# CONTRIBUTIONS TO THE THEORY OF FUNCTIONS OF A BICOMPLEX VARIABLE[1)]

JAMES D. RILEY

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## INTRODUCTION

**1.** The customary axiomatic definition of the system of ordinary complex numbers may be given as follows: (see Dickson, [1][2)])

Let $a$, $b$, $c$, $d$ be real numbers. Two couples $(a, b)$ and $(c, d)$ are called equal if and only if $a = c$ and $b = d$. Addition and multiplication of two couples are defined by the formulas:

$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b)\ (c, d) = (ac - bd,\ ad + bc).$$

Addition and multiplication are commutative and associative, and the distributive law holds.

Subtraction is defined as the operation inverse to addition. It is always possible and unique.

Division is defined as the operation inverse to multiplication. Division, except by $(0, 0)$ is possible and unique:

$$\frac{(c, d)}{(a, b)} = \left(\frac{ac+bd}{a^2+b^2},\ \frac{ad-bc}{a^2+b^2}\right).$$

Now let $(a, 0)$ be $a$, and $(0, 1)$ be $i$. Then

$$i^2 = (0, 1)\ (0, 1) = (-1, 0) = -1$$
$$(a, b) = (a, 0) + (0, b) = (a, 0) + (b, 0)\ (0, 1) = a + bi.$$

Thus the set of all real couples, with the above definitions, becomes the field of all complex numbers. The theory of complex-valued analytic functions of a complex variable has been extensively developed.

**2.** The question next arises as to what occurs if the above definitions are applied to couples of complex numbers, and the corresponding function theory investigated. This new system permits the same definition of the four fundamental operations, except that division will not be possible by the couple $(a, b)$

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2) Numbers in brackets refer to the bibliography at the end of the paper.

if $a^2 + b^2 = 0$. This occurs if $b = \pm ai$, and the system is therefore not a field. Furthermore, the product of the couples $(a, ai)$ and $(a, -ai)$ is $(a^2 - a^2, -a^2 i + a^2 i) = (0, 0)$. Thus nil-factors or divisors of zero occur. However, the system is a linear algebra.

Futagawa, [2] and [3], has published two articles on the theory of functions of quadruples, which are equivalent to the couples of complex numbers. Scorza [4] and Spampinato [5] each have presented results concerning a system equivalent to these couples of complex numbers except for notation. From the extensive literature concerning analytic functions on linear algebras in general, mention is made here only of the papers by Scheffers [6] and Ringleb [7] and several sections in a book by Hille [8]. Takasu [9] has presented a theory of functions on an algebra which is a generalization of, and includes as a special case, the algebra presently being discussed. An article by Ward [10] includes an extensive bibliography which eliminates the necessity of including a complete set of references here.

**3.** A simplified notation is obtained by introducing a new unit $j = (0, 1)$. Then

$$j^2 = (0,1)\ (0,1) = (-1,0) = -1.$$

The couple $(a, b) = (a, 0) + (0, b) = (a, 0) + (b, 0)\ (0, 1) = a + bj$ will be termed a bicomplex number. This number may also be written as a real linear combination of the four units $1, i, j, ij$. A geometric interpretation is afforded by the four-dimensional Euclidean space.

**4.** By squaring the numbers $\frac{1}{2}(1+ij)$ and $\frac{1}{2}(1-ij)$ it is found that they are idempotent elements. A result of Scheffers (see Dickson [1] p. 26-27) then states that this system is reducible. In fact the numbers $e_1 = \frac{1}{2}(1+ij)$, $e_2 = ie_1$, $e_3 = \frac{1}{2}(1-ij)$, $e_4 = ie_3$ form a basis if real coefficients are used, and $e_1 e_3 = e_1 e_4 = e_2 e_3 = e_2 e_4 = 0$. Then if complex coefficients are permitted, $e_1$ and $e_3$ alone form the basis. The bicomplex number $a+bj$ is uniquely represented as $(a-bi)e_1 + (a+bi)e_3$.

Now consider the bicomplex variable $z = x + jy$, $x$ and $y$ complex. Then $z = (x-iy)e_1 + (x+iy)e_3$. For convenience let $x - iy = z_1$ and $x + iy = z_3$. Then $z = z_1 e_1 + z_3 e_3$. A fundamental result of Ringleb [7] (which he proves for reducible linear algebras in general) then states that an analytic function $f(z)$ (the analyticity of a function of a bicomplex variable will be defined in section I.) can be decomposed uniquely into the sum of functions analytic in the separate sub-algebras, i.e., $f(z) = g(z_1)e_1 + h(z_3)e_3$, where $g(z_1)$ is an analytic functon of $z_1$ and $h(z_3)$ is an analytic function of $z_3$, and that conversely if $g(z_1)$ is an analytic function of $z_1$ and $h(z_3)$ is an analytic function of $z_3$, then $f(z) = g(z_1)e_1 + h(z_3)e_3$ is an analytic function of $z$. Here $f(z)$ takes bicomplex values, while $g(z_1)$ and $h(z_3)$ take only complex values. A proof of this directly based on the definition of an analytic function and not using differentials, as Ringleb's proof does, will be given in section I. This provides a powerful method for the study of analytic functions of a bicomplex variable. This decomposition actually occurs in Futagawa's work for the special case where $f(z)$ is $\sin z$ or $\cos z$, but he makes no use of the decomposition.

**5.** Let $z=z_1e_1+z_3e_3$ and $w=w_1e_1+w_3e_3$. Since $e_1e_3=0$, $e_1^2=e_1$, and $e_3^2=e_3$, $zw=z_1w_1e_1+z_3w_3e_3$. Thus $zw=0$ if and only if $z_1w_1=0$ and $z_3w_3=0$. Thus the product of two non-zero numbers is zero if and only if one of them is a complex multiple of $e_1$ and the other is a complex multiple of $e_3$. Note also that $ze_1=z_1e_1$ and $ze_3=z_3e_3$. Thus it will suffice to say multiple instead of complex multiple. The set of numbers which are multiples of $e_1$ will be termed the first nil-plane. Similarly the set of numbers which are multiples of $e_3$ will be termed the second nil-plane[3]. A non-zero number which is a multiple of $e_1$ will be termed a first nil-factor and a non-zero number which is a multiple of $e_3$ will be termed a second nil-factor. By these conventions, the origin belongs to both nil-planes, but is not a nil-factor.

**6.** The elementary functions have been discussed by Futagawa. However they may well be defined by the formula $f(z) = f(z_1)e_1 + f(z_3)e_3$, where, in the right member, $f$ denotes the elementary function whose generalization to bicomplex values is desired, since for $z$ complex, $z=ze_1+ze_3$, and thus $z_1=z_3=z$. In fact, this formula provides a natural way of extending every complex-valued function of a complex variable into the bicomplex space.

