_i overlap at most 54-times.*

PROOF. We cover the z-plane by a net of regular triangles, whose vertices are $z_{mn} = m\rho e^{\pi i/3} + n\rho$ $(m, n = 0, \pm 1, \pm 2, \cdots)$. Let $\Delta_1, \Delta_2, \cdots, \Delta_N$ be the triangles, which contain points of E, then since Δ_i is contained in $|z| \leqq 1 + \rho$ and the area of Δ_i is $\sqrt{3}\,\rho^2/4$ and $0 < \rho < 1$,

$$N \leqq \pi(1+\rho)^2 \Big/ \frac{\sqrt{3}\rho^2}{4} = \frac{4\pi}{\sqrt{3}}\left(1 + \frac{1}{\rho}\right)^2 \leqq \frac{4\pi}{\sqrt{3}}\left(\frac{1}{\rho} + \frac{1}{\rho}\right)^2 = \frac{16\pi}{\sqrt{3}\rho^2}.$$

We take a point z_i $(\in E)$ in Δ_i and draw a circle C_i of radius ρ with z_i as its center, then C_i contains Δ_i, so that $C_1, \cdots, C_N$ cover E. Let C_i^0 be a circle of radius 2ρ with z_i as its center, then it is easily seen that C_i^0 overlap at most 54-times.

PROOF OF THEOREM 4. Let $k > 1$ and $\Delta_\nu(\delta)$ be the part of $\Delta(\delta)$ contained in $k^{\nu-1} \leqq |z| \leqq k^\nu$ $(\nu = 0, 1, 2, \cdots)$ and $\Delta_\nu^0(3\delta)$ be the part of $\Delta(3\delta)$ contained in $k^{\nu-2} \leqq |z| \leqq k^{\nu+1}$, so that $\Delta_\nu(\delta) \subset \Delta_\nu^0(3\delta)$. By transforming $\Delta_\nu(\delta)$ into a closed set in $|\zeta| \leqq 1$ by $\zeta = \frac{z}{k^\nu}$ and applying the lemma, with $\rho = \frac{\delta}{k}$, we see easily that $\Delta_\nu(\delta)$ can be covered by N circles $C_\nu^{(i)}$ $(i = 1, 2, \cdots, N)$ of radius $k^{\nu-1}\delta$ and center $z_\nu^{(i)}$ $(\in \Delta_\nu(\delta))$, such that

$$N \leqq \frac{16\pi}{\sqrt{3}} \frac{k^2}{\delta^2}$$

and circles $C_\nu^{0(i)}$ of radius $2k^{\nu-1}\delta$ with center $z_\nu^{(i)}$ overlap at most 54-times.

Let a_1, a_2, a_3 be any three values and $S_\nu, S_\nu^{(i)}$ be the area on the w-sphere generated by $w = w(z)$, when z varies in $\Delta_\nu(\delta)$, C_ν^i and $n_\nu^0, n_\nu^{0(i)}$ be the number of zero points of $\prod_{i=1}^{3}(w(z) - a_i)$ in $\Delta_\nu(3\delta)$, $C_\nu^{0(i)}$ respectively, then by Theorem 1,

$$S_\nu^{(i)} \leqq n_\nu^{0(i)} + A,$$

where A depends on a_1, a_2, a_3, k, δ only.

Since $C_\nu^{0\,(i)}$ is contained in $\Delta_\nu(3\delta)$ and overlap at most 54-times and $S_\nu \leqq \sum_{i=1}^{N} S_\nu^{(i)}$, we have

$$S_\nu \leqq 54 n_\nu^0 + NA.$$

From this we have the similar theorem as Theorem 2, where $\Delta = \Delta(\delta)$, $\Delta_0 = \Delta(3\delta)$ and from this we can prove Therem 4 as Theorem 3.

