ON THE UNIFORM MEROMORPHIC FUNCTIONS WITH THE SET OF CAPACITY ZERO OF ESSENTIAL SINGULARITIES

TADASHI KURODA

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Introduction

Recently many interesting results on the singularities and value-distributions of uniform meromorphic functions with the set of capacity zero[1] of essential singularities have been obtained by S. Kametani [6], K. Noshiro [9], [10] and M. Tsuji [12], [13]. But, as far as I know, the relation between the order of such meromorphic functions and their inverse functions is not yet obtained.

First we shall state Evans' theorem [5] without proof in §1. We, in §2, prove an extension of Noshiro's result, from which some results already proved by Messrs. K. Noshiro and M. Tsuji are obtained as corollaries. In §3, by the method due to M. Tsuji [14], we shall give an extension of Ahlfors' distortion theorem [1] on the conformal mapping. By this method, we get, in §4, the relation between the order of functions belonging to a certain class and their inverse functions.

§1. Preparation.

1. Let E be a non-empty, bounded and closed set of capacity zero in the z-plane. We suppose that the function $w = f(z)$ is uniform and meromorphic outside the set E and has an essential singularity at every point of E. We denote by $\mathfrak{F}$ the class of such functions.

Since E is a bounded and closed set of capacity zero, there exists a positive mass distribution $d\mu(a)$ on E by Evans' theorem [5] or by Selberg's [11] such that the potential

$$u(z) = \int_E \log \frac{1}{|z-a|}\, d\mu(a) \qquad \left(\int_E d\mu(a) = 1\right)$$

is harmonic at every finite point except all the points belonging to E and that $u(z)$ tends to $+\infty$ when z tends to any points of E and tends to $-\infty$ when z tends to infinity.

Let $v(z)$ be its conjugate harmonic function and we put

$$\zeta = \chi(z) = e^{u(z)+iv(z)} = r(z)e^{iv(z)} \qquad (0 \leqq v(z) < 2\pi).$$

This function is called Evans' function associated with the set E. It may be easily seen that the niveau curve C_r: $r(z) = \text{const.} = r$ associated with

1) Throughout this paper we mean by "Capacity" the logarithmic capacity.

the set E consists of a finite number of simple closed and analytic curves surrounding the set E and that

$$\int_{C_r} dv(z) = \int_{C_r} \frac{\partial u}{\partial n}\, ds = 2\pi,$$

where n is the inner normal of C_r and ds is the arc length of C_r.

§2. An extension of Noshiro's theorem.

2. Let $w = f(z)$ be a function belonging to the class $\mathfrak{F}$ and E be the set of its essential singularities. Denote by $z = \varphi(w)$ the inverse function of $w = f(z)$ and suppose that this inverse function $z = \varphi(w)$ has at least one transcendental singularity and Ω is such a one with the projection $w = \omega$.

Let Δ_ρ be the set of all the values taken by the branch $z = \varphi_\rho(w)$ corresponding to the ρ-neighbourhood Φ_ρ ($\subset \Phi$) of the accessible boundary point Ω of Φ, where Φ denotes the Riemann covering surface which has the Riemann sphere as its basic surface and is associated with the inverse function $z = \varphi(w)$. Then, obviously, Δ_ρ is a connected domain and its boundary consists of at most an enumerable number of analytic contours γ_ρ and the non-empty closed subset E_ρ of E. It is immediate that the function $w = f(z)$ is meromorphic in the closed domain Δ_ρ excluding the set E_ρ and satisfies the relation: $[f(z), \omega] < \rho$ inside Δ_ρ and $[f(z), \omega] = \rho$ on γ_ρ, where

$$[f(z), \omega] = \frac{|f(z) - \omega|}{\sqrt{1 + |f(z)|^2}\,\sqrt{1 + |\omega|^2}}$$

represents the spherical distance between the points $w = f(z)$ and $w = \omega$.

Since E_ρ is a closed subset of a bounded set E, E_ρ is of capacity zero. Hence there exists Evans' function

$$\zeta = X(z) = e^{u(z)+iv(z)} = r(z)e^{iv(z)} \qquad (0 \leqq v(z) < 2\pi)$$

associated to the set E_ρ. If C_r represents the niveau curve $C_r: r(z) = \text{const.} = r$ associated to E_ρ, then we have

$$\int_{C_r} dv(z) = 2\pi.$$

We denote by $\nu(r)$ the number of simple closed and analytic curves of the niveau curve C_r.

Let θ_r be the intersection of the domain Δ_ρ and the niveau curve C_r and $\Delta_\rho(r)$ be the intersection of Δ_ρ and the domain exterior to C_r. $\Delta_\rho(r)$ consists of a finite number of components $\Delta_\rho^{(1)}(r), \cdots, \Delta_\rho^{(m)}(r)$ $(m = m(r) \geqq 1)$ for all sufficiently large r. Suppose that $\Phi_\rho(r)$ and $\Phi_\rho^{(i)}(r)$ are the Riemannian images of $\Delta_\rho(r)$ and $\Delta_\rho^{(i)}(r)$, respectively, on Φ_ρ by $w = f(z)$.