**7.** Two immediate generalizations to the bicomplex case of the concept of absolute value of a complex number will be employed extensively. They are the norm of $z=x+jy$, denoted by $\|z\|$, and defined as $\|z\|\equiv\sqrt{|x|^2+|y|^2}$ and the absolute value of $z$, denoted by $|z|$, and defined as $|z|\equiv\sqrt{|x^2+y^2|}$. The norm of $z$ is readily seen to be the Euclidean distance norm, and thus satisfies the properties required of a norm. The absolute value of $z$ does not satisfy the triangle inequality and is zero for the class of numbers by which division is not permitted, i.e., when $z$ is zero or a nil-factor. (This absolute value is the first modulus in Futagawa's polar representation of $z$.) It is frequently convenient to express $|z|$ and $\|z\|$ in terms of $|z_1|$ and $|z_3|$. Thus

$$|z| =\sqrt{|x^2+y^2|} =\sqrt{|(x-iy)(x+iy)|}= \sqrt{|z_1z_3|} = \sqrt{|z_1|\cdot|z_3|}\,.$$

Then $|z|\neq 0$ and division by $z$ is possible if and only if $z_1$ and $z_3$ are both non-zero. In the representation of the bicomplex number system based on its reducibility, division by $z$ takes a particularly simple form, since

3) See Futagawa [2] for a geometric interpretation of the nil-planes.

$$\frac{1}{z}=\frac{1}{z_1e_1+z_3e_3}=\frac{1}{z_1e_1+z_3e_3}\cdot\frac{\frac{1}{z_1}e_1+\frac{1}{z_3}e_3}{\frac{1}{z_1}e_1+\frac{1}{z_3}e_3}=\frac{\frac{1}{z_1}e_1+\frac{1}{z_3}e_3}{e_1+e_3}=\frac{1}{z_1}e_1+\frac{1}{z_3}e_3.$$

The identity

$$\|z\|=\frac{1}{\sqrt{2}}\sqrt{|z_1|^2+|z_3|^2}$$

is easily verified by expressing both members in terms of the four real components of $z$, or somewhat more conveniently by employing polar coordinates in the $z_1$- and $z_3$- planes.

Note that if $z$ is in a nil-plane, say the first, then $\|z\|=\frac{1}{\sqrt{2}}|z_1|$, so that distance in a nil-plane, if measured by the absolute value of the complex number $z_1$, differs from distance in the bicomplex space by a constant factor.

Note also that if $z = x + jy$ is a complex number, so that $y = 0$ and $z = x$, then both $|z|$ and $\|z\|$ are equal to $|x|$.

## I. ANALYTIC FUNCTIONS-DECOMPOSITION

8. Analyticity will now be defined and the decomposition theorem proved. The definition and the first part of the proof bear considerable similarity to the corresponding definition and the derivation of the Cauchy-Riemann equations in the theory of functions of a complex variable. It will also be discovered that differentiation of an analytic function with respect to $z$ will be equivalent to differentiation of the separate components with respect to their respective variables, $z_1$ and $z_3$.

9. Throughout this paper, the topological concepts employed for sets of bicomplex numbers will be those of four-dimensional Euclidean space. For example, a set of points $S$ will be called open if for every $z_0$ in $S$ there exists a $K>0$ such that every $z$ for which $\|z-z_0\|<K$ is also in $S$. An open connected set will be called a region. The set of all bicomplex numbers with this topology will be called the bicomplex space. If $T$ is a region, and if each $z$ in $T$ is written in the form $z = z_1e_1 + z_3e_3$, (where $e_1=\frac{1}{2}(1+ij)$, $e_3=\frac{1}{2}(1-ij)$, see Introduction, parts 4 and 5), then the set $T_1$ of values of $z_1$ is a region in the $z_1$-plane (in the topology of that plane) and the set $T_3$ of values of $z_3$ is a region in the $z_3$-plane. These regions $T_1$ and $T_3$ will be termed the component regions of $T$. If the regions $T_1$ and $T_3$ are given, the largest region $T$ whose component regions are $T_1$ and $T_3$ will be termed the product-region of $T_1$ and $T_3$.

It should be observed that for convenience the regions $T_1$ and $T_3$ have been chosen in the complex $z_1$- and $z_3$-planes, which are not planes of the bicomplex space. If component-regions in the space itself are desired, the components $z_1e_1$ and $z_3e_3$ of the number $z$, located in the first and second nil-planes, respectively, should be considered.

**10.** Let $z_0 = x_0 + jy_0$ be a bicomplex number. The bicomplex variable $z = x + jy$ will be said to approach $z_0$, and $z_0$ will be termed the limit of $z$ if $\|z-z_0\|$ approaches zero. It may be verified that $z$ approaches $z_0$ if and only if $x$ approaches $x_0$ and $y$ approaches $y_0$.

DEFINITION. Let $f(z)$ be a bicomplex-valued function of the bicomplex variable $z=x+jy$, defined in a region $T$. Let $z_0$ be a point in $T$. Then $f(z)$ will be termed *analytic at* $z_0$ if and only if there exists a bicomplex number $f'(z_0)$ such that for any $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that

$$\left\| \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) \right\| < \varepsilon$$

whenever $\|z-z_0\| < \delta_\varepsilon$ and $|z-z_0| \neq 0$.

DEFINITION. A function $f(z)$ will be termed *analytic in a region* $T$ if it is analytic at each point of $T$.

THEOREM. (*Decomposition theorem of Ringleb*). *Let $f(z)$ be analytic in a region $T$, and let $T_1$ and $T_3$ be the component regions of $T$, in the $z_1$- and $z_3$-planes, respectively. Then there exists a unique pair of complex-valued analytic functions, $g(z_1)$ and $h(z_3)$, defined in $T_1$ and $T_3$, respectively, such that*

$$(A) \qquad f(z) = g(z_1)e_1 + h(z_3)e_3$$

*for all $z$ in $T$. Conversely, if $g(z_1)$ is any complex-valued analytic function in a region $T_1$ and $h(z_3)$ any complex-valued analytic function in a region $T_3$, then the bicomplex-valued function $f(z)$ defined by the formula (A) is an analytic functon of the bicomplex variable $z$ in the product-region $T$ of $T_1$ and $T_3$.*

PROOF. Let $f(z) = u(x,y) + jv(x,y)$. Let $z_0 = x_0 + jy_0$ be an arbitrary point in $T$, and let $z$ approach $z_0$ in such a way that $y$ is always equal to $y_0$, i.e., $z = x + jy_0$. Then $z-z_0=x-x_0$, hence the assumption that $|z-z_0| \neq 0$ is satisfied for all $x \neq x_0$, and

$$\frac{f(z)-f(z_0)}{z-z_0} = \frac{[u(x,y_0)+jv(x,y_0)] - [u(x_0,y_0)+jv(x_0,y_0)]}{x-x_0}$$

$$= \frac{u(x,y_0) - u(x_0,y_0)}{x-x_0} + j\,\frac{v(x,y_0) - v(x_0,y_0)}{x-x_0}.$$

This tends to a limit if and only if each term tends separately to a limit. But this means simply that the complex-valued functions $u$ and $v$ of the two complex variables $x$ and $y$ possess partial derivatives with respect to $x$, $\partial u/\partial x$ and $\partial v/\partial x$, at $x=x_0$, $y=y_0$, and that

$$f'(z_0) = \left(\frac{\partial u}{\partial x}+j\frac{\partial v}{\partial x}\right)_{x=x_0,y=y_0}.$$

Similarly if $z$ approaches $z_0$ so that $z-z_0=(y-y_0)j$, i.e., $z=x_0+jy$, it is found that $\partial u/\partial y$ and $\partial v/\partial y$ exist at $x=x_0$, $y=y_0$ and

$$f'(z_0) = \left(\frac{\partial v}{\partial y}-j\frac{\partial u}{\partial y}\right)_{x=x_0,\ y=y_0}.$$

Comparison of the two expressions for $f'(z_0)$ gives

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ ; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

for any point $z_0=x_0+jy_0$ of $T$. These are the generalized Cauchy-Riemann equations. Now, using the representation mentioned in the introduction

$$w = f(z) = u + jv = (u-iv)e_1 + (u+iv)e_3$$
$$z = x + jy = (x-iy)e_1 + (x+iy)e_3$$

and with the notation

$$w_1 = u-iv, \quad w_3 = u+iv,$$
$$z_1 = x-iy, \quad z_3 = x+iy,$$

then $w = w_1e_1+w_3e_3$ and $z = z_1e_1+z_3e_3$.