REMARK. From (3) in the proof of Theorem 3, we see that there exists $r_1 < r_2 < \cdots < r_n \to \infty$, such that

$$\lim_{n\to\infty} S(r_n; \Delta)/\log r_n = \infty. \tag{4}$$

Let

$$N = [\log r_n / \log k], \quad k^N \leqq r < k^{N+1},$$

then from (4), there exists a certain curvilinear quadrilateral Q_n: $|\arg z - \alpha| \leqq \delta$, $k^{\nu_n - 1} \leqq |z| \leqq k^{\nu_n} (\nu_n \leqq N)$, such that

$$S_n = \frac{1}{\pi} \iint_{Q_n} \left(\frac{|w'(te^{i\theta})|}{1+|w(te^{i\theta})|^2} \right)^2 t\,dt\,d\theta \to \infty \quad (n \to \infty).$$

Let Q_n^0: $|\arg z - \alpha| \leqq 2\delta$, $k^{\nu_n - 2} \leqq |z| \leqq k^{\nu_n + 1}$. We map Q_n^0 conformally on $|\zeta| < 1$ by $w = w(\zeta)$, such that the center of Q_n becomes $\zeta = 0$, then the image of Q_n lies in $|\zeta| \leqq \eta < 1$, where η depends on k, δ only. We put $w(z) = v(\zeta)$ and put

$$S(r) = \frac{1}{\pi} \int_0^r \int_0^{2\pi} \left(\frac{|v'(te^{i\theta})|}{1+|v(te^{i\theta})|^2} \right)^2 t\,dt\,d\theta \qquad (0 \leqq r \leqq 1),$$

$$L(r) = \int_0^{2\pi} \frac{|v'(re^{i\theta})|}{1+|v(re^{i\theta})|^2} r\,d\theta,$$

then $S_n \leqq S(\eta)$ and

$$(L(r))^2 \leqq 2\pi^2 r \frac{dS(r)}{dr}.$$

Suppose that

$$L(r) \geqq (S(r))^{3/4} \qquad \text{for } \eta \leqq r \leqq 1,$$

then

$$(S(r))^{3/2} \leqq 2\pi^2 r \frac{dS(r)}{dr},$$

$$1-\eta \leqq \int_\eta^1 \frac{dr}{r} \leqq 2\pi^2 \int_\eta^1 \frac{dS(r)}{(S(r))^{3/2}} \leqq \frac{4\pi^2}{S(\eta)^{1/2}}, \text{ or}$$

$$S_n \leqq S(\eta) \leqq \left(\frac{4\pi^2}{1-\eta}\right)^2.$$

Hence if $S_n > \left(\frac{4\pi^2}{1-\eta}\right)^2$, then there exists a certain r_n $(\eta \leqq r_n \leqq 1)$, such that $L(r_n) < (S(r_n))^{3/4}$, or

$$L(r_n) / S(r_n) < 1 / S(r_n)^{1/4} \leqq 1 / S_n^{1/4} \to 0 \quad (n \to \infty).$$

From this we conclude by Ahlfors' theorem on covering surfaces, the following theorem:

Let J: $\arg z = \alpha$ be a Borel's direction, then for any $\delta > 0$, the image of Δ; $|\arg z - \alpha| \leqq \delta$ by $w = w(z)$ on the w-shere covers schlicht infinitely often one of any five disjoint simply connected domains on the w-sphere.

4. Borel's directions of meromorphic functions of zero order.

We consider meromorphic functions of zero order, such that

$$\lim_{r\to\infty} \log T(r) / \log r = 0, \qquad \overline{\lim_{r\to\infty}}\, T(r) / (\log r)^2 = \infty.$$

First we will prove a lemma.