We put

$$S(r, \Delta_\rho) = \frac{1}{\pi\rho^2}\iint_{\Delta_\rho(r)} \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2}\, d\sigma$$

and

$$L(r, \Delta_\rho) = \int_{\theta_r} \frac{|f'(z)|}{1+|f(z)|^2} |dz|,$$

where $d\sigma$ is the area element of the z-plane. These quantities have the geometrical meaning:

$S(r, \Delta_\rho)$ is the average number of sheets of $\Phi_\rho(r)$ and $L(r, \Delta_\rho)$ is the length of the boundary of $\Phi_\rho(r)$ relative to the disc (c_ρ): $[w, \omega] < \rho$.

3. We shall now state two important lemmas without proofs.

LEMMA 1 (Tsuji [12], Noshiro[6]).

$$\lim_{r\to\infty} S(r, \Delta_\rho) = \infty \text{ and } \liminf_{r\to\infty} \frac{L(r, \Delta_\rho)}{S(r, \Delta_\rho)} = 0.$$

LEMMA 2. *Let F be a finite covering surface having F_0 as its basic surface and $D_1, D_2, \cdots, D_q (q \geqq 2)$ be q closed discs such that each lies entirely inside F_0 and no two of them have any point in common and let $\overline{F_0}$ be the domain obtained by excluding all the discs $D_1, D_2, \cdots, D_q$ from F_0. For each D_j $(j = 1, 2, \cdots, q)$ we denote by $n(D_j)$ the number of sheets of all the islands above D_j and by $n_1(D_j)$ the number of orders of all branch points of all the islands above D_j. Finally we denote by $S(F_0)$ the average number of sheets of F and by $L(F_0)$ the length of the boundary of F relative to F_0. Then*

$$\sum_{j=1}^{q} n(D_j) - \sum_{j=1}^{q} n_1(D_j) \geqq \eta(\overline{F_0})S(F_0) - \overset{+}{\eta}(F) - hL(F_0),$$

where $\eta(\overline{F_0})$ is Euler's characteristic of $\overline{F_0}$, $\overset{+}{\eta} = \mathrm{Max}(\eta, 0)$ and h is a constant depending only upon $D_1, D_2, \cdots, D_q$ and F_0.

This lemma was proved by J. Dufresnoy [4] and Y. Tumura [15] independently and was used by K. Kunugui [7] and K. Noshiro [9], [10]. This is also an extension of Ahlfors' fundamental theorem [2] on a finite covering surface.

4. K. Noshiro [9] proved the following theorem.

If Φ_ρ is simply connected, then Φ_ρ covers every point infinitely often inside the disc (c_ρ): $[w, \omega] < \rho$ except at most one point.

We can now generalize this theorem in the following form:

THEOREM 1. *Φ_ρ covers every point infinitely often inside (c_ρ): $[w, \omega] < \rho$ except at most $2 + \xi_1 + \xi_2$ points, where*

$$\xi_1 = \limsup_{r\to\infty} \frac{\nu(r)}{S(r, \Delta_\rho)}, \quad \xi_2 = k(\rho) \limsup_{r\to\infty} \frac{m(r)\nu(r)}{S(r, \Delta_\rho)}$$

and $k(\rho) = (4/\pi\rho^2) \sin^{-1}(\rho/2)$ is the constant depending only on ρ.

In this theorem we assume nothing about the connectivity of Φ_ρ.

In order to prove this, it is sufficient to show the following theorem:

THEOREM 2. *Denote by* $D_1, D_2, \cdots, D_q (q \geqq 2)$ q *closed discs lying entirely inside* (c_ρ): $[w, \omega] < \rho$. *Let* $n^{(i)}(D_j)$ *be the number of sheets of all islands* $\{D_j^{(i)}\}$ *contained in* $\Phi_\rho^{(i)}(r)$ *and lying above* D_j *and* $n_1^{(i)}(D_j)$ *be the number of orders of all branch points of all islands* $\{D_j^{(i)}\}$. *If we put*

$$\sum_{i=1}^{m} n^{(i)}(D_j) = n(r, D_j; \Delta_\rho), \qquad \sum_{i=1}^{m} n_1^{(i)}(D_j) = n_1(r, D_j; \Delta_\rho)$$

and

$$\delta(D_j; \Delta_\rho) = \liminf_{r\to\infty}\left(1 - \frac{n(r, D_j; \Delta_\rho)}{S(r, \Delta_\rho)}\right), \quad \theta(D_j; \Delta_\rho) = \liminf_{r\to\infty} \frac{n_1(r, D_j; \Delta_\rho)}{S(r, \Delta_\rho)},$$

then

$$\sum_{j=1}^{q} \delta(D_j; \Delta_\rho) + \sum_{j=1}^{q} \theta(D_j; \Delta_\rho) \leqq 2 + \xi_1 + \xi_2.$$