Then since the partial derivatives of $u$ and $v$ with respect to $x$ and $y$ exist in $T$, the partial derivatives of $w_1$ and $w_3$ with respect to $x$ and $y$ exist in $T$ and

$$\frac{\partial w_1}{\partial x}=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial x}, \quad \frac{\partial w_1}{\partial y}=\frac{\partial u}{\partial y}-i\frac{\partial v}{\partial y},$$
$$\frac{\partial w_3}{\partial x}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}, \quad \frac{\partial w_3}{\partial y}=\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}.$$

Using the generalized Cauchy-Riemann equations

$$\frac{\partial w_1}{\partial y}=-i\frac{\partial w_1}{\partial x}; \quad \frac{\partial w_3}{\partial y}=i\frac{\partial w_3}{\partial x}.$$

Also, since $x = \frac{1}{2}(z_1+z_3)$ and $y = \frac{i}{2}(z_1-z_3)$,

$$\frac{\partial x}{\partial z_1}=\frac{\partial x}{\partial z_3}=\frac{1}{2}, \quad \frac{\partial y}{\partial z_1}=\frac{i}{2}, \quad \frac{\partial y}{\partial z_3}=-\frac{i}{2}.$$

Since $w_1$ and $w_3$ are analytic functions of $x$ and $y$, and $x$ and $y$ are analytic functions of $z_1$ and $z_3$, for any point $z_0=x_0+jy_0$ of $T$, $w_1$ and $w_3$ are analytic functions of $z_1$ and $z_3$. Further

$$\frac{\partial w_1}{\partial z_1} = \frac{\partial w_1}{\partial x}\frac{\partial x}{\partial z_1}+\frac{\partial w_1}{\partial y}\frac{\partial y}{\partial z_1} = \frac{\partial w_1}{\partial x}\cdot\frac{1}{2}-i\frac{\partial w_1}{\partial x}\cdot\frac{i}{2} = \frac{\partial w_1}{\partial x}$$

$$\frac{\partial w_1}{\partial z_3} = \frac{\partial w_1}{\partial x}\frac{\partial x}{\partial z_3}+\frac{\partial w_1}{\partial y}\frac{\partial y}{\partial z_3} = \frac{\partial w_1}{\partial x}\cdot\frac{1}{2}-i\frac{\partial w_1}{\partial x}\left(-\frac{i}{2}\right) = 0$$

$$\frac{\partial w_3}{\partial z_1} = \frac{\partial w_3}{\partial x}\frac{\partial x}{\partial z_1}+\frac{\partial w_3}{\partial y}\frac{\partial y}{\partial z_1} = \frac{\partial w_3}{\partial x}\frac{1}{2}+i\frac{\partial w_3}{\partial x}\cdot\frac{i}{2} = 0$$

$$\frac{\partial w_3}{\partial z_3} = \frac{\partial w_3}{\partial x}\frac{\partial x}{\partial z_3}+\frac{\partial w_3}{\partial y}\frac{\partial y}{\partial z_3} = \frac{\partial w_3}{\partial x}\frac{1}{2}+i\frac{\partial w_3}{\partial x}\left(-\frac{i}{2}\right) = \frac{\partial w_3}{\partial x}.$$

Since these equations hold at all points of the region $T$, it follows that $w_1$ is an analytic function of $z_1$ alone and $w_3$ is an analytic function of $z_3$ alone, in the region $T$. Then placing $w_1(z_1) = g(z_1)$ for $z_1$ in $T_1$ and $w_3(z_3) = h(z_3)$ for $z_3$ in $T_3$,

$$f(z) = g(z_1)e_1 + h(z_3)e_3$$

and the representation is unique, since $w_1$ is uniquely determined as $u-iv$ and $w_3$ is uniquely determined as $u+iv$.

Conversely if $g(z_1)$ is an analytic function of $z_1$ in a region $T_1$ of the $z_1$-plane and $h(z_3)$ is an analytic function of $z_3$ in a region $T_3$ of the $z_3$-plane, then $g(z_1)e_1 + h(z_3)e_3$ is defined as a function of $z=z_1e_1+z_3e_3$ in the product-region of the bicomplex space having the components $T_1$ and $T_3$. Denoting this function by $f(z)$ then, if $z_0=z^0{}_1e_1+z^0{}_3e_3$ is a point of $T$, and $z$ a point of $T$ for which $|z-z_0|\neq 0$,

$$\frac{f(z)-f(z_0)}{z-z_0} = \frac{g(z_1)-g(z_1^0)}{z_1-z_1^0}e_1 + \frac{h(z_3)-h(z_3^0)}{z_3-z_3^0}e_3 .$$

Since the right side approaches a limit as $z_1\to z^0{}_1$, $z_1\neq z^0{}_1$, and $z_3\to z^0{}_3$, $z_3\neq z^0{}_3$, then the left side approaches a limit as $z\to z_0$ and $|z-z_0|\neq 0$, since $|z-z_0|=0$ if and only if $z_1=z^0{}_1$ or $z_3=z^0{}_3$.

**11.** COROLLARY 1. *Let $f(z)$ be analytic in a region $T$ which intersects the complex plane. Let $S$ be a set of points in the intersection of $T$ and the complex plane and let $S$ have a limit point in this intersection. Suppose that for all $z$ in $S$, $f(z)$ assumes complex values. Then $f(z)$ assumes complex values for every $z$ in the intersection of $T$ and the complex plane and $f(z)$ may be defined for every value $z$ in the components $T_1$ and $T_3$ of $T$ so that the Ringleb decomposition formula becomes*

$$f(z) = f(z_1)e_1 + f(z_3)e_3$$

*for all $z=z_1e_1+z_3e_3$ in $T$.*

PROOF. By the Ringleb decomposition theorem $f(z)=g(z_1)e_1+h(z_3)e_3$ for $z$ in $T$. For $z=x+jy$ in $S$, $z$ is complex and $y=0$. Then $z_1=x-iy=x$, $z_3=x+iy=x$, and $z=z_1=z_3=t$, where $t$ *is* a new complex variable introduced for convenience. For each $z$ in $S$, $f(z)$ is a complex number, and thus $g(z_1)=f(z)$ and $h(z_3)=f(z)$, or $f(t)=g(t)=h(t)$. Now $S$ has a limit point in the complex plane. Thus $g(t)\equiv h(t)$. Thus $g(z_1)=h(z_3)$ whenever $z_1=z_3$. But $z_1=z_3$ for all $z$ in the complex plane. Thus if $z$ has a value $t$ in the intersection of $T$ and the complex plane, $f(t)=g(t)e_1+g(t)e_3=g(t)$ or $f(z)=f(z)e_1+f(z)e_3$, and $f(z)$ assumes complex values there.

Now for every value of $z_1$ in $T_1$ for which $f(z_1)$ is not already defined (recall that the complex values for which $f(z)$ is defined are those of $T$; compare Section I, part 9), define $f(z_1)=g(z_1)$; and for every value of $z_3$ in $T_3$ for which $f(z_3)$ is not already defined, define $f(z_3)=h(z_3)$. Then $f(z_1)$ is an analytic function of $z_1$ in $T_1$ and $f(z_3)$ is an analytic fuuction of $z_3$ in $T_3$. Thus $f(z)=f(z_1)e_1+f(z_3)e_3$ for $z$ in $T$.