LEMMA. *Let $T(r) > 0$ be an increasing function, such that*

$$\lim_{r\to\infty} \log T(r) / \log r = 0, \qquad \overline{\lim_{r\to\infty}}\, T(r) / (\log r)^2 = \infty,$$

then for any $\lambda > 1$, $k > 1$, there exists $r_1 < r_2 \cdots < r_n \to \infty$, such that

$$\lim_{n\to\infty} T(r_n) / (\log r_n)^2 = \infty, \; T(\lambda r_n) \leqq kT(r_n) \; (n = 1, 2, \cdots).$$

PROOF. First we will prove that for any $M > 0$, there exists $\nu_1 < \nu_2 < \cdots < \nu_n \to \infty$, such that

$$T(\lambda^\nu) \geqq M (\log \lambda^\nu)^2 \tag{1}$$

holds for $\nu = \nu_n$ $(n = 1, 2, \ldots\ldots)$.

For, if for $\nu \geqq \nu_0$, $T(\lambda^\nu) < M(\log \lambda^\nu)^2$, then for $\lambda^\nu \leqq r < \lambda^{\nu+1}$, $T(r) \leqq T(\lambda^{\nu+1}) < M(\log \lambda^{\nu+1})^2 = M((\nu+1)/\nu)^2 (\log \lambda^\nu)^2 \leqq M((\nu+1)/\nu)^2 (\log r)^2$, so that

$$\overline{\lim_{r\to\infty}}\, T(r) / (\log r)^2 \leqq M < \infty,$$

which contradicts the hypothesis, hence (1) holds for an infinite number of ν.

Next we will prove that there exists an infinite number of ν, for which (1) and

$$T(\lambda^{\nu+1}) \leqq kT(\lambda^{\nu}) \tag{2}$$

hold simultaneously.

For, suppose that for all $\nu \geqq \nu_0$, for which (1) holds,

$$T(\lambda^{\nu+1}) > kT(\lambda^{\nu}), \tag{3}$$

then since $k > 1$,

$$T(\lambda^{\nu+1}) > kT(\lambda^{\nu}) \geqq kM(\log\lambda^{\nu})^2 = kM(\nu/(\nu+1))^2 (\log\lambda^{\nu+1})^2$$
$$\geqq M(\log\lambda^{\nu+1})^2, \qquad (\nu \geqq 1/(\sqrt{k}-1)),$$

so that $\lambda^{\nu+1}$ satisfies (1), hence by the hypothesis,

$$T(\lambda^{\nu+2}) > kT(\lambda^{\nu+1}).$$

Hence (3) holds for all sufficiently large ν, so that

$$\overline{\lim_{r\to\infty}} \log T(r)/\log r \geqq \log k/\log\lambda > 0,$$

which contradicts the hypothesis, hence there exists an infinite number of ν, which satisfy (1) and (2) simultaneously. If we take $M_1 < M_2 < \cdots < M_n \to \infty$ for M, then we have the lemma.

THEOREM 5[3]. *Let $w(z)$ be a meromorphic function of order zero, such that*

$$\overline{\lim_{r\to\infty}} T(r)/(\log r)^2 = \infty,$$

then there exists a direction J: $\arg z = \alpha$, such that for any angular domain Δ: $|\arg z - \alpha| \leqq \delta$, which contains J,

$$\overline{\lim_{n\to\infty}} N(r_n, a; \Delta)/T(r_n) \geqq |\Delta|/(72\pi), \qquad (|\Delta| = 2\delta)$$

for any a, with two possible exceptions, where the sequence $\{r_n\}$ is independent of a and Δ, such that

$$\lim_{n\to\infty} T(r_n)/(\log r_n)^2 = \infty.$$

PROOF. By the lemma, for any $\lambda > 1$, $k > 1$, there exists $\{r_n\}$, such that

$$\lim_{n\to\infty} T(r_n)/(\log r_n)^2 = \infty, \; T(\lambda r_n) \leqq kT(r_n), \; (n = 1, 2, \cdots). \tag{1}$$

3) G. Valiron: Sur les directions de Borel des fonctions méromorphes d'ordre nul, Bul. Sci. Math. **39** (1935).

