ON THE DIRECT-PRODUCT OF OPERATOR ALGEBRAS I

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1. Introduction. Recently, R Schatten and J. von Neumann have defined the direct-product of Banach spaces, and obtained many interesting results ——for example, they have proved that the double conjugate space of Banach space of all completely continuous operators on the Hilbert space coincides with the space of all bounded operators on that space (cf. [4 : Theorem 5.15]).

As they say, the direct-product of Banach spaces is seen to be an effective tool in dealing with Banach spaces whose elements are operators on some Banach space. On the other hand, from the algebraic standpoint, the direct-product of Banach spaces is considered to be a generalization of the Kronecker product of vector spaces to the infinite dimensional cases; so the author suppose that it is significant to consider the direct-product of the C^*-algebras as a generalization of the Kronecker product of rings to the infinite dimensional cases.

In the present paper, we shall define the direct-product of operator algebras as Kronecker product of rings, and introduce a suitable norm in this product space; and finally completing this ring by using the norm defined above, we shall construct a new C^*-algebra as the direct-product of C^*-algebras.

In § 2, we recall R. Schatten-J. von Neumann's definition of the direct-product and define the product of expressions. In § 3, we consider about the states on the direct-product and introduce a suitable norm (cross-norm in R. Schatten's sense) on this direct-product ring. Finally in § 4, we apply the above consideration to the commutative C^*-algebras, and prove that direct-product of the commutative C^*-algebras of all continuous functions defined on compact Hausdorff spaces Ω and Γ, is isometrically isomorphic to the C^*-algebra of all continuous functions defined on the product space $\Omega\times\Gamma$; this result may justify our norm, and is related to Dunford-Schatten's results [1].

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2. Definitions and notations. We begin with recalling Schatten-J. von Neumann's definitions and notations.

For any C^*-algebras A_1, A_2 (with unit), let X be a set of all formal "*expressions*"

$$\sum_{i=1}^{n} x_i \times y_i = x_1 \times y_1 + \dots + x_n \times y_n$$

where $x_i \in A_1$, $y_i \in A_2$, $i = 1, 2, \dots, n$; $n = 1, 2, \dots$.

In X, we introduce a relation $\sim$ subjects to the following rules:

(i) $\sum_{i=1}^{n} x_i \times y_i \sim \sum_{i=1}^{n} x_{p(i)} \times y_{p(i)}$, where $p(1), \dots, p(n)$ denotes any permutation of the integers $1, 2, \dots, n$.

(ii) $(x_1' + x_1'') \times y_1 + \sum_{i=2}^{n} x_i \times y_i \sim x_1' \times y_1 + x_1'' \times y_1 + \sum_{i=2}^{n} x_i \times y_i$.

(ii′) $x_1 \times (y_1' + y_1'') + \sum_{i=2}^{n} x_i \times y_i \sim x_1 \times y_1' + x_1 \times y_1'' + \sum_{i=2}^{n} x_i \times y_i$.

(iii) $\sum_{i=1}^{n} (a_i x_i) \times y_i \sim \sum_{i=1}^{n} x_i \times (a_i y_i)$, where a_i denotes any scalar.

And finally, two expressions $\sum_{i=1}^{n} x_i \times y_i$ and $\sum_{j=1}^{m} s_j \times t_j$ in X will be termed *equivalent* if one can be transformed into the other by a finite number of successive applications of rules (i)-(iii); and we write

$$\sum_{i=1}^{n} x_i \times y_i \simeq \sum_{j=1}^{m} s_j \times t_j.$$

Then we can easily verify that the relation $\simeq$ is reflexive, symmetric, transitive; so we define the linear set $A_1 \odot A_2$ as a set of residue classes of X by this relation $\simeq$.

Now, we define the product and involution * in X as follows:

(*product*) $$\left(\sum_{i=1}^{n} x_i \times y_i\right) \cdot \left(\sum_{j=1}^{m} s_j \times t_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i s_j \times y_i t_j,$$

(*involution*) $$\left(\sum_{i=1}^{n} x_i \times y_i\right)^* = \sum_{i=1}^{n} x_i^* \times y_i^*.$$

Then the following Lemma can be proved:

LEMMA 1. *The product, and the involution * defined above are invariant under the relation* $\simeq$.