PROOF. We can find a positive number r_0 such that for all $r \geqq r_0$, $\Delta_\rho(r)$ consists of a finite number of components $\Delta_\rho^{(1)}(r), \cdots, \Delta_\rho^{(m)}(r)$ $(m = m(r) \geqq 1)$. Since $\Phi_\rho^{(i)}(r)$ is a finite covering surface having the disc (c_ρ): $[w, \omega] < \rho$ as its basic surface, we have by lemma 2

$$\sum_{j=1}^{q} n^{(i)}(D_j) - \sum_{j=1}^{q} n_1^{(i)}(D_j) \geqq (q-1)S^{(i)}(r, \Delta_\rho) - \overset{+}{\eta}(\Phi^{(i)}(r)) - hL^{(i)}(r, \Delta_\rho),$$

where $S^{(i)}(r, \Delta_\rho)$ is the average number of sheets of $\Phi_\rho^{(i)}(r)$ and $L^{(i)}(r, \Delta_\rho)$ is the length of boundary of $\Phi_\rho^{(i)}(r)$ relative to (c_ρ): $[w, \omega] < \rho$ and h is a constant depending only on $D_1, D_2, \cdots D_q$, and (c_ρ): $[w, \omega] < \rho$. Since we can easily see then

$$S(r, \Delta_\rho) = \sum_{i=1}^{m} S^{(i)}(r, \Delta_\rho) \text{ and } L(r, \Delta_\rho) = \sum_{i=1}^{m} L^{(i)}(r, \Delta_\rho),$$

we have

$$(1) \qquad \sum_{j=1}^{q} (S(r, \Delta_\rho) - n(r, D_j; \Delta_\rho)) + \sum_{j=1}^{q} n_1(r, D_j; \Delta_\rho)$$
$$\leqq S(r, \Delta_\rho) + \sum_{i=1}^{m} \overset{+}{\eta}(\Phi_\rho^{(i)}(r)) + hL(r, \Delta_\rho).$$

On the other hand we have easily

$$(2) \qquad \overset{+}{\eta}(\Phi_\rho^{(i)}(r)) = \overset{+}{\eta}(\Delta_\rho^{(i)}(r)) \leqq \eta(\Delta_\rho^{(i)}(r)) + 1 \leqq \mu^{(i)}(r),$$

where $\mu^{(i)}(r)$ denotes the number of component of boundary of $\Delta_\rho^{(i)}(r)$. Hence there are two classes of such components, namely:

i) Components consisting of only one component of niveau curve C_r, whose number will be denoted by $\nu^{(i)}(r)$. Obviously

$$\sum_{i=1}^{m} \nu^{(i)}(r) \leqq \nu(r).$$

ii) Components consisting of at least one part of contours γ_ρ. We denote by $\kappa^{(i)}(r)$ the number of such components.

Now let the variable z vary on the part of γ_ρ which belongs to a component of the second class such that the corresponding point $w=f(z)$ varies on the circle $c_\rho: [w, \omega]=\rho$ at most once. Thus we obtain function elements lying on the circle $c_\rho: [w, \omega]=\rho$. We prolonge every function element along a radius of the circle $c_\rho: [w, \omega]=\rho$ to its centre $w=\omega$. These prolongations are continued until they meet a branch point or the boundary of $\Phi_\rho^{(i)}(r)$ relative to $(c_\rho): [w, \omega]<\rho$. Let A' be the area of the schlicht domain e just obtained above and L' be the length of the boundary of this domain e, which are contained in the relative boundary of $\Phi_\rho^{(i)}(r)$. Then, by using Ahlfors' first covering theorem [2], we get

$$\pi\rho^2 - A' < h'L' + k(\rho), \tag{3}$$

where h' is a constant depending only on $(c_\rho): [w, \omega]<\rho$ and $k(\rho)=(4/\pi\rho^2)\sin^{-1}(\rho/2)$.

The term $k(\rho)$ in (3) appeared in virtue of the components consisting of the parts of γ_ρ and C_r.