By dividing $(0, 2\pi)$ into 2^m equal parts, we see that there exists an angular domain Δ_m of magnitude $2\pi/2^m$, such that $\Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_m \supset \cdots$,

$$T(r_n; \Delta_m) \geqq T(r_n)/2^m \tag{2}$$

holds for an infinite number of n.

Let Δ_m converge to a direction J: $\arg z = \alpha$ and Δ: $|\arg z - \alpha| \leqq \delta(1-\varepsilon)$ $(\varepsilon > 0)$ be any angular domain, which contains J.

Let m be such that $2\pi/2^m \leqq \delta(1-\varepsilon) < 2\pi/2^{m-1}$, then $\Delta \supset \Delta_m$, so that by (2), (1),

$$T(r_n; \Delta) \geqq T(r_n; \Delta_m) \geqq 2^{-m} T(r_n) \geqq k^{-1} 2^{-m} T(\lambda r_n) \tag{3}$$

holds for an infinite number of n.

Let Δ_0: $|\arg z - \alpha| \leqq \delta$, then

$$|\Delta_0| = 2\delta < 8\pi/(2^m(1-\varepsilon)). \tag{4}$$

We apply Theorem 2 for Δ_0, Δ and r_n, then

$$T(\lambda r_n)/k2^m \leqq T(r_n; \Delta) \leqq 3\sum_{i=1}^{3} N(\lambda r_n, a_i; \Delta_0) + A(\log r_n)^2,$$

hence by (1), (4),

$$|\Delta_0|(1-\varepsilon)/(24k\pi) \leqq \sum_{i=1}^{3} \overline{\lim_{n\to\infty}}\, N(\lambda r_n, a_i; \Delta_0)/T(\lambda r_n).$$

If we make $\varepsilon \to 0$, $k \to 1$, we have

$$|\Delta_0|/(24\pi) \leqq \sum_{i=1}^{3} \overline{\lim_{n\to\infty}}\, N(\lambda r_n, a_i; \Delta_0)/T(\lambda r_n).$$

Hence

$$\overline{\lim_{n\to\infty}}\, N(\lambda r_n, a; \Delta_0)/(T\lambda r_n) \geqq |\Delta_0|/(72\pi),$$

with two possible exceptions. If we write r_n, Δ instead of λr_n, Δ_0, then we have the theorem.

5. Meromorphic functions in a half-plane.

1. First fundamental theorem.

Let $w(z)$ be meromorphic in $\Re z \geqq 0$ and let $z = \rho e^{i\theta}$ $(|\theta| \leqq \pi/2)$,

$$\zeta = -1/z = \sigma + it, \qquad \sigma = -\cos\theta/\rho, \quad t = \sin\theta/\rho, \tag{1}$$

then the niveau curve $\Re(1/z) = \text{const.} = 1/r$, or

$$\sigma = \text{const.} = -1/r \qquad (r > 0) \tag{2}$$

is a circle: $r\cos\theta = \rho$, whose diameter is r and which touches the imaginary axis at the origin and the niveau curve

$$t = \text{const.} = 1/t_0 \tag{3}$$

is a circle, whose diameter is $|t_0|$ and which touches the real axis at the origin. Hence to a rectangle Q_σ on the ζ-plane, which is bounded by four lines: $t = \pm\pi$, $\sigma = \sigma_0 = -1/r_0$, $\sigma = -1/r\,(r > r_0)$, there corresponds on the z-plane a domain Δ_r, which is bounded by four circles.