PROOF. To complete the proof of this lemma, it is sufficient to show that

$$\sum_{i=1}^{n} x_i \times y_i \simeq \sum_{i=1}^{n'} x_i' \times y_i', \quad \text{and} \quad \sum_{j=1}^{m} s_j \times t_j \simeq \sum_{j=1}^{m'} s_j' \times t_j'$$

imply

$$\sum_{i=1}^{n} \sum_{j=1}^{m} x_i s_j \times y_i t_j \simeq \sum_{i=1}^{n'} \sum_{j=1}^{m'} x_i' s_j' \times y_i' t_j',$$

and

$$\sum_{i=1}^{n} x_i^* \times y_i^* \simeq \sum_{i=1}^{n'} x_i'^* \times y_i'^*.$$

Firstly, by [4: Lemma 1.1] we can assume, without loss of generality, that the sets $\{x_i\}$, $\{y_i\}, \cdots, \{s_j'\}$ and $\{t_j'\}$ are linearly independent respectively, then $n = n'$ and $m = m'$ by [4: Lemma 1.1], and furthermore, there exist matrices (A_{ik}), (a_{ik}), (B_{pj}) and (b_{pj}) such that

$$x_i' = \sum_{k=1}^{n} A_{ki}\, x_k, \quad y_i' = \sum_{k=1}^{n} a_{ik} y_k, \quad \text{and} \quad (A_{ik})(a_{ik}) = 1_n \ ;$$

$$s_j' = \sum_{p=1}^{m} B_{pj}\, s_p, \quad t_j' = \sum_{p=1}^{m} b_{jp}\, t_p, \quad \text{and} \quad (B_{pj})(b_{pq}) = 1_m,$$

where 1_r denotes the $r \times r$ unit matrix.

Thus,

$$x_i' s_j' = \Big(\sum_{k=1}^{n} A_{ki}\, x_k\Big)\Big(\sum_{p=1}^{m} B_{pj}\ s_p\Big)$$
$$= \sum_{k=1}^{n}\sum_{p=1}^{m} A_{ki}\, B_{pj}\, x_k\, s_p.$$

Similarly,

$$y_i' t_j' = \sum_{h=1}^{n}\sum_{q=1}^{m} a_{ih}\, b_{jq} y_h\, t_q.$$

Therefore,

$$\sum_{i=1}^{n}\sum_{j=1}^{m} x_i' s_j' \times y_i' t_j' = \sum_{i=1}^{n}\sum_{j=1}^{m}\Big(\sum_{k,p=1}^{n,m} A_{ki}\, B_{pj}\, x_k\, s_p\Big) \times \Big(\sum_{h,q=1}^{n,m} a_{ih}\, b_{jq} y_h\, t_q\Big)$$
$$\simeq \sum_{i,k,h=1}^{n}\ \sum_{j,p,q=1}^{m} (A_{ki}\, a_{ih}\, B_{pj}\, b_{jq}) x_k\, s_p \times y_h\, t_q$$
$$\simeq \sum_{k=1}^{n}\sum_{p=1}^{m} x_k\, s_p \times y_k\, t_p.$$

This proves the first part of the Lemma.

The proof of the second part is similar.

By this lemma, $A_1 \odot A_2$ can be considered as a $*$-algebra with unit 1×1.

3. **Norms on the space $\mathbf{A}_1 \odot \mathbf{A}_2$.** In this section, we consider about a norm on $A_1 \odot A_2$. For any expression $\sum_{i=1}^{n} x_i \times y_i \in X$, R. Schatten defined the *greatest cross-norm*:

$$\gamma\Big(\sum_{i=1}^{n} x_i \times y_i\Big) = \inf\Big[\sum_{j=1}^{m} \|x_j'\| \, \|y_j'\| \ \Big|\ \sum_{j=1}^{m} x_j' \times y_j' \simeq \sum_{i=1}^{n} x_i \times y_i\Big]$$

and he proved that $\gamma(\cdot)$ has following properties:

(1) $\sum_{i=1}^{n} x_i \times y_i \simeq 0 \times 0$ if and only if $\gamma\left(\sum_{i=1}^{n} x_i \times y_i\right) = 0$,

(2) $\gamma\left(\sum_{i=1}^{n} x_i \times y_i + \sum_{j=1}^{m} s_j \times t_j\right) \leq \gamma\left(\sum_{i=1}^{n} x_i \times y_i\right) + \gamma\left(\sum_{j=1}^{m} s_j \times t_j\right)$,

(3) $\gamma\left(a \cdot \sum_{i=1}^{n} x_i \times y_i\right) = |a| \cdot \gamma\left(\sum_{i=1}^{n} x_i \times y_i\right)$, where a denotes a scalar,

(4) $\sum_{i=1}^{n} x_i \times y_i \simeq \sum_{j=1}^{m} s_j \times t_j$ implies $\gamma\left(\sum_{i=1}^{n} x_i \times y_i\right) = \gamma\left(\sum_{j=1}^{m} s_j \times t_j\right)$,

(5) $\gamma(x \times y) = \|x\| \cdot \|y\|$.

Furthermore we can prove the following:

LEMMA 2. *For any* $\sum_{i=1}^{n} x_i \times y_i$, *and* $\sum_{j=1}^{m} s_j \times t_j$ *in* X,

(6) $\gamma\left[\left(\sum_{i=1}^{n} x_i \times y_i\right) \cdot \left(\sum_{j=1}^{m} s_j \times t_j\right)\right] \leq \gamma\left(\sum_{i=1}^{n} x_i \times y_i\right) \cdot \gamma\left(\sum_{j=1}^{m} s_j \times t_j\right)$,

(7) $\gamma\left[\left(\sum_{i=1}^{n} x_i \times y_i\right)^{*}\right] = \gamma\left(\sum_{i=1}^{n} x_i \times y_i\right)$.

PROOF. Since (7) is similarly proved, we prove (6) only.

$$\gamma\left[\left(\sum_{i=1}^{n} x_i \times y_i\right)\left(\sum_{j=1}^{m} s_j \times t_j\right)\right] = \inf\left[\sum_{k=1}^{p} \|u_k\| \|v_k\| \,\Big|\, \sum_{k=1}^{p} u_k \times v_k \simeq \sum_{i=1}^{n}\sum_{j=1}^{m} x_i s_j \times y_i t_j\right]$$

$$\leq \inf\left[\sum_{k=1}^{p}\sum_{h=1}^{q} \|x'_k\| \|y'_k\| \|s'_h\| \|t'_h\| \,\Big|\, \sum_{k=1}^{p} x'_k \times y'_k \simeq \sum_{i=1}^{n} x_i \times y_i,\ \sum_{h=1}^{q} s'_h \times t'_h \simeq \sum_{j=1}^{m} s_j \times t_j\right]$$

$$= \inf\left[\sum_{k=1}^{p} \|x'_k\| \|y'_k\| \sum_{h=1}^{q} \|s'_h\| \|t'_h\| \,\Big|\, \cdots\cdots\cdots\cdots\right]$$

$$= \inf\left[\sum_{k=1}^{p} \|x'_k\| \|y'_k\| \,\Big|\, \sum_{k=1}^{p} x'_k \times y'_k \simeq \sum_{i=1}^{n} x_i \times y_i\right]$$

$$\times \inf\left[\sum_{h=1}^{q} \|s'_h\| \|t'_h\| \,\Big|\, \sum_{h=1}^{q} s'_h \times t'_h \simeq \sum s_j \times t_j\right]$$

$$= \gamma\left(\sum_{i=1}^{n} x_i \times y_i\right) \cdot \gamma\left(\sum_{j=1}^{m} s_j \times t_j\right).$$

Now, if we consider a non-complete normed space $A_1 \odot A_2$ with norm γ, we obtain a non-complete normed $*$-algebra by the properties (1)–(7) of γ; we denote this $*$-algebra by $A_1 \odot_\gamma A_2$.