COROLLARY 2.

$$\frac{df}{dz} = \frac{dg}{dz_1}e_1 + \frac{dh}{dz_3}e_3.$$

PROOF. By direct substitution of results obtained in the course of proving the theorem

$$\frac{df}{dz} = f'(z) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\right)e_1 + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)e_3$$
$$= \frac{\partial w_1}{\partial x}e_1 + \frac{\partial w_3}{\partial x}e_3 = \frac{\partial w_1}{\partial z_1}e_1 + \frac{\partial w_3}{\partial z_3}e_3 = \frac{dg}{dz_1}e_1 + \frac{dh}{dz_3}e_3.$$

REMARK. If $f(z)$ is analytic in a region $T$, then the decomposition formula $f(z) = g(z_1)e_1 + h(z_3)e_3$ will automatically define an analytic function, coinciding with $f(z)$ in $T$, at all points of the product region of $T_1$ and $T_3$, which in general will include points not in $T$. This trait of the bicomplex function theory has no counterpart in the theory of functions of a complex variable.

**12.** The decomposability of a function is in itself a rather strong requirement, as shown by the following.

REMARK. Let $f(z)$ be any bicomplex-valued function of a complex variable, i.e., a bicomplex-valued function defined if and only if $z$ is in the complex plane. Then if it be required that $f(z)$ be extended into the bicomplex space in such a way that $f(z)$ is decomposable as $f(z) = g(z_1)e_1 + h(z_3)e_2$, then the extension is already uniquely determined. This follows from the fact that for $z$ in the complex plane, $z = z_1 = z_3$ and the definition of $f(z)$ for these values determines $g(z_1)$ and $h(z_3)$ in their entire domains of definition, the complex $z_1$- and $z_3$-planes, respectively. Thus $f(z)$ is determined in the entire bicomplex space. Of course if $f(z)$ is analytic the continuation will be analytic.

COROLLARY. *If $f(z)$ and $F(z)$ are two analytic function of the bicomplex variable $z$ which are equal for all complex values of $z$, the functions are equal for all bicomplex valves of $z$.*

The above remark and its corollary could be generalized in various ways.

## II. POWER SERIES AND TAYLOR'S THEOREM

**13.** DEFINITION. Let $\sum_{n=0}^{\infty} a_n$ be a series of bicomplex terms, and let $s_k = \sum_{n=0}^{k} a_n$. The series will be said to *converge* if for $\varepsilon > 0$ there exists an integer $N$ such that for all $m, n > N$, $\| s_m - s_n \| < \varepsilon$.

Let $a_n = b_n e_1 + c_n e_3$, where $b_n$ and $c_n$ are complex. It will be useful to show that $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} b_n$ and $\sum_{n=0}^{\infty} c_n$ converge in the ordinary sense. Therefore let $\sum_{n=0}^{\infty} a_n$ converge. Let $s_k = p_k e_1 + q_k e_3$. Since $a_n = b_n e_1 + c_n e_3$, $s_k = \sum_{n=0}^{k} a_n = \left(\sum_{n=0}^{k} b_n\right)e_1 + \left(\sum_{n=0}^{k} c_n\right)e_3$. Thus $p_k = \sum_{n=0}^{k} b_n$ and $q_k = \sum_{n=0}^{k} c_n$. Now $s_m - s_n = (p_m - p_n)e_1 + (q_m - q_n)e_3$ and $\| s_m - s_n \| = \frac{1}{\sqrt{2}}\sqrt{|p_m - p_n|^2 + |q_m - q_n|^2} < \varepsilon$ for $m, n > N$. Thus $|p_m - p_n| < \varepsilon\sqrt{2}$ and $|q_m - q_n| < \varepsilon\sqrt{2}$. Therefore $\sum_{n=0}^{\infty} b_n$ and $\sum_{n=0}^{\infty} c_n$ converge.

Conversely let $\sum_{n=0}^{\infty} b_n$ and $\sum_{n=0}^{\infty} c_n$ converge. Then for $\varepsilon > 0$, there exists $N_1$ such that $|p_m - p_n| < \varepsilon$ for $m, n > N_1$ and $N_2$ such that $|q_m - q_n| < \varepsilon$ for $m, n > N_2$. Then for $N = \max(N_1, N_2)$, these inequalities both hold for $m, n > N$. Thus $\| s_m - s_n \| < \varepsilon$ for $m, n > N$ and $\sum_{n=0}^{\infty} a_n$ converges.

DEFINITION. Let $\sum_{n=0}^{\infty} a_n$ be a series of bicomplex terms, and let $s_k = \sum_{n=0}^{k} a_n$. The series will be said to *converge to the sum* $S$ if for $\varepsilon > 0$ there exists an integer $N$ such that for $k > N$, $\| s_k - S \| < \varepsilon$.

It is easily verified that a series is convergent if and only if the series converges to a sum $S$, and that $S = Pe_1 + Qe_3$, where $P = \sum_{n=0}^{\infty} b_n$, $Q = \sum_{n=0}^{\infty} c_n$.

Now let $\sum_{n=0}^{\infty} a_n z^n$ denote a power series. It converges if and only if $\sum_{n=0}^{\infty} b_n z_1^n$ and $\sum_{n=0}^{\infty} c_n z_3^n$ converge, from the above analysis, since $a_n z^n = (b_n e_1 + c_n e_3) \times (z_1 e_1 + z_3 e_3)^n = (c_n z_1^n)e_1 + (c_n z_3^n)e_3$.

DEFINITION. The set of all interior points of the set of points at which a power series is convergent will be termed the *region of convergence* of the power series.

It follows from the Ringleb decomposition theorem that a convergent power series represents an analytic function in its region of convergence.

14. Since $\sum_{n=0}^{\infty} b_n z_1^n$ and $\sum_{n=0}^{\infty} c_n z_3^n$ are complex power series they will have radii of convergence, which may be zero or infinite. Let the radius of convergence of $\sum_{n=0}^{\infty} b_n z_1^n$ be $R_1$ and the radius of convergence of $\sum_{n=0}^{\infty} c_n z_3^n$ be $R_3$. If $R_1 = 0$ then the set of points of convergence of $\sum_{n=0}^{\infty} a_n z^n$ is restricted to the

second nil-plane, and if $R_3=0$ to the first nil-plane. In these two cases the region of convergence as defined above will be empty. If $R_1, R_3 > 0$ however, then the set of points of convergence contains a hypersphere, $||z|| < k$ for some $k > 0$, centered at the origin, and the region of convergence is not empty. Let

$$\xi = \left(\frac{e_1}{R_1}+\frac{e_3}{R_3}\right)z = \frac{z_1}{R_1}e_1+\frac{z_3}{R_3}e_3 = \xi_1 e_1+\xi_3 e_3$$

or

$$z = (R_1e_1+R_3e_3)\xi = R_1\xi_1e_1+R_3\xi_3e_3.$$

Then the series $\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n(R_1{}^n e_1+R_3{}^n e_3)\,\xi^n$ converges for $|\xi_1| < 1$ and $|\xi_3| < 1$. Thus a power series may be normalized to unit radii of convergence of both component power series.

**15.** It seems desirable to be able to describe the region of convergence in terms of $z$ itself, particularly in terms of a norm such that the series converges when the norm is less than some constant and diverges when the norm is greater than this constant. Since $||z|| = \frac{1}{\sqrt{2}}\sqrt{|z_1|^2+|z_3|^2}$, it is seen that this norm fails to describe the region of convergence for the normalized power series. For if $|z_1| < 1$ and $|z_3| < 1$, then $||z|| < 1$, and if $|z_1| = |z_3| = 1$, then $||z|| = 1$. But if $|z_3| = 0$ and $1 < |z_1| < \sqrt{2}$, then $||z|| < 1$. The problem is solved by the following theorem, which is applicable when $R_1=R_3=R$, which is the case when the power series has been normalized, and which is in particular the case whenever the series represents a function which is complex whenever $z$ is complex, for then $f(z)=f(z_1)e_1+f(z_3)e_3$ and $a_n=b_n=c_n$ for all $n$. This refers, of course, to the generalization of any function studied in classical complex variable theory.