We put $w(z) = \mathfrak{w}(\zeta)$ and let $\mathfrak{n}(\sigma, a)$ be the number of zero points of $\mathfrak{w}(\zeta) - a$ in Q_σ and

$$\mathfrak{m}(\sigma, a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log \frac{1}{[\mathfrak{w}(\sigma + it), a]}\, dt, \tag{4}$$

$$\mathfrak{N}(\sigma, a) = \int_{\sigma_0}^{\sigma} \mathfrak{n}(\sigma, a)\, d\sigma, \tag{5}$$

where

$$[a, b] = |a - b|/[(1 + |a|^2)(1 + |b^2|)]^{1/2}. \tag{6}$$

Since $w(z)$ is meromorphic on three circles, which correspond to three lines; $\sigma = \sigma_0$, $t = \pm\pi$, we have by the argument princ ple, if $\mathfrak{w}(\zeta) \neq a, \neq b$ on $\Re\zeta = \sigma$,

$$\frac{\partial \mathfrak{m}(\sigma, a)}{\partial \sigma} - \frac{\partial \mathfrak{m}(\sigma, b)}{\partial \sigma} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial}{\partial \sigma} \log\left|\frac{\mathfrak{w} - b}{\mathfrak{w} - a}\right| dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\arg\left(\frac{\mathfrak{w} - b}{\mathfrak{w} - a}\right) = \mathfrak{n}(\sigma, b) - \mathfrak{n}(\sigma, a) + O(1),$$

so that

$$\mathfrak{m}(\sigma, a) + \mathfrak{N}(\sigma, a) = \mathfrak{m}(\sigma, b) + \mathfrak{N}(\sigma, b) + O(1). \tag{7}$$

Returning to the z-plane, if we write

$$\mathfrak{m}(\sigma, a) = m(r, a),\ \mathfrak{n}(\sigma, a) = n(r, a),\ \mathfrak{N}(\sigma, a) = N(r, a),$$

then we have easily

$$m(r, a) = \frac{1}{2\pi r}\int_{-\tan^{-1}\pi r}^{\tan^{-1}\pi r} \log(1/[w(z), a])\sec^2\theta\, d\theta, \tag{8}$$

$$N(r, a) = \int_{r_0}^{r} \frac{n(r, a)}{r^2}\, dr, \tag{9}$$

where the right hand side of (8) is integrated on a circle $\Re(1/z) = 1/r$ and $n(r, a)$ is the number of zero points of $w(z) - a$ in Δ_r. If we put

$$T(r, a) = m(r, a) + N(r, a), \tag{10}$$

then (7) becomes

$$T(r, a) = T(r, b) + O(1). \tag{11}$$

From this we have easily the following

THEOREM 6. (*First fundamental theorem*).

$$T(r, a) = T(r) + O(1),$$

where

$$T(r) = \int_{r_0}^{r} \frac{S(r)}{r^2}\, dr,$$

$$S(r) = \frac{1}{\pi} \int_{\Delta}\int \left(\frac{|w'(\rho e^{i\theta})|}{1 + |w(\rho e^{i\theta})|^2}\right)^2 \rho\, d\rho\, d\theta.$$

Hence $T(r)$ is an increasing convex function of $\sigma = -1/r$. We call $T(r)$ the characteristic function of $w(z)$ for $\Re z \geqq 0$.

2. It can easily be proved:

THEOREM 7. $\int^{\infty} \frac{T(r)}{r^{\lambda+1}}\, dr$ *and* $\int^{\infty} \frac{S(r)}{r^{\lambda+2}}\, dr$ $(\lambda > 0)$ *converge simultaneously and*

$$\int^{\infty} \frac{N(r, a)}{r^{\lambda+1}}\, dr, \quad \int^{\infty} \frac{n(r, a)}{r^{\lambda+2}}\, dr, \quad \sum_{\nu} [\Re(1/z_{\nu}(a))]^{\lambda+1} \qquad (\lambda>0)$$

converge simultaneouly, where $z_{\nu}(a)$ *are zero points of* $w(z) - a$.