Next, we consider a linear functional $\varphi \times \psi$ on the space $A_1 \odot A_2$, where φ and ψ denote the states on A_1 and A_2, respectively. That $\varphi \times \psi$ is considered as a functional on $A_1 \odot A_2$; that is

$$[\varphi \times \psi]\left(\sum_{i=1}^{n} x_i \times y_i\right) = \sum_{i=1}^{n} \varphi(x_i)\psi(y_i)$$

is invariant under the equivalence $\simeq$, is easily verified. We shall prove the following two essential lemmas.

LEMMA 3. *$\varphi \times \psi$ is a state on $A_1 \odot A_2$: that is, $\varphi \times \psi$ is a positive type functional with $[\varphi \times \psi](1 \times 1) = 1$.*

PROOF. It is sufficient to prove the following inequality:

$$[\varphi \times \psi]\left(\left(\sum_{i=1}^{n} x_i \times y_i\right)\left(\sum_{i=1}^{n} x_i \times y_i\right)^*\right) = \sum_{i,j=1}^{n} \varphi(x_i x_j^*)\psi(y_i y_j^*) \geq 0,$$

for any expression $\sum_{i=1}^{n} x_i \times y_i \in X$.

Now, we recall I. E. Segal's results (cf. [3]); by his theorem, for states φ, ψ, there correspond the representations of A_1 and A_2 respectively; we denote this representation spaces by $A_1^\varphi = \{x^\varphi | x \in A_1\}$ and $A_2^\psi = \{y^\psi | y \in A_2\}$, and their inner products by $< .,. >_\varphi$ and $< .,. >_\psi$ respectively. Then,

$$[\varphi \times \psi]\left(\left(\sum_{i=1}^{n} x_i \times y_i\right)\left(\sum_{i=1}^{n} x_i^* \times y_i^*\right)\right) = \sum_{i,j=1}^{n} \varphi(x_i x_j^*)\psi(y_i y_j^*)$$

$$= \sum_{i,j=1}^{n} < x_i^\varphi, x_j^\varphi >_\varphi < y_i^\psi, y_j^\psi >_\psi = < \sum_{i=1}^{n} x_i^\varphi \times y_i^\psi, \sum_{j=1}^{n} x_j^\varphi \times y_j^\psi >_\sigma$$

in $A_1^\varphi \odot_\sigma A_2^\psi$,

where $A_1^\varphi \odot_\sigma A_2^\psi$ is used in Murray-Neumann's sense [4]. Therefore $[\varphi \times \psi]\left(\left(\sum_{i=1}^{n} x_i \times y_i\right)\left(\sum_{i=1}^{n} x_i \times y_i\right)^*\right) \leq 0$, as desired.

LEMMA 4. *The set $S = \{\varphi \times \psi\}$, where φ and ψ are pure states on A_1 and A_2 respectively, is complete on $A_1 \odot A_2$ in the following sense: For any expression $\sum_{i=1}^{n} x_i \times y_i \neq 0 \times 0$, there exists $\varphi \times \psi$ in S such that*

$$[\varphi \times \psi]\left(\left(\sum_{i=1}^{n} x_i \times y_i\right)\left(\sum_{i=1}^{n} x_i \times y_i\right)^*\right) > 0.$$

PROOF. Without loss of generality, we can assume that $\{x_i\}$ are linearly independent. Since A_2 is a C^*-algebra, there exists a pure state ψ on A_2 such that $\psi(y_1 y_1^*) > 0$.

Now, let $\overline{y_1^\psi} = y_1^\psi (\neq 0)$, $\overline{y_2^\psi}, \ldots, \overline{y_k^\psi}$ be linearly independent elements among the elements $y_1^\psi, \ldots, y_n^\psi$ of A_2^ψ, and furthermore let

$$y_i^{\psi} = \sum_{p=1}^{k} a_{ip}\overline{y_p^{\psi}} \quad i = 1, 2, \ldots, n$$

be their representations by the base $\{\overline{y_p^{\psi}}\}$. Now, define the element x in A_1:

$$x = x_1 + a_{21}x_2 + \cdots + a_{n1}x_n$$

then by the linearly independency of $\{x_i\}$, there exists a pure state φ on A_1 such that $\varphi(xx^*) > 0$. Then

$$[\varphi \times \psi]\left(\left(\sum_{i=1}^{n} x_i \times y_i\right)\left(\sum_{i=1}^{n} x_i \times y_i\right)^*\right) = \sum_{i,j=1}^{n} \varphi(x_i x_j^*)\psi(y_i y_j^*)$$

$$= \sum_{i,j=1}^{n} < x_i^{\varphi}, x_j^{\varphi} >_{\varphi} < y_i^{\psi}, y_j^{\psi} >_{\psi} = < \sum_{i=1}^{n} x_i^{\varphi} \times y_i^{\psi}, \sum_{j=1}^{n} x_j^{\varphi} \times y_j^{\psi} >_{\sigma}$$

in $A_1^{\varphi} \odot_{\sigma} A_2^{\psi}$.

Therefore, if we can prove $\sum_{i=1}^{n} x_i^{\varphi} \times y_i^{\psi} \neq 0 \times 0$ in $A_1^{\varphi} \odot A_2^{\psi}$, then the right-hand side of the last equation is positive, consequently the proof is completed.

Now,

$$\sum_{i=1}^{n} x_i^{\varphi} \times y_i^{\psi} = x_1^{\varphi} \times y_1^{\psi} + \cdots + x_n^{\varphi} \times y_n^{\psi}$$

$$\simeq x_1^{\varphi} \times \overline{y_1^{\psi}} + x_2^{\varphi} \times \left(\sum_{p=1}^{k} a_{2p}\overline{y_p^{\psi}}\right) + \cdots + x_n^{\varphi} \times \left(\sum_{p=1}^{k} a_{np}\overline{y_p^{\psi}}\right)$$

$$\simeq (x_1^{\varphi} + a_{21}x_2^{\varphi} + \cdots + a_{n1}x_n^{\varphi}) \times \overline{y_1^{\psi}} + (a_{22}x_2^{\varphi} + \cdots + a_{n2}x_n^{\varphi}) \times \overline{y_2^{\psi}}$$
$$+ \cdots + (a_{nk}x_n^{\varphi}) \times \overline{y_k^{\psi}}$$

$$\simeq x^{\varphi} \times \overline{y_1^{\psi}} + \cdots + (a_{nk}x_n^{\varphi}) \times \overline{y_k^{\psi}}.$$

Since $\overline{y_1^{\psi}}, \ldots, \overline{y_k^{\psi}}$ are linearly independent in A_2^{ψ}, and $x^{\varphi} \neq 0$ in A_1^{φ}, consequently $\sum_{i=1}^{n} x_i^{\varphi} \times y_i^{\psi} \neq 0 \times 0$ by [4: Lemma 1.1]. q. e. d.

Now, let $\mathfrak{S}$ be the set of positive type functional Φ such that

$$\Phi\left(\sum_{j=1}^{m} s_j \times t_j\right) = \frac{[\varphi \times \psi]\left(\left(\sum_{i=1}^{n} x_i \times y_i\right)\left(\sum_{j=1}^{m} s_j \times t_j\right)\left(\sum_{i=1}^{n} x_i \times y_i\right)^*\right)}{[\varphi \times \psi]\left(\left(\sum_{i=1}^{n} x_i \times y_i\right)\left(\sum_{i=1}^{n} x_i \times y_i\right)^*\right)}$$

where $\varphi \times \psi \in S$, and $\sum_{i=1}^{n} x_i \times y_i$ is an arbitrary element of $A_1 \odot A_2$, and we introduce a new norm in $A_1 \odot A_2$:

$$(*) \quad \alpha\left(\sum_{i=1}^{n} x_i \times y_i\right) = \sup\left[\Phi\left(\left(\sum_{i=1}^{n} x_i \times y_i\right)\left(\sum_{i=1}^{n} x_i \times y_i\right)^*\right)^{1/2} \,\middle|\, \Phi \in \mathfrak{S}\right],$$

then we can prove the following:

LEMMA 5. *The functional on X,* $\alpha\left(\sum_{i=1}^{n} x_i \times y_i\right)$ *defined by* (*) *has the following properties, (that is* $\alpha(\cdot)$ *is a cross-norm on* $A_1 \odot A_2$ *in Schatten's sense* [4: Chap. II, § 2]):

$$(8)\quad \alpha\left(a\cdot\sum_{i=1}^{n} x_i \times y_i\right) = |a|\alpha\left(\sum_{i=1}^{n} x_i \times y_i\right) \text{ where } a \text{ denotes a scalar;}$$

$$\alpha\left(\sum_{i=1}^{n} x_i \times y_i\right) \geq 0; \qquad \alpha(1\times 1) = 1;$$

$$\sum_{i=1}^{n} x_i \times y_i \simeq \sum_{j=1}^{m} s_j \times t_j \text{ implies } \alpha\left(\sum_{i=1}^{n} x_i \times y_i\right) = \alpha\left(\sum_{j=1}^{m} s_j \times t_j\right),$$

$$(9)\quad \alpha\left(\sum_{i=1}^{n} x_i \times y_i + \sum_{j=1}^{m} s_j \times t_j\right) \leq \alpha\left(\sum_{i=1}^{n} x_i \times y_i\right) + \alpha\left(\sum_{j=1}^{m} s_j \times t_j\right),$$

$$(10)\quad \alpha\left(\left(\sum_{i=1}^{n} x_i \times y_i\right)\left(\sum_{j=1}^{m} s_j \times t_j\right)\right) \leq \alpha\left(\sum_{i=1}^{n} x_i \times y_i\right)\alpha\left(\sum_{i=1}^{m} s_j \times t_j\right),$$

$$(11)\quad \alpha\left(\left(\sum_{i=1}^{n} x_i \times y_i\right)^*\right) = \alpha\left(\sum_{i=1}^{n} x_i \times y_i\right),$$

$$(12)\quad \alpha\left(\left(\sum_{i=1}^{n} x_i \times y_i\right)\left(\sum_{i=1}^{n} x_i \times y_i\right)^*\right) = \left(\alpha\left(\sum_{i=1}^{n} x_i \times y_i\right)\right)^2,$$

$$(13)\quad \alpha\left(\sum_{i=1}^{n} x_i \times y_i\right) = 0 \text{ is equivalent to } \sum_{i=1}^{n} x_i \times y_i \simeq 0 \times 0,$$

$$(14)\quad \alpha(x \times y) = \|x\|\,\|y\|.$$

PROOF. Although the argument is similar to that of Fukamiya's paper [2], we give a sketch of proof. In the course of the proof we denote $\sum_{i=1}^{n} x_i \times y_i$, $\sum_{j=1}^{m} s_j \times t_j$ by ξ, η respectively, for convenience.

Ad. (8): Except the last one, all parts are clear. On the other hand, the last one follows from [4].

Ad. (9): By the definition of $\alpha(\cdot)$, for any positive ε there exists a $\Phi \in \mathfrak{S}$ such that

$$\begin{aligned}\alpha^2(\xi+\eta) - \varepsilon &< \Phi((\xi+\eta)(\xi+\eta)^*) \\ &= \Phi(\xi\xi^*) + 2Re\,\Phi(\xi\eta^*) + \Phi(\eta\eta^*) \leq \alpha^2(\xi) + 2\alpha(\xi)\alpha(\eta) + \alpha^2(\eta) \\ &= (\alpha(\xi)+\alpha(\eta))^2.\end{aligned}$$

This proves (9), since ε is arbitrary.