THEOREM. *Let*

$$N(z)=\sqrt{||z||^2+\sqrt{||z||^4-|z|^4}}.$$

*Then $N(z)$ is a norm, and if $\sum^{\infty} a_n z^n$ is a power series whose component series*

Hence, if $N(z) < R$, both $|z_1| < R$ and $|z_3| < R$ and $\sum_{n=0}^{\infty} a_n z^n$ converges, whereas if $N(z) > R$, either $|z_1|$ or $|z_3|$ will be greater than $R$, hence the series must diverge.

Using the above representation of $N(z)$, it is immediately verified that $N(z)$ is a norm and also that $N(zw) \leqq N(z)N(w)$. Also the following inequalities may be verified

$$||z|| \leqq N(z) \leqq ||z|| \sqrt{2} \text{ and } |z| \leqq N(z).$$

Each of the equations $||z|| = N(z)$ and $|z| = N(z)$ is satisfied if and only if $|z_1| = |z_3|$ (which is then also equal to $|z|$), hence in particular when $z$ is complex. The equation $N(z) = ||z|| \sqrt{2}$ is satisfied if and only if $z$ is a nil-factor (i.e., $z_1 = 0$ or $z_3 = 0$), or if $z = 0$.

**16.** The existence of a Taylor series is demonstrated by Futagawa [3], without the use of the Ringleb decomposition. His conclusion is that the series is absolutely convergent in the hypersphere of radius one-fourth the distance from the point of expansion (the center of the hypersphere) to the boundary of the region $T$ of analyticity of the function. Taylor series in more general systems are discussed by several authors. With the use of the decomposition theorem and the above norm, $N(z)$, it is possible to show that the region of convergence not only contains the hypersphere of radius equal to the distance from the point of expansion to the boundary of the region $T$ but actually extends outside of this hypersphere in certain directions.

The actual process of expanding an analytic function as a power series may be carried out without employing the decomposition directly, as is evidenced by

TAYLOR'S THEOREM. *Let $f(z)$ be analytic in a four-dimensional region $T$, and let $\alpha$ be a point of $T$. Then $f(z)$ may be expanded as a generalized Taylor series about the point $\alpha$:*

$$f(z) = f(\alpha) + \frac{z-\alpha}{1!} f'(\alpha) + \frac{(z-\alpha)^2}{2!} f''(\alpha) + \cdots + \frac{(z-\alpha)^n}{n!} f^{(n)}(\alpha) + \cdots,$$

*wherever $f(z)$ is defined and the series is convergent. If $d$ is the greatest lower bound of $||z-\alpha||$ for $z$ a boundary point of $T$, then the above series for $f(z)$ converges for $N(z) < d\sqrt{2}$. In particular this implies convergence in the hypersphere $||z-\alpha|| < d$.*

PROOF. By the Ringleb decomposition theorem

$$f(z) = g(z_1)e_1 + h(z_3)e_3$$

and also

$$f'(z) = g'(z_1)e_1 + h'(z_3)e_3$$

in $T$. Then

$$f^{(n)}(z) = g^{(n)}(z_1)e_1 + h^{(n)}(z_3)e_3$$

in $T$. Let $\alpha=\alpha_1 e_1+\alpha_3 e_3$ be a point in $T$. Then, by Taylor's theorem for the complex case,

$$\text{(A)}\quad g(z_1)=g(\alpha_1)+\frac{z_1-\alpha_1}{1!}g'(\alpha_1)+\cdots+\frac{(z_1-\alpha_1)^n}{n!}g^{(n)}(\alpha_1)+\cdots$$

and

$$\text{(B)}\quad h(z_3)=h(\alpha_3)+\frac{z_3-\alpha_3}{1!}h'(\alpha_3)+\cdots+\frac{(z_3-\alpha_3)^n}{n!}h^{(n)}(\alpha_3)+\cdots$$

for $|z_1-\alpha_1|<R_1$ and $|z_3-\alpha_3|<R_3$, where the radii of convergence, $R_1$ and $R_3$, are not zero because of the openness of $T$. There exists a point $\beta_1$ in the $z_1$-plane such that $|\beta_1-\alpha_1|=R_1$ and $g(z_1)$ has a singularity for $z_1=\beta_1$. Then $f(z)$ is singular for $z=\beta_1 e_1+\alpha_3 e_3$. Thus $||(\beta_1 e_1+\alpha_3 e_3)-\alpha||\geqq d$ or $||(\beta_1-\alpha_1)e_1||\geqq d$. But $||(\beta_1-\alpha_1)e_1||=\frac{1}{\sqrt{2}}|\beta_1-\alpha_1|=\frac{R_1}{\sqrt{2}}$. Therefore $\frac{R_1}{\sqrt{2}}\geqq d$ or $R_1\geqq d\sqrt{2}$. By similar reasoning, $R_3\geqq d\sqrt{2}$.

Now consider the series

$$\text{(C)}\quad f(\alpha)+\frac{z-\alpha}{1!}f'(\alpha)+\cdots+\frac{(z-\alpha)^n}{n!}f^{(n)}(\alpha)+\cdots.$$

The series (C) has the component series (A) and (B), and thus (C) converges for $z=z_1e_1+z_3e_3$ if and only if (A) and (B) both converge. Then (A), (B), and hence (C), certainly converge for $N(z)<d\sqrt{2}$, and for those points of the set $N(z)<d\sqrt{2}$ which belong to $T$, the sum of the series (C) is $g(z_1)e_1+h(z_3)e_3=f(z)$. Moreover, since $N(z)\leqq||z||\sqrt{2}$, the inequality $N(z)<d\sqrt{2}$ holds in particular for $||z||<d$.

REMARKS. This conclusion does not mean that (C) diverges for $N(z)>d\sqrt{2}$. This is the case, however, if $R_1=R_3=d\sqrt{2}$, which shows that no better general conclusion is possible. Convergence for $||z||<d$ is not the best possible conclusion, which brings out the fact that a power series with a bounded region of convergence, hence representing a function which cannot be analytic for all values of $z$, always converges for some points which are more distant from the point of expansion than the nearest singularity, if distance is measured in the sense of the Euclidean norm.

**17.** DEFINITION. The series $\sum_{n=0}^{\infty}a_n$, where the $a_n$ are bicomplex numbers, is termed *absolutely convergent* if $\sum_{n=0}^{\infty}||a_n||$ is convergent.

It may be verified that a necessary and sufficient condition for absolute convergence of the series $\sum_{n=0}^{\infty}a_n$ is the absolute convergence of the component series, $\sum_{n=0}^{\infty}b_n$, and $\sum_{n=0}^{\infty}c_n$, since

$|b_n| \leqq \sqrt{|b_n|^2 + |c_n|^2} = ||a_n||\sqrt{2}$ and $|c_n| \leqq \sqrt{|b_n|^2 + |c_n|^2} = ||a_n||\sqrt{2}$, and conversely

$$||a_n|| \leqq \sqrt{|b_n|^2 + |c_n|^2} \leqq |b_n| + |c_n|.$$

A power series is absolutely convergent in its region of convergence.

18. DEFINITION. Let $\sum_{n=0}^{\infty} a_n$ be a series of bicomplex terms. The series will be said to be *quasi-absolutely convergent* if $\sum_{n=0}^{\infty} |a_n|$ converges.