Hence, if for every component of the second class belonging to $\Delta_\rho^{(i)}(r)$ we carry out the process just stated and add the inequalities (3), it is easily seen that

$$\pi\rho^2\kappa^{(i)}(r) < \sum_e A' + h'\sum_e L' + k(\rho)\nu(r). \tag{4}$$

Since each domain e has no common part with each other, it follows that

$$\sum_e A' \leqq \pi\rho^2 S^{(i)}(r, \Delta_\rho) \quad \text{and} \quad \sum_e L' \leqq L^{(i)}(r, \Delta_\rho),$$

whence we obtain, from (4),

$$\sum_{i=1}^{m} \kappa^{(i)}(r) \leqq S(r, \Delta_\rho) + h''L(r, \Delta_\rho) + k(\rho)m(r)\nu(r),$$

where $h''=h'/\pi\rho^2$. Consequently it follows that

$$\sum_{i=1}^{m} \mu^{(i)}(r) \leqq S(r, \Delta_\rho) + h''L(r, \Delta_\rho) + \nu(r) + k(\rho)\,m(r)\,\nu(r).$$

From this and (1), (2) we get

$$\sum_{j=1}^{q} (S(r, \Delta_\rho) - n(r, D_j;\ \Delta_\rho)) + \sum_{j=1}^{q} n_1(r, D_j;\ \Delta_\rho)$$
$$\leqq 2S(r, \Delta_\rho) + h'''L(r, \Delta_\rho) + \nu(r) + k(\rho)m(r)\nu(r),$$

where $h'''=h+h''$ depends only on $D_1, D_2, \cdots, D_q$ and $(c_\rho): [w, \omega]<\rho$.

By using lemma 1, we obtain

$$\sum_{j=1}^{q} \delta(D_j;\ \Delta_\rho) + \sum_{j=1}^{q} \theta(D_j;\ \Delta_\rho) \leqq 2 + \xi_1 + \xi_2\ ,$$

where $\xi_1 = \limsup_{r\to\infty} \frac{\nu(r)}{S(r, \Delta_\rho)}$, $\xi_2 = \limsup_{r\to\infty} k(\rho) \frac{m(r)\nu(r)}{S(r, \Delta_\rho)}$ and $k(\rho) = (4/\pi\rho^2)$ $\sin^{-1}(\rho/2)$.

5. From Theorem 2 Noshiro's theorem stated in the preceding section is deduced and moreover the following theorems are deduced:

COROLLARY 1 (Noshiro [10]). *If the set E_ρ lies on one component of γ_ρ, then Φ_ρ covers every point inside $(c_\rho): [w, \omega] < \rho$ infinitely often except at most two points.*

COROLLARY 2 (Tsuji[13]). *If Δ_ρ, consequently Φ_ρ, is finitely connected, Φ_ρ covers every point inside $(c_\rho): [w, \omega] < \rho$ infinitely often except at most one point. Moreover we denote by $\bar{\Delta}_\rho$ the domain Δ_ρ with addition of the inner parts of closed componens γ_ρ of boundary of the domain Δ_ρ. We call $\bar{\Delta}_\rho$ the associated domain of Δ_ρ. If the associatetd domain $\bar{\Delta}_\rho$ of Δ_ρ is finitely connected, Φ_ρ covers every point inside $(c_\rho): [w, \omega] < \rho$ except at most two points.*

COROLLARY 3. *If the number of contours γ_ρ extended to certain points belonging to the set E_ρ is finite, then Φ_ρ covers infinitely often every point inside $(c_\rho): [w, \omega] < \rho$ except at most $2 + \xi_1$ points.*

6. By the similar way as the proof of theorem 2 we can show the following

THEOREM 3. *Suppose that the branch $z = \varphi_\rho(w)$ has a transcendental singularity Ω_0 with projection $w = \omega_0$ such that $[\omega, \omega_0] < \rho$. And we denote by Φ_0 the ρ_0-neighbourhood (on Φ_ρ) of the accessible boundary point Ω_0 of Φ_ρ such that Φ_0 lies above the disc $[w, \omega_0] < \rho_0$, which lies entirely inside the disc $(c_\rho): [w, \omega] < \rho$. If Δ_ρ, namely Φ_ρ, is finitely connected, then Φ_0 covers infinitely often every point inside its basic surface $(c_0): [w, \omega_0] < \rho_0$ except at most two points.*

§3. An extension of Ahlfors' distortion theorem.

7. M. Tsuji [14] extended the famous Ahlfors' distortion theorem [1]. We shall extend it a little more.

Let D be a simply connected domain in the z-plane. Suppose that the bounded and closed set E of capacity zero lies on the boundary Γ of the domain D.

We construct Evans' function $\zeta = e^{u(z)+iv(z)} = r(z)\, e^{iv(z)} (0 \leqq v(z) < 2\pi)$ associated to E and describe the niveau curve C_r: $r(z) = \text{const.} = r$ surrounding the set E. We put $\theta_r = D \cap C_r$. We shall show the following

THEOREM 4. *If we map the domain D conformally on the unite circle $|w| < 1$ by a function $w = f(z)$, then, for all sufficiently large r, the image λ_r of θ_r in $|w| < 1$ can be enclosed in a finite number of circles $k_r^{(i)}$ $(i = 1, \cdots,$*

$n = n(r)$), which cut $|w| = 1$ orthogonally, such that the sum of their radii is less than

$$\text{const. } \exp\left(-\pi\int_{r_0}^{kr}\frac{dr}{r\theta(r)}\right),$$

where $r_0 < kr < r, r_0$ a certain positive number and $\theta(r) = \int_{\theta_r} dv(z)$.