Ad. (10): Put $\Psi(\eta) = \Phi(\xi\eta\xi^*)/\Phi(\xi\xi^*)$ for $\Phi \in \mathfrak{S}$ then $\Psi(\cdot) \in \mathfrak{S}$, so $\Psi(\eta\eta^*) \leq \alpha^2(\eta)$. Therefore by the definition of $\alpha(\cdot)$

$$\alpha(\xi\eta) \leq \alpha(\xi)\alpha(\eta).$$

Ad. (11): For any $\varepsilon > 0$, there exists a $\Phi \in \mathfrak{S}$ such that

$$\alpha^2(\xi) - \varepsilon < \Phi(\xi\xi^*).$$

Put again $\Psi(\eta) = \Phi(\xi\eta\xi^*)/\Phi(\xi\xi^*)$, then $\Psi(\cdot) \in \mathfrak{S}$. Therefore,

$$\alpha^2(\xi^*) \geq \Psi(\xi^*\xi) = \Phi(\xi\xi^*\xi\xi^*)/\Phi(\xi\xi^*) \geq \Phi(\xi\xi^*) > \alpha^2(\xi) - \varepsilon.$$

Since ε is arbitrary, $\alpha(\xi^*) \geq \alpha(\xi)$.

By the symmetry, $\alpha(\xi^*) = \alpha(\xi)$.

Ad. (12): Since $\alpha(\xi\xi^*) \leq \alpha(\xi)\alpha(\xi^*) = \alpha(\xi)^2$ by (10), (11), it is sufficient to show $\alpha(\xi\xi^*) \geq \alpha(\xi)^2$.

For arbitrarily given $\varepsilon > 0$, choose $\Phi \in \mathfrak{S}$ such that

$$\alpha^2(\xi) - \varepsilon < \Phi(\xi\xi^*).$$

Then,

$$(\alpha^2(\xi) - \varepsilon)^2 < \Phi(\xi\xi^*)^2 \leq \Phi(\xi\xi^*\xi\xi^*) \leq \alpha^2(\xi\xi^*).$$

Therefore $\alpha^2(\xi) \leq \alpha(\xi\xi^*)$ by the arbitrariness of ε.

Ad. (13): This immediately follows from Lemma 4.

Ad. (14): Firstly we shall prove $\alpha(\xi) \leq \gamma(\xi\xi^*)^{1/2}$. Indeed, for any real number $k > \gamma(\xi\xi^*)$, the element

$$\xi' = \left(1 \times 1 - \frac{\xi\xi^*}{k}\right)^{1/2}$$

exists in the complete $*$-algebra $A_1 \times_\gamma A_2$ (=the completion of $A_1 \odot_\gamma A_2$ by $\gamma(\cdot)$), and $\xi' = \xi'^*$. Therefore for any $\Phi \in \mathfrak{S}$,

$$0 \leq \Phi(\xi'\xi'^*) = \Phi(\xi'^2) = \Phi\left(1 \times 1 - \frac{1}{k}\xi\xi^*\right) = 1 - \frac{1}{k}\Phi(\xi\xi^*),$$

Consequently, $\qquad \Phi(\xi\xi^*) \leq \gamma(\xi\xi^*).$

Then by the definition of $\alpha(.)$, $\alpha(\xi) \leq \gamma(\xi\xi^*)^{1/2}$.

Using this fact,

$$\alpha(x \times y) \leq \gamma((x \times y)(x \times y)^*)^{1/2} = (\|xx^*\| \|yy^*\|)^{1/2} = \|x\| \cdot \|y\|.$$

On the other hand,

$$\alpha(x \times y) = \sup[\Phi(xx^* \times yy^*)^{1/2} | \Phi \in \mathfrak{S}] \geq \sup[(\varphi(xx^*)\psi(yy^*))^{1/2} | \varphi \times \psi \in S]$$
$$= \|x\| \cdot \|y\|.$$

This completes the proof.

From the above considerations, $A_1 \odot_\alpha A_2$ is a non-complete C^*-algebra with unit 1×1, so completing $A_1 \odot_\alpha A_2$, we obtain a new C^*-algebra, and we denote this algebra by $A_1 \times_\alpha A_2$. Thus the following main theorem has been completely proved.

THEOREM 1. *Let A_1 and A_2 be any C^*-algebras (with unit 1), and define the cross-norm $\alpha(\cdot)$ on the direct product $*$-algebra $A_1 \odot A_2$ by* (*), *then the* *completion* $A_1 \times_\alpha A_2$ *is also a* C^**-algebra* (*with unit*).

4. Direct-product of commutative algebras. In this section, we consider the commutative case which may justify our definition of norm on the direct-product.