If for some integer $N$ all $a_n$ for $n>N$ are nil-factors, then the series $\sum_{n=0}^{\infty} a_n$ will be quasi-absolutely convergent even though the series itself may diverge. However, if for some $N$ there exists $k \geqq 0$ such that $||a_n|| \leqq k|a_n|$ for $n > N$, then the quasi-absolute convergence will imply absolute convergence. This condition will be shown to hold if the $a_n$ are in a plane through the origin (for definition of a plane see Section IV, part 36) which does not have any other point in common with a nil-plane. Of course if the plane intersects a nil-plane in a line, then the $a_n$ may lie along the line and quasi-absolute convergence will not imply convergence.

The proof will use the following:

LEMMA. *Let $a, b, c, d$ be complex numbers and let $cx+dy$ be different from zero for all real $(x,y) \neq (0,0)$. Then $F(x,y) = (ax+by)/(cx+dy)$ is bounded for all real $(x,y) \neq (0,0)$.*

PROOF. If $x=0$, $F=b/d$. For $x \neq 0$, let $y/x=\lambda$. Then $F=(a+b\lambda)/(c+d\lambda)$, where $\lambda$ is real. Consider $\lambda$ as a complex variable. Then F is a linear fractional transformation which takes the real axis into a straight line or a circle. Since $c+d\lambda \neq 0$ for real $\lambda$, the real axis transforms into a circle. Thus $F$ is bounded for $(x,y) \neq (0,0)$.

Now let the points $a_n$ lie in a plane through the origin which has no other point in common with a nil-plane. Then each $a_n$ may be expressed in the form $m\alpha + n\beta$, where $m$ and $n$ are real, $\alpha$ and $\beta$ are fixed bicomplex numbers which are not zero or a nil-factor, and $m\alpha + n\beta$ is not zero or a nil-factor except for $m = n = 0$. Let $\alpha = \alpha_1 e_1 + \alpha_3 e_3$ and $\beta = \beta_1 e_1 + \beta_3 e_3$. Then

$$||m\alpha + n\beta|| = \frac{1}{\sqrt{2}} \sqrt{|m\alpha_1 + n\beta_1|^2 + |m\alpha_3 + n\beta_3|^2}$$

and

$$|m\alpha + n\beta| = \sqrt{|m\alpha_1 + n\beta_1| \cdot |m\alpha_3 + n\beta_3|}.$$

Then

$$\frac{||m\alpha + n\beta||}{|m\alpha + n\beta|} = \frac{\frac{1}{\sqrt{2}}\sqrt{|m\alpha_1 + n\beta_1|^2 + |m\alpha_3 + n\beta_3|^2}}{\sqrt{|m\alpha_1 + n\beta_1| \cdot |m\alpha_3 + n\beta_3|}}.$$

This will be bounded if and only if

$$\frac{|m\alpha_1 + n\beta_1|}{|m\alpha_3 + n\beta_3|} + \frac{|m\alpha_3 + n\beta_3|}{|m\alpha_1 + n\beta_1|}$$

is bounded. Since $m\alpha+n\beta$ is never a nil-factor or zero except when $m=n=0$, each term is bounded by the lemma for either $m$ or $n$ different from zero. Let the bound be $k$. If $m = n = 0$, then $\|m\alpha + n\beta\| = |m\alpha + n\beta| = 0$. Thus in all cases $\|m\alpha + n\beta\| > k|m\alpha + n\beta|$ and quasi-absolute convergence implies absolute convergence, which in turn implies convergence.

**19.** DEFINITION. A series of the form $\sum_{n=-\infty}^{\infty} a_n z^n$ will be termed a *Laurent series.*

Under the previous notation, $\sum_{n=-\infty}^{\infty} a_n z^n = \left[\sum_{n=-\infty}^{\infty} b_n z_1^n\right] e_1 + \left[\sum_{n=-\infty}^{\infty} c_n z_3^n\right] e_3$ will be convergent if and only if $\sum_{n=-\infty}^{\infty} b_n z_1^n$ and $\sum_{n=-\infty}^{\infty} c_n z_3^n$ are convergent. If the series represents the extension of a complex-valued function of a complex variable to the bicomplex space, then $b_n$ will equal $c_n$ for all $n$, and the series in brackets will converge for $r < |z_{1,3}| < R$. Then the region of convergence of the series may be represented by $N(1/z) < 1/r$ and $N(z) < R$, since $N(1/z) = \max[1/|z_1|, 1/|z_3|] = 1/\min[|z_1|, |z_3|] < 1/r$ if and only if $\min[|z_1|, |z_3|] > r$. This, of course, bounds $z$ away from the nil-planes.

## III. SINGULARITIES AND ZEROS

**20.** DEFINITION. A point $z_0$ will be called a *singularity of a function* $f(z)$ if $z_0$ is a boundary point of a region $T$ in which $f(z)$ is analytic and if there does not exist a region $T'$ including $T$ and containing $z_0$ and a function $g(z)$ analytic in $T'$ and coinciding with $f(z)$ in $T$. The point $z_0$ will be called a *removable singularity* if the described $T'$ and $g(z)$ do exist.

Note that a removable singularity is not a singularity.

The decomposition theorem of Ringleb leads at once to the result that $f(z) = g(z_1)e_1 + h(z_3)e_3$ can have a singularity at $z = z_0 = z_1^0 e_1 + z_3^0 e_3$ if and only if $g(z_1)$ has a singularity at $z_1 = z_1^0$ or $h(z_3)$ has a singularity at $z_3 = z_3^0$. But then it follows that $f(z)$ has a singularity at every point of the intersection of the closure of the region of analyticity of $f(z)$ and one of the nil-planes with respect to $z_0$, i.e., the set of points which are of the form $z_0+\alpha$ for $\alpha$ in a nil-plane.

*Thus there are no isolated singularities.*

Since a functon $f(z)$ with a singularity at the origin can have no point of analyticity in one of the nil-planes, $f(z)$ will be said to be singular in that nil-plane, even though all points of the nil-plane may not be boundary points of the region $T$ in which $f(z)$ is analytic,

**21.** Definition. A hypersphere $||z-z_0||<k$, for some $k>0$, will be referred to as a *neighborhood* $P'$ of the point $z_0$.

Definition. A neighborhood $P'$ of a point minus the nil-planes with respect to the point will be called a *deleted neighborhood* $P$ of the point. The neighborhood $P'$ will be called the *associated neighborhood of* $P$.

Definition. Let $f(z)$ be analytic in a deleted neighborhood of $z=z_0\equiv z_1{}^0e_1+z_3{}^0e_3$. Then $f(z)$ will be said to have *a pole of order at most* $n$, where $n$ is a non-negative integer, in the nil-planes with respect to $z_0$ if both $g(z_1)$ has a pole of order at most $n$ at $z_1=z_1{}^0$ and $h(z_3)$ has a pole of order at most $n$ at $z_3=z_3{}^0$. If both $g(z_1)$ and $h(z_3)$ have a pole of order $n$ at these points, then $f(z)$ will be said to have a *pole of order* $n$.

Here a point of analyticity in the complex planes, and hence also in the bicomplex space, has been referred to as a pole of order zero.

**22.** In the theory of functions of a complex variable, the inequalities $|f(z)|<M$, $|f(z)|\cdot|z|^n<M$, and $|f(z)|\leqq M|z|^n$ holding in a neighborhood of the origin were shown to imply that $f(z)$ has at the origin a removable singularity, a pole, and a zero, respectively.

These results motivate using the norm and the absolute value in assuming similar inequalities in the bicomplex space to hold in a deleted neighborhood of the origin and investigating the effect on the function. Because $||z||\leqq N(z)\leqq ||z||\sqrt{2}$, the norm $N(z)$ may be substituted in any of these inequalities for $||z||$ and the conclusion will be unchanged. This of course includes substituting $N[f(z)]$ for $||f(z)||$.