PROOF. There exists a positive number r_0 such that, for all $r \geqq r_0$, θ_r is not empty. We denote by $\theta_r^{(i)}$ $(i = 1, \cdots, m = m(r))$ the components of θ_r. We map D conformally into the unit circle $|w| < 1$, then we can suppose without loss of generality that a certain point on the boundary of D not belonging to E corresponds to the point $w = 1$. Then the set E corresponds to a set E^* of measure zero on $|w| = 1$ and the point $w = 1$ does not belong to this set E^*.

Let $\lambda_r^{(i)}$ be the image of $\theta_r^{(i)}$ $(i = 1, \cdots, m)$ and $\lambda_r = \bigcup_{i=1}^{m}\lambda_r^{(i)}$. Then obviously λ_r converges to the set E^* when r tends to ∞. Denote by $k_r^{(i)}$ $(i = 1, \cdots, n = n(r) \leqq m)$ the system of circles enclosing the Jordan arcs $\lambda_r^{(i)}(i = 1, \cdots, m)$ and cutting $|w| = 1$ orthogonally.

We map again the disc $|w| < 1$ conformally on the upper semi-plane $\Im\sigma > 0$ of the σ-plane by the linear transformation $\sigma = \sigma(w) = i(1+w)/(1-w)$ and we denote by Λ_r, $\Lambda_r^{(i)}$ and $K_r^{(i)}$ the image of λ_r, $\lambda_r^{(i)}$ and $k_r^{(i)}$ in $\Im\sigma > 0$, respectively. Then we get the domains $e_1, \cdots, e_q$ $(m \geqq q = q(r) \geqq 1)$, each of which is bounded by some $\Lambda_r^{(i)}$ and the segments lying on the real axis $\Im\sigma = 0$ and which correspond to the common parts of the domain D and the interior of C_r. We represent by $L^{(j)}(r)$ the length of the boundary $\{\Lambda_r^{(i)}\}$ of e_j and by $A^{(j)}(r)$ the area of e_j. Moreover let

$$L(r) = \sum_{j=1}^{q} L^{(j)}(r) \quad \text{and} \quad A(r) = \sum_{j=1}^{q} A^{(j)}(r).$$

Then we can see without difficulty that

$$A^{(j)}(r) \leqq (L^{(j)}(r))^2/2\pi. \tag{5}$$

Let $z = \chi^{-1}(\zeta)$ be the inverse function of Evans' function $\zeta = \chi(z)$ associated to the set E and put $\sigma = F(\zeta) = \sigma(f(\chi^{-1}(\zeta)))$. Then it is also clear that

$$L(r) = \int_{\Theta_r} |F'(\zeta)| r d\theta \quad \text{and} \quad A(r) = \int_r^{\infty}\int_{\Theta_r} |F'(\zeta)|^2 r d\theta\, dr,$$

where Θ_r is the image of θ_r by $\zeta = \chi(z)$ $(0 \leqq v(z) < 2\pi)$ on the ζ-plane and $\zeta = re^{i\theta}$. Using the Schwarz inequality and

$$\theta(r) = \int_{\Theta_r} d\theta = \int_{\Theta_r} dv(z) \leqq \int_{C_r} dv(z) = 2\pi,$$

it follows that

$$L^2(r) \leqq r\theta(r) \int_{\Theta_r} |F'(\zeta)|^2 r d\theta,$$

from which we get, by (5),

$$(6) \qquad \int_r^\infty \frac{L^2(r)}{r\theta(r)}\, dr \leqq A(r) = \sum_{j=1}^q A^{(j)}(r) \leqq \frac{1}{2\pi} \sum_{j=1}^p (L^{(j)}(r))^2 \leqq \frac{1}{2\pi} L^2(r).$$

If we put $\eta(r) = \int_r^\infty \frac{L^2(r)}{r\theta(r)}\, dr$, then we have

$$-L^2(r) \leqq r\theta(r)\eta'(r).$$

Hence, from (6),

$$2\pi \int_{r_0}^r \frac{dr}{r\theta(x)} \leqq \log \frac{\eta(r_0)}{\eta(r)},$$

whence it follows that

$$\eta(r) \leqq \text{const. } \exp\left(-2\pi \int_{r_0}^r \frac{dr}{r\theta(r)}\right).$$

Since the sum $l(r)$ of radii of circles $K_r^{(i)}$ $(i = 1, \cdots, n)$ is not greater than $\frac{1}{2} L(r)$, we can see

$$\int_r^\infty \frac{l^2(r)}{r\theta(r)}\, dr \leqq \text{const. } \exp\left(-2\pi \int_{r_0}^r \frac{dr}{r\theta(r)}\right).$$