THEOREM 8. *Let* $w(z)$ *be regular for* $\Re z \geqq 0$ *and* $\Delta: |\arg z| \leqq \alpha < \pi/2$,

$$M(r; \Delta) = \operatorname*{Max}_{|\theta| \leqq \alpha} |w(re^{i\theta})|,$$

then

$$\overset{+}{\log} M(r; \Delta) \leqq Ar\,(T(\lambda r) + O(1)),$$

where

$$A = 2(1 + \sin\alpha)/\{\cos\alpha\,(1 + \sin\alpha)\}, \qquad \lambda = 2/\cos\alpha.$$

PROOF. Let $M(r, \Delta) = \operatorname*{Max.}_{|\theta| \leqq \alpha} |w(re^{i\theta})|$ be attained at $z_0 = re^{i\theta_0}$ $(|\theta_0| \leqq \alpha)$, which lies in a circle $|z - \rho| = \rho \sin\alpha$ $(\rho = r/\cos\alpha)$, which touches two lines $\arg z = \pm\alpha$, so that

$$z_0 = re^{i\theta_0} = \rho + t_0 e^{i\varphi_0}, \quad |t_0| \leqq \rho \sin\alpha.$$

Since $\overset{+}{\log} |w(z)|$ is subharmonic, we have by means of Poisson integral on $|z - \rho| = \rho$,

$$\log^+ M(r;\Delta) = \log^+ |w(z_0)| \leqq \frac{\rho+|t_0|}{\rho-|t_0|}\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+|w(\rho+\rho e^{i\theta})|\,d\theta$$
$$\leqq \frac{1+\sin\alpha}{1-\sin\alpha}\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(1+|w(\rho+\rho e^{i\theta})|^2\right)^{1/2}d\theta$$
$$= \frac{1+\sin\alpha}{1-\sin\alpha}\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left[1/[w(\rho+\rho e^{i\theta}),\ \infty]\right]d\theta$$
$$\leqq \frac{1+\sin\alpha}{1-\sin\alpha}2\rho(m(2\rho,\ \infty)+O(1)) = \frac{1+\sin\alpha}{1-\sin\alpha}2\rho\,(T(2\rho,\ \infty)+O(1))$$
$$\leqq \frac{1+\sin\alpha}{1-\sin\alpha}2\rho(T(2\rho)+O(1)) = Ar(T(\lambda r)+O(1)),$$

where

$$A = 2(1+\sin\alpha)/\{\cos\alpha(1-\sin\alpha)\}, \qquad \lambda = 2/\cos\alpha.$$

THEOREM 9. *Let* $w(z)$ *be meromorphic in* $\Re(z)\geqq 0$ *and* $T(r)=O(1)$, *then* $w(z)=g(z)/h(z)$, *where* $g(z)$, $h(z)$ *are regular and* $|g(z)|\leqq 1$, $|h(z)|\leqq 1$ *for* $\Re(z)>0$.

PROOF. By $x=(z-1)/(z+1)$, we map $\Re(z)\geqq 0$ on $|x|<1$ and put $w(z)=w_1(x)$ and $T_1(\rho)$ be the Nevanlinna's characteriatic function of $w_1(x)$ in $|x|<1$,

$$T_1(\rho)=\int_0^\rho \frac{S_1(\rho)}{\rho}\,d\rho \qquad (0\leqq\rho<1),$$
$$S_1(\rho)=\frac{1}{\pi}\int_0^\rho\int_0^{2\pi}\left(\frac{|w_1'(re^{i\theta})|}{1+|w_1(re^{i\theta})|^2}\right)^2 r\,dr\,d\theta.$$

Since the circle $\Re(1/z)=1/r\ (r>1)$ is mapped on a circle, which contains a circle $|x|=(r-1)/(r+1)=\rho$,

$$S_1(\rho)\leqq S(r)+O(1) \qquad (\rho=(r-1)/(r+1)),$$

and since $d\rho/\rho = 2/(r^2-1)\,dr \leqq 4/r^2\,dr\ (r\geqq\sqrt{2})$, we have

$$\int^1 \frac{S_1(\rho)}{\rho}\,d\rho \leqq 4\int^\infty \frac{S(r)}{r^2}\,dr+O(1)=O(1).$$

Hence $T_1(\rho)=O(1)$, so that by Nevanlinna's theorem, $w_1(x)=g_1(x)/h_1(x)$, where $g_1(x)$, $h_1(x)$ are regular and $|g_1(x)|\leqq 1$, $|h_1(x)|\leqq 1$ in $|x|<1$. Returning to the z-plane, we have the theorem.