**23.** The first step might be to assume that $f(z)$ is analytic in a deleted neighborhood $P$ of the origin and that $||f(z)||<M$ in $P$, then prove that $g(z_1)$ and $h(z_3)$ have removable singularities at their respective origins, and that therefore $f(z)$ has a removable singularity at all points of the intersection of the nil-planes with the associated neighborhood $P$. This can be done. However it is included in the following perhaps unexpected more general theorem, since for $||z||<1$, then $||f(z)||\cdot||z||^n<||f(z)||<M$.

Theorem. *Let $f(z)$ be analytic in a deleted neighborhood $P$ of the origin and let*

$$||f(z)||\cdot||z||^n<M$$

*in $P$ for a positive integer $n$, where $M$ is a positive constant. Then $f(z)$ has a removable singularity at all points of the intersection of the nil-planes with the associated neighborhood $P'$.*

This theorem may be proved directly, but will be proved as a corollary of the next theorem

**24.** Since the boundedness of $\|f(z)\| \cdot \|z\|^n$ in a deleted neighborhood of the origin is too strong a restriction on $f(z)$ to permit singularities, the next step might be to weaken this assumption by replacing one of the norms or both by the absolute value, first checking the restriction on $f(z)$ of the boundedness of $|f(z)|$ in a deleted neighborhood of the origin. Again this is included in a more general

THEOREM. *Let $f(z)$ be analytic in a deleted neighborhood $P$ of the origin and let*

$$|f(z)| \cdot \|z\|^n < M$$

*in $P$, where $n$ is a positive integer and $M$ is a positive constant. Then there are two cases:*

*Case* 1. *$|f(z)| \equiv 0$ in $P$, but $f(z)$ may be singular in a nil-plane.*

*Case* 2. *$f(z)$ has a removable singularity at all points of the intersection of the nil-planes with the associated neighborhood $P'$.*

PROOF. If $|f(z)| \not\equiv 0$ in $P$, then there exists a point $z = ae_1 + be_3$ in $P$ such that $a \neq 0$, $b \neq 0$, $h(b) = c \neq 0$. Then for $z_3 = b$ and $0 < |z_1| < |a|$,

$$|f(z)| \cdot \|z\|^n = \sqrt{|g(z_1)| \cdot |c|}\left[\frac{1}{\sqrt{2}}\sqrt{|z_1|^2 + |b|^2}\right]^n < M$$

or

$$|g(z_1)| < \frac{M^2 2^n}{|c|\,[\,|z_1|^2 + |b|^2]^n} \leqq \frac{M^2 2^n}{|c| \cdot |b|^{2n}}$$

for $0 < |z_1| < |a|$. Therefore $g(z_1)$ has a removable singularity at $z_1 = 0$. Similarly $h(z_3)$ has a removable singularity at $z_3 = 0$. Therefore $f(z)$ has a removable singularity at all points of the intersection of the nil-planes with the associated neighborhood $P'$.

Proof of the preceding theorem: Since $|f(z)| \cdot \|z\|^n \leqq \|f(z)\| \cdot \|z\|^n < M$, either the conclusion is already proved or else $|f(z)| \equiv 0$ in $P$. But then $f(z) \equiv g(z_1)e_1$ or $f(z) \equiv h(z_3)e_3$, say $g(z_1)e_1$. Then for $z_3 = b \neq 0$,

$$|g(z_1)| \cdot \|b\|^n = \sqrt{2}\,|f(z)| \cdot \|b\|^n < M\sqrt{2}$$

or

$$|g(z_1)| < M\sqrt{2}/\|b\|^n.$$

Therefore $g(z_1)$ has a removable singularity at the origin and the conclusion is proved in any case.

**25.** THEOREM. *Let $f(z)$ be analytic in a deleted neighborhood $P$ of the origin and let*

$$\|f(z)\| \cdot |z|^n < M$$

*in $P$, where $n$ is a positive integer and $M$ is a positive constant. Then $f(z)$ has a pole of order at most $[n/2]$ in the nil-planes, where $[n/2] = n/2$ if $n$ is even and $[n/2] = (n-1)/2$ if $n$ is odd. Further, if $n$ is odd,*

$$\|f(z)\| \cdot |z|^{n-1}$$

*is also bounded in some neighborhood of the origin.*

PROOF.

$$\|f(z)\|\cdot|z|^n = (1/\sqrt{2})\sqrt{|g(z_1)|^2 + |h(z_3)|^2}\sqrt{|z_1|^n|z_3|^n} < M$$

or

$$[|g(z_1)|^2 + |h(z_3)|^2]\cdot|z_1|^n\cdot|z_3|^n < 2M^2$$

for $z$ in $P$. Let $z = ae_1 + be_3$ be a point in $P$. Then $a \neq 0$ and $b \neq 0$. Let $z_3 = b$ and $0 < |z_1| < |a|/2$, then $z = z_1e_1 + z_3e_3$ is in $P$. Then

$$|[g(z_1)]^2|\cdot|z_1|^n = |g(z_1)|^2|z_1|^n \leqq [|g(z_1)|^2 + |h(b)|^2]|z_1|^n < 2M^2/|b|^n$$

for $0 < |z_1| < |a|/2$. Thus $[g(z_1)]^2$ has a pole of order at most $n$ at $z_1 = 0$. Therefore $g(z_1)$ has a pole of order at most $[n/2]$ at $z_1 = 0$. Similarly $h(z_3)$ has a pole of order at most $[n/2]$ at $z_3 = 0$. Thus $f(z)$ has a pole of order at most $[n/2]$ in the nil-planes.

Further if $n = 2r + 1$, where $r$ is an integer, then $[n/2] = r$ and $|g(z)|\cdot|z_1|^r < N$ and $|h(z_3)|\cdot|z_3|^r < R$ in some neighborhood of their respective origins, where $N$ and $R$ are positive constants. Let $|z_1| < 1$ and $|z_3| < 1$. Then

$$\begin{aligned}\|f(z)\|\cdot|z|^{n-1} &= \|f(z)\|\cdot|z|^{2r} = (1/\sqrt{2})\sqrt{|g(z_1)|^2 + |h(z_3)|^2}[\sqrt{|z|\cdot|z_3|}]^{2r} \\ &= (1/\sqrt{2})\sqrt{|g(z_1)|^2|z_1z_2|^{2r} + |h(z_3)|^2|z_1z_3|^{2r}} \\ &< (1/\sqrt{2})\sqrt{|g(z_1)|^2|z_1|^{2r} + |h(z_3)|^2|z_3|^{2r}} < (1/\sqrt{2})\sqrt{N^2+R^2}.\end{aligned}$$

This last part of the proof also serves to show that the conclusion of the theorem is the best possible.

26. THEOREM. *Let $f(z)$ be analytic in a deleted neighborhood $P$ of the origin and let*

$$|f(z)|\cdot|z|^n < M$$

*in $P$, where $n$ is a positive integer and $M$ is a positive constant. Then there are two cases.*

*Case* 1. *$|f(z)| \equiv 0$ in $P$, but $f(z)$ may be singular in one nil-plane.*

*Case* 2. *$f(z)$ has a pole of order at most $n$ in the nil-planes in the associated neighborhood $P'$.*

PROOF. If $|f(z)| \equiv 0$, then $|f(z)|\cdot|z|^n \equiv 0 < M$.