Let Ω and Γ be compact Hausdorff spaces and, $C(\Omega)$ and $C(\Gamma)$ be the C^*-algebras of all complex-valued continuous functions on the space Ω and Γ with usual norm and involution, respectively. Then we can prove the following:

THEOREM 2. $C(\Omega)\times_\alpha C(\Gamma)$, *the direct-product of* $C(\Omega)$ *and* $C(\Gamma)$ *in the sense of Theorem* 1, *can be represented isometrically isomorphic as the* C^**-algebra of all continuous functions on the product space* $\Omega\times\Gamma$.

LEMMA 6. *The mapping* $\sum_{i=1}^n x_i\times y_i \to \sum_{i=1}^n x_i(\cdot)y_i(\cdot)$ *is the isometric homomorphism from* $C(\Omega)\odot_\alpha C(\Gamma)$ *into the* $C(\Omega\times\Gamma)$.

PROOF. Firstly we prove this mapping is invariant under the equivalence $\simeq$. Indeed, if $\sum_{i=1}^n x_i\times y_i \simeq \sum_{j=1}^m s_j\times t_j$, then without loss of generality, we can assume that $\{x_i\},\dots,\{t_j\}$ are linearly independent respectively, and $n=m$, furthermore there exist matrices (A_{ij}) and (a_{ij}) such that

$$s_k=\sum_{j=1}^n A_{jk}x_j,\quad t_k=\sum_{j=1}^n a_{kj}y_j,\ \text{ and } (A_{jk})(a_{jk})=1_n.$$

Then,

$$\begin{aligned}\sum_{k=1}^n s_k(\cdot)t_k(\cdot) &= \sum_{k=1}^n\Big(\sum_{j=1}^n A_{jk}x_j(\cdot)\Big)\Big(\sum_{h=1}^n a_{kh}y_h(\cdot)\Big)\\ &= \sum_{k=1}^n\sum_{j,h=1}^n A_{jk}a_{kh}x_j(\cdot)y_h(\cdot) = \sum_{j=1}^n x_j(\cdot)y_j(\cdot).\end{aligned}$$

Since that this mapping is a homomorphism is clear, it is sufficient to show the isometric property. While by the definition of $\alpha(\cdot)$,

$$\begin{aligned}\alpha\Big(\sum_{i=1}^n x_i\times y_i\Big) &= \sup\Big[[\omega\times\gamma]\Big(\Big(\sum_{i=1}^n x_i\times y_i\Big)\Big(\sum_{i=1}^n x_i\times y_i\Big)^*\Big)^{1/2}\Big|\ \omega\in\Omega,\ \gamma\in\Gamma\Big]\\ &= \sup\Big[\Big(\sum_{i,j=1}^n x_i x_j^*(\omega)y_i y_j^*(\gamma)\Big)^{1/2}\Big|\ \omega\in\Omega,\ \gamma\in\Gamma\Big]\\ &= \sup\Big[\Big|\sum_{i=1}^n x_i(\omega)y_i(\gamma)\Big|\ \Big|\ \omega\in\Omega,\ \gamma\in\Gamma\Big]\\ &= \Big\|\sum_{i=1}^n x_i(\cdot)y_i(\cdot)\Big\| \qquad \text{in } C(\Omega\times\Gamma).\end{aligned}$$

PROOF OF THEOREM 2. By the above Lemma 6, $C(\Omega) \times_\alpha C(\Gamma)$ is embeded in $C(\Omega \times \Gamma)$ by the isometrically isomorphic mapping, so the image R of $C(\Omega) \times_\alpha C(\Gamma)$ is a closed $*$-subring of $C(\Omega \times \Gamma)$. Furthermore for any distinct points (ω_1, γ_1) and (ω_2, γ_2) there exists clearly an element r of R such that $r(\omega_1, \gamma_1) \neq r(\omega_2, \gamma_2)$, so $R = C(\Omega \times \Gamma)$.

REMARK. Concerning the algebraic properties of the direct-product— for example, simplicity, factoriality of $A_1 \times_\alpha A_2$—we shall discuss in the later paper in this journal.

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