3. SECOND FUNDAMENTAL THEOREM.

In Ahlfors' proof of Nevanlinna's second fundamental theorem,[4] if we

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4) L. Ahlfors: Über eine Methode in der Theorie der meromorphen Funktionen, Soc. Sci. Fenn. Comment. Phys-Math. 8, No. 10 (1932).

replace $\log z = \log r + i\theta$ by $\zeta = -1/z = \sigma + it$, we have the following

THEOREM 10. (*Second fundamental theorem*).

$$(q-2)T(r) \leqq \sum_{i=1}^{q} N(r, a_i) - N_1(r) + O(\log r + \log T(r)),$$

outside certain intervals $\{J_\nu\}$, *such that*

$$\sum_{\nu} \int_{J_\nu} r^{\lambda-1}\, dr < \infty \qquad (0 \leqq \lambda < 1),$$

where $N_1(r)$ *is formed similarly as* $N(r, a)$ *with respect to all multiple values,* a*-ple value being counted* $(a-1)$*-times.*

Especially if we take $q=3$, $\lambda=0$,

$$T(r) \leqq \sum_{i=1}^{3} N(r, a_i) + O(\log r + \log T(r)), \tag{1}$$

outside intervals $\{J_\nu\}$, such that

$$\sum_{\nu} \int_{J_\nu} d \log r < \infty. \tag{2}$$

From this we have

THEOREM 11. *If* $\varlimsup_{r\to\infty} T(r)/\log r = \infty$, *then* $w(z)$ *takes any value infinitely often with two possible exceptions.*

6. Theorems of Valiron and Nevanlinna.

As an application of the theorems proved in § 5, we will prove theorems of Valiron and Nevanlinna as follows.

THEOREM 12 (VALIRON)[5]. *Let* $w(z)$ *be meromorphic in* Δ_0; $|\arg z| \leqq \alpha_0$, $(|\Delta_0| = 2\alpha_0)$ *and* Δ: $|\arg z| \leqq \alpha < \alpha_0$ *be an angular domain contained in* Δ_0. *If for a certain value* a *and* $\rho > \pi/|\Delta_0|$,

$$\sum_{\nu} 1/|z_\nu(a, \Delta)|^\rho = \infty,$$

then

$$\sum_{\nu} 1/|z_\nu(a, \Delta_0)|^\rho = \infty$$

5) G. Valiron : Sur les directions de Borel des fonctions méromorphes d'ordre fini, Journ. de Math. **9** séries 10 (1931).

for any a, with two possible exceptions and Δ_0 contains a Borel's direction of order ρ of divergence type.

PROOF. We choose

$$\Delta_1: |\arg z| \leqq \alpha_1 \qquad (\alpha < \alpha_1 < \alpha_0),$$

such that $\rho > k_1 = \pi/|\Delta_1|$.

By $z^{k_1} = x$, we map Δ_1 on $\Re(x) \geqq 0$, then Δ is mapped on $\omega: |\arg x| \leqq \beta < \pi/2$. We put $w(z) = w_1(x)$, $|z| = r$, $|x| = R\,(=r^{k_1})$,

$$(z_\nu(a, \Delta))^{k_1} = x_\nu(a, \omega) = R_\nu e^{i\varphi_\nu}, \qquad (|\varphi_\nu| \leqq \beta),$$

so that

$$\Re(1/x_\nu(a,\omega)) = \cos\varphi_\nu/R_\nu \geqq \cos\beta/R_\nu = \cos\beta/|z_\nu(a, \Delta)|^{k_1}.$$

Hence $\sum_\nu (\Re(1/x_\nu(a, \omega)))^{\rho/k_1} = \infty$, a fortiori, $\sum_\nu (\Re(1/x_\nu(a)))^{\rho/k_1} = \infty$, where $x_\nu(a)$ are zero points of $w_1(z) - a$ in $\Re(x) > 0$.