If we notice that $l(r)$ is a monotone decreasing function of r, it follows

$$l^2(r) \int_{kr}^r \frac{dr}{r\theta(r)} \leqq \int_{kr}^\infty \frac{l^2(r)}{r\theta(r)}\, dr \leqq \text{const. } \exp\left(-2\pi \int_{r_0}^{kr} \frac{dr}{r\theta(r)}\right),$$

where $r_0 < kr < r$. However, we can easily see that

$$\int_{kr}^r \frac{dr}{r\theta(r)} \geqq \frac{1}{2\pi} \log \frac{1}{k},$$

from which we obtain

$$l(r) \leqq \text{const. } \exp\left(-\pi \int_{r_0}^{kr} \frac{dr}{r\theta(r)}\right).$$

The radius of $k_r^{(i)}$ is less than constant multiple of the radius of $K_r^{(i)}$. Consequently the sum of radii of $k_r^{(i)}$ $(i = 1, \cdots, n)$ is less than constant multiple of $l(r)$. Therefore, from the above inequality, the sum of radii of $k_r^{(i)}$ $(i = 1, \cdots, n)$ is less than

$$\text{const. } \exp\left(-\pi\int_{r_0}^{kr}\frac{dr}{r\theta(r)}\right),\quad r_0 < kr < r,$$

which proves the theorem.

8. By applying Theorem 4, we can prove the following theorem of Phragmén-Lindelöf type.

THEOREM 5. *Let $w = f(z)$ be regular in the simply connected domain D and suppose that* $\limsup_{z\to\zeta} |f(z)| \leqq 1$, *where ζ denotes an arbitrary point on the boundary Γ of D not belonging to the bounded and closed set E of capacity zero lying on Γ. We put $\theta_r = D \cap C_r$ (where C_r is the niveau curve associated to the set E). If* $\liminf_{r\to\infty} r^{-\pi/\theta} \log M(r) = 0$, *then $|f(z)| \leqq 1$ at every point of D, where* $M(r) = \operatorname*{Max}_{z\in\theta r} |f(z)|$ *and $\theta(\leqq 2\pi)$ is the upper bound of* $\theta(r) = \int_{\theta_r} dv(z)$ *for all sufficiently large r.*

PROOF. Let z_0 be a point in D. We can choose $r_1(>0)$ such that, for all $r \geqq r_1$, z_0 is contained in a component D_r of domains which are the parts of D_r lying outside C_r. The boundary of D_r consists of the part $\bar{\theta}_r$ of θ_r and the parts of boundary of D, and it contains no point belonging to the set E. We denote by $\omega(z, \bar{\theta}_r, D_r)$ the harmonic measure of $\bar{\theta}_r$ with respect to D_r, namely the harmonic function in D_r such that it equals to 1 on $\bar{\theta}_r$ and to zero on the other boundary of D_r. If we notice that log $|f(z)|$ is harmonic in D_r except zero-points of $f(z)$, by using the maximum principle or Nevanlinna's "*Zweikonstantensatz*" [8], we have

$$\log|f(z)| \leqq \omega(z, \bar{\theta}_r, D_r) \log M(r),$$

whence at the point $z = z_0$ we have

$$\log|f(z_0)| \leqq \omega(z_0, \theta_r, D_r) \log M(r). \tag{7}$$

We shall now map D conformally on the unite circle $|\tau| < 1$ in the τ-plane by the function $\tau = \tau(z)$ $(\tau(z_0) = 0)$. Similarly as in the proof of the preceding theorem, we can enclose the image of θ_r by system of circles $k_r^{(i)}$ $(i = 1, \cdots, n)$ which cut $|\tau| = 1$ orthogonally. Denote by α_i and β_i two edge points of $k_r^{(i)}$ on the circle $|\tau| = 1$ and suppose that $\alpha_i < \beta_i$.

We can find $r_2(>0)$ such that, for all $r \geqq r_2$, the point $\tau = 0$ lies outside these circles $k_r^{(i)}$ $(i = 1, \cdots, n)$. We denote by $\Omega_r^{(i)}$ the domain, which contains the image $\tau = 0$ of the point $z = z_0$ and whose boundary consists of $k_r^{(i)}$ and $|\tau| = 1$. If we put $\psi_i = \arg(\beta_i/\alpha_i)$, we can easily see from Theorem 4 that

$$\sum_{i=1}^{n} \psi_i \leqq \text{const. } \exp\left(-\pi\int_{r_0}^{kr}\frac{dr}{r\theta(r)}\right) \tag{8}$$