Let $T_1(R)$, $N_1(R, a)$ be the functions defined in § 5 for $w_1(x)$, then since $\rho/k_1 > 1$, we have by Theorem 7,

$$\int^\infty \frac{S_1(R)}{R^{\rho/k_1+1}}\, dR = \infty. \tag{1}$$

If $S(r, \Delta_1)$ is defined as in § 2, then $S_1(R) \leqq S(r, \Delta_1)$ $(R = r^{k_1})$, so that from (1),

$$\int^\infty \frac{S(r, \Delta_1)}{r^{\rho+1}}\, dr = \infty.$$

Since $T(2r, \Delta_1) \geqq S(r, \Delta_1)\log 2$, we have

$$\int^\infty \frac{T(r, \Delta_1)}{r^{\rho+1}}\, dr = \infty. \tag{2}$$

Hence by Theorem 2,

$$\int^\infty \frac{N(r, a; \Delta_0)}{r^{\rho+1}}\, dr = \infty, \quad \text{or} \sum_\nu |z_\nu(a, \Delta_0)|^{-\rho} = \infty,$$

with two possible exceptions. From (2) we conclude as Theorem 3 that Δ_0 connains a Borel's direction of order ρ of divergence type.

THEOREM 13 (NEVANLINNA-VALIRON). *Let $w(z)$ be regular in $\Delta_0: |\arg z| \leqq \alpha_0$ and $\Delta: |\arg z| \leqq \alpha < \alpha_0$ be an angular domain contained in Δ_0. If for some $\rho > \pi/|\Delta_0|$ $(\geqq 1/2)$*

$$\int^{\infty} \frac{\log^+ M(r, \Delta)}{r^{\rho+1}} dr = \infty,$$

then

$$\sum_{\nu} 1/|z_{\nu}(a, \Delta_0)|^{\rho} = \infty$$

for any a, with two possible exceptions[6] *and* Δ_0 *contains a Borel's direction of order* ρ *of divergence type*[7].

PROOF. Let $\Delta_1 : |\arg z| \leqq \alpha_1 (\alpha < \alpha_1 < \alpha_0)$ be so chosen that $\rho > k_1 = \pi/|\Delta_1|$ and by $z^{k_1} = x$, we map Δ_1 on $\Re x \geqq 0$, then Δ is mapped on $\omega : |\arg x| \leqq \beta < \pi/2$. We put $w(z) = w_1(x)$, then

$$M_1(R, \omega) = \max_{\theta \leqq \beta} |w_1(Re^{i})| = M(r, \Delta) \qquad (R = r^{k_1}),$$

so that

$$\int^{\infty} \frac{\log^+ M_1(R, \omega)}{R^{\rho\, k_1+1}} dR = k_1 \int^{\infty} \frac{\log^+ M(r, \Delta)}{r^{\rho+1}} dr = \infty. \qquad (1)$$

Let $T_1(R)$ be the characteristic function of $w_1(x)$ defined in §5, then by Theorem 8

$$\log^+ M_1(R, \omega) \leqq AR(T_1(\lambda R) + O(1)), \ (\lambda > 1),$$

so that from (1),

$$\int^{\infty} \frac{T_1(R)}{R^{\rho/k_1}} dR = \infty, \text{ hence} \int^{\infty} \frac{S_1(R)}{R^{\rho\, k_1+1}} dR = \infty.$$

From this we proceed similarly as Thorem 12 and have the theorem.

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6) R. Nevanlima: Untersuchungen über Picard'schen Satz. Acta Soc. Sci. Fenn. **50** (1924).
7) G. Valiron. l. c. (7)