$$r_0 < kr < r, \quad \theta(r) = \int_{\theta_r} dv(z),$$

where $r_0 = \mathrm{Max}(r_1, r_2)$. We put $v_i(\tau) = \arg(\tau - \beta_i)/(\tau - \alpha_i)$ and $V_i(\tau) = 2(v_i(\tau) - \psi_i/2)/\pi$. Then $V_i(\tau)$ is the harmonic function in the domain $\Omega_r^{(i)}$ and equals to 1 on $k_r^{(i)}$ and to zero on the other boundary of $\Omega_r^{(i)}$. If we put $\Omega_r = \bigcap_{i=1}^{n} \Omega_r^{(i)}$, then Ω_r is contained in the image of D_r and the function $V(\tau) = \sum_{i=1}^{n} V_i(\tau)$ is harmonic in Ω_r and is greater than 1 on $k_r^{(i)}$ $(i = 1, \cdots, n)$ and equals to zero on the other boundary of Ω_r. Since harmonic property is invariant by a conformal mapping, we have

$$\omega(z_0, \overline{\theta_r}, D_r) \leqq V(0).$$

Hence, from (7), it follows

$$\log |f(z_0)| \leqq V(0) \log M(r).$$

On the other hand, it is easy to see that $V(0) = \sum_{i=1}^{n} \psi_i/\pi$ and so from (8)

$$\log |f(z_0)| \leqq \text{const.} \exp\left(-\pi \int_{r_0}^{kr} \frac{dr}{r\theta(r)}\right) . \log M(r).$$

Since $\theta(r) \leqq 2\pi$, there exists $\theta\,(>0)$ such that $\theta(r) \leqq \theta \leqq 2\pi$ for all $r \geqq r_0$. Hence it is easily seen that

$$\log |f(z_0)| \leqq \text{const.}\, r^{-\pi/\theta} . \log M(r).$$

Accordingly, if $\liminf_{r\to\infty} r^{-\pi/\theta} \log M(r) = 0$, then we have

$$|f(z_0)| \leqq 1.$$

Since z_0 is an arbitrary point in D, we obtain the theorem.

Especially we have

COROLLARY. *If* $\liminf_{r\to\infty} r^{-1/2} . \log M(r) = 0$, *then we have* $|f(z)| \leqq 1$ *at every point in* D.

REMARK. Theorem 5 can be proved by the Beurling's distortion theorem [3].

From the proof of the preceding theorem, the following theorem is obtained without difficulties.

THEOREM 6. *If the point* z *lies in the domain* D_r, *then*

$$\omega(z_0, \overline{\theta_r}, D_r) \leqq h(r') \exp\left(-\pi \int_{r_0}^{kr} \frac{dr}{r\theta(r)}\right), \qquad (r' < r),$$

where $h(r')$ *is a constant depending only on* r'.

§4. The relation between the order of functions belonging to the class $\mathfrak{F}$ and their inverse function.

9. Let $w = f(z)$ be the function belonging to the class $\mathfrak{F}$ and Ω be the transcendental singularity of its inverse function. Denote by ω the projection of Ω. M. Tsuji [12] proved the following theorem:

The set of projections of the direct transcendental singularities of the inverse function on the w-plane is of capacity zero.

We denote by Δ_ρ the set of all values taken by the branch $z = \varphi_\rho(w)$ of the inverse function $z = \varphi(w)$ of $w = f(z)$, where $z = \varphi_\rho(w)$ corresponds to the ρ-neighbourhood Φ_ρ of an accessible boundary point Ω of Φ which is the Riemann covering surface having w-plane as its basic surface[2]).

Let Ω_0 be any transcendental singularity of $z = \varphi_\rho(w)$ and $w = \omega_0$ be its projection on the w-plane such that ω_0 lies inside the disc (c_ρ): $|w| > 1/\rho$ $(\omega = \infty)$ or $|w - \omega| < \rho\,(\omega \neq \infty)$. We call Ω_0 the direct transcendental singularity of the branch $z = \varphi_\rho(w)$, when the point $w = \omega_0$ is lacunary with respect to the ρ_0-neighbourhood $\Phi_0\,(\subset\Phi_\rho)$ of Ω_0. Then, by similary as the argument of Tsuji [12], we can easily show the following theorem:

THEOREM 7. *The set of projections of the direct transcendental singularities of the branch $z = \varphi_\rho(w)$ on the disc (c_ρ) is of capacity zero.*

10. We describe the niveau curve C_r associated with the set E_ρ by constructing Evans' function associated with the set E_ρ, where E_ρ is the closed subset of E which is the bounded closed set of essential singularities of the function $w = f(z)$ and E_ρ belongs to the boundary of Δ_ρ. We put $C_r \cap \Delta_\rho = \theta_r$ and

$$M_\rho(r) = \begin{cases} \displaystyle\max_{z\in\theta_r} |f(z)| & \text{for } \omega = \infty, \\ \displaystyle\max_{z\in\theta_r} 1/|f(z) - \omega| & \text{for } \omega \neq \infty, \end{cases}$$

and

$$\limsup_{r\to\infty} \frac{\log\log M_\rho(r)}{\log r} = p(\rho).$$

We call $p(\rho)$ the M-order of $w = f(z)$ with respect to Δ_ρ. Then we shall prove

THEOREM 8. *Suppose that the M-order $p(\rho)$ of $w = f(z)$ with respect to Δ_ρ is finite. If Δ_ρ is simply connected, then the number of direct transcendental singularities of the branch $z = \varphi(w)$ lying above $w = \omega$ is not greater than $2p(\rho)$.*

PROOF. Without loss of generality we can suppose $\omega = \infty$. Let Ω_0 be a certain direct transcendental singularity of $z = \varphi_\rho(w)$ lying above the point $w = \omega$. We denote by Φ_0 the ρ_0-neighbourhood of Ω_0 lying entirely inside Φ_ρ and by Δ_0 the set of values taken by the branch $z = \varphi_0(w)$ corres-

2) If $\omega = \infty$, then we take a certain connected piece Φ_ρ lying above the disc $|w| > 1/\rho$. If $\omega \neq \infty$, we take a disc $|w - \omega| < \rho$ instead of $|w| > 1/\rho$. In the following we consider Φ n this sense. And we denote by (c_ρ) the basic disc of Φ_ρ.

ponding to Φ_0. The boundary of Δ_0 consists of an enumerable number of analytic curves γ_0 and the closed subset E_0 of E_ρ. The function $w = f(z)$ is meromorphic in the closed domain $\overline{\Delta}_0$ excluding the set E_0 and satisfies the relation: $|f(z)| > 1/\rho_0$ inside Δ_0 and $|f(z)| = 1/\rho_0$ on γ_0.

We can choose $\rho_0\,(>0)$ such that $f(z) \neq \omega = \infty$ in Δ_0, or $f(z)$ is regular in Δ_0. Since Δ_ρ is simply connected by the assumption, the associated domain $\overline{\Delta}_0$ (See Theorem 2, Corollary 2) is also simply connnected. The function $\overset{+}{\log}\, |f(z)| \cdot \rho_0$ is subharmonic and is equal to 0 on γ_0. We shall extend the definition of this subharmonic function in the domain Δ_0 by putting $\overset{+}{\log}\, |f(z)| \cdot \rho_0 = 0$ inside holes of Δ_0, then $\overset{+}{\log}\, |f(z)| \cdot \rho_0$ is subharmonic in $\overline{\overline{\Delta}}_0$. We put

$$\overline{M}(r) = \operatorname*{Max}_{z\in\overline{\theta}_r} |f(z)|,$$

where $\overline{\theta}_r = C_r \cap \Delta_0 \;(\subset \theta_r)$. We can use the argument used in § 3 for θ_r and, deforming the proof of Theorem 5, we can see that there exists a certain number $\varepsilon > 0$ such that for all sufficiently large r

$$r^{-\pi/\overline{\theta}} \log \overline{M}(r) \geqq \varepsilon > 0,$$

where $\overline{\theta}$ is the upper bound of $\displaystyle\int_{\theta_r} dv(z)$.

On the other hand it is immediate that $\overline{M}(r) \leqq M(r)$. Accordingly, for all sufficiently large r, we see

$$r^{-\pi/\overline{\theta}} \log M(r) \geqq \varepsilon$$

or

$$\log \log M(r) \geqq (\pi/\overline{\theta}) \log r + \text{const.}$$

If we suppose that there exist n direct transcendental singularities of Φ_ρ lying above $w = \omega$, then we get similar n inequalities as above. We can, however, choose n ρ_0-neighbourhoods disjoint with each other. Hence there exists at least one such that $\overline{\theta} \leqq \theta/n \leqq 2\pi/n$. Then we have for such $\overline{\theta}$

$$\log \log M(r) \geqq \frac{n}{2} \log r + \text{const.},$$

or

$$p(\rho) = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} \geqq \frac{n}{2},$$

which proves the theorem.

11. Now let $w = f(z)$ be a regular function belonging to the class $\mathfrak{F}$. For the niveau curve C_r associated with the set E, we put

$$M(r) = \operatorname*{Max}_{z\in C_r} |f(z)|$$

and

$$p = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r}$$

We call p the order of a regular function $w = f(z)$ belonging to the class $\mathfrak{F}$.

Then, by the similar way as the proof of Theorem 8, we can show the following

THEOREM 9. *Suppose that $w = f(z)$ is a regular function outside the bounded closed set E of capacity 0 and it has an essential singularity at every point of E. We suppose moreover that its order p is finite. If its inverse function has n distinct asymptotic values on a point $w = \omega$ and all their ρ-neighbourhoods are simply connected, then $n \leqq 2p$.*

We can state easily the analogues of other theorem of the above type.

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MATHEMATICAL INSTITUTE, TÔHOKU UNIVERSITY, SENDAI.