If $|f(z)| \not\equiv 0$, then $g(z_1) \not\equiv 0$, and $h(z_3) \not\equiv 0$. Then there exists $z = ae_1 + be_3$ in $P$ such that $a \neq 0$, $b \neq 0$, and $h(b) = c \neq 0$. Let $z_3 = b$ and $0 < |z_1| < |a|/2$, then $z = z_1e_1 + z_3e_3$ is in $P$. Now

$$|f(z)|\cdot|z|^n = \sqrt{|g(z_1)|\cdot|h(z_3)|\cdot|z_1|^n\cdot|z_3|^n} < M$$

for $z$ in $P$, or

$$|g(z_1)|\cdot|z_1|^n < M^2/(|c|\cdot|b|^n),$$

$0 < |z_1| < |a|/2$. Thus $g(z_1)$ has a pole of order at most $n$ at $z_1 = 0$. Similarly $h(z_3)$ has a pole of order at most $n$ at $z_3 = 0$. Therefore $f(z)$ has a pole of order at most $n$ in the nil-planes.

REMARK. If $f(z) = 1/z^n$, then $|f(z)|\cdot|z|^n = 1$ for all $z$ in the deleted neighborhood of the origin. Thus the conclusion of the theorem is the best possible,

**27.** THEOREM. *Let $f(z)$ be analytic in a deleted neighborhood $P$ of the origin. Then $f(z)$ has a pole of order $n$ in the nil-planes if and only if $|f(z)z^n|$ approaches a limit $K \neq 0$ as $z$ approaches the origin through points of $P$.*

PROOF. In $P$, $|f(z)z^n| = \sqrt{|g(z_1)| \cdot |z_1|^n \cdot |h(z_3)| \cdot |z_3|^n}$. As $z$ approaches zero, $z_1$ and $z_3$ approach zero independently. Thus $|f(z)z^n|$ approaches a finite non-zero limit if and only if $|g(z_1)| \cdot |z_1|^n$ and $|h(z_3)| \cdot |z_3|^n$ approach finite non-zero limits. This means $g(z_1)$ has a pole of order $n$ at $z_1 = 0$ and $h(z_3)$ has a pole of order $n$ at $z_3 = 0$, or $f(z)$ has a pole of order $n$ in the nil-planes.

**27.** DEFINITION. Let $f(z)$ be analytic in a neighborhood of the origin. Then $f(z)$ will be said to have a *zero of order at least n*, where $n$ is a positive integer, at the origin if and only if both $g(z_1)$ has a zero of order at least $n$ at $z_1 = 0$ and $h(z_3)$ has a zero of order at least $n$ at $z_3 = 0$.

THEOREM. *Let $f(z)$ be analytic in a deleted neighborhood $P$ of the origin and let*

$$\|f(z)\| \leqq M\|z\|^n$$

*in $P$, where $n$ is a positive integer and $M$ is a positive constant. Then $f(z)$ has a zero of order at least $n$ at the origin.*

PROOF. Since for $\|z\| < 1$, $\|f(z)\| \leqq M\|z\|^n < M$, $f(z)$ has a removable singularity in the nil-planes. As $z$ approaches zero, $f(z)$ approaches zero. Therefore redefine $f(0) = g(0) = h(0) = 0$, and $f(z)$ is then analytic in the associated neighborhood $P'$. Let $z_3 = 0$ and $z = z_1 e_1$ be in $P$. Then $f(z) = g(z_1)e_1$, and $\|z\| = (1/\sqrt{2})|z_1|$. Therefore $\|f(z)\| = (1/\sqrt{2})|g(z_1)| \leqq M\|z\|^n = 2^{-n/2} M \cdot |z_1|^n$ or $|g(z_1)| \leqq 2^{-(n-1)/2} M |z_1|^n$. Thus $g(z_1)$ has a zero of order at least $n$ at $z_1 = 0$. Similarly $h(z_3)$ has a zero of order at least $n$ at $z_3 = 0$. Thus $f(z)$ has a zero of order at least $n$ at $z = 0$.

The conclusion is best possible, for if $f(z) = z^n$, then $\|f(z)\| = (1/\sqrt{2}) \cdot \sqrt{|z_1|^{2n} + |z_3|^{2n}} \leqq M\|z\|^n = M[(1/\sqrt{2})\sqrt{|z_1|^2 + |z_3|^2}]^n$ for $M = (\sqrt{2})^{n-1}$, since $|z_1|^{2n} + |z_3|^{2n} \leqq [|z_1|^2 + |z_3|^2]^n$.

**28.** THEOREM. *Let $f(z)$ be analytic in a neighborhood $P$ of the origin and let*

$$\|f(z)\| \leqq M|z|^n$$

*in $P$, where $n$ is a positive integer and $M$ is a positive constant. Then $f(z) \equiv 0$ for $z$ in $P$.*

PROOF. Let $z_3 = 0$ and $z_1 e_1$ be in $P$, then $|z| = 0$ and thus $\|f(z)\| = 0$. Therefore $h(0) = 0$ and $g(z_1) \equiv 0$ for $z_1 e_1$ in $P$. Similarly $h(z_3) \equiv 0$ for $z_3 e_3$ in $P$. Thus $f(z) \equiv 0$ for $z$ in $P$.

**29.** THEOREM. *Let $f(z)$ be analytic in a neighborhood $P$ of the origin and let*

$$|f(z)| \leqq M|z|^n$$

*in P, where n is a positive integer and M is a positive constant. Then there are two cases:*

*Case* 1. $|f(z)| \equiv 0$

*Case* 2. *$f(z)$ has a zero of order at least n at the origin.*

PROOF. If $|f(z)| \equiv 0$, then $|f(z)| \leqq M|z|^n$.

If $|f(z)| \not\equiv 0$, then $g(z_1) \not\equiv 0$ and $h(z_3) \not\equiv 0$. Then for some $z = ae_1 + ce_3$ in $P$, $a \neq 0$, $c \neq 0$, $g(a) = b \neq 0$ and $h(c) = d \neq 0$. Since for $z$ in $P$

$$|f(z)| = \sqrt{|g(z_1)| \cdot |h(z_3)|} \leqq M\sqrt{|z_1|^n \cdot |z_3|^n}$$

or

$$|g(z_1)| \cdot |h(z_3)| \leqq M^2 |z_1|^n \cdot |z_3|^n.$$

Now let $z_3 = c$ and $|z_1| < |a|/2$, then $z = z_1e_1 + z_3e_3$ is in $P$ and $|g(z_1)| \cdot |d| \leqq M^2 |z_1|^n \cdot |c|^n$ or $|g(z_1)| \leqq [M^2 |c|^n / |d|] |z_1|^n$. Thus $g(z_1)$ has a zero of order at least $n$ at $z_1 = 0$. Similarly $h(z_3)$ has a zero of order at least $n$ at $z_3 = 0$. Then the conclusion follows.

The conclusion is best possible, for if $f(z) = z^n$, then

$$|f(z)| = \sqrt{|z_1|^n \cdot |z_3|^n} = [\sqrt{|z_1| \cdot |z_3|}\,]^n = |z|^n.$$

**30.** THEOREM. *Let $f(z)$ be analytic in a neighborhood P of the origin and let*

$$|f(z)| \leqq M\|z\|^n$$

*in P, where n is a positive integer and M is a positive constant. Then* $|f(0)| = 0$ *and the sum of the orders of the zeros of the component functions at their respective origins is at least* $2n$.

PROOF. Suppose first that $g(0) = b \neq 0$. Let $z_1 = 0$ and $z = z_3e_3$ be in $P$. Then

$$|f(z)| = \sqrt{|g(z_1)| \cdot |h(z_3)|} \leqq M[(1/\sqrt{2})\sqrt{|z_1|^2 + |z_3|^2}\,]^n$$

or

$$|h(z_3)| \leqq \frac{M^2}{2^n |b|} |z_3|^{2n}.$$

Thus $h(z_3)$ has a zero of order at least $2n$ at $z_3 = 0$.

Now suppose $g(z_1)$ has a zero of order $m < 2n$. Then $\lim_{z_1 \to 0} |g(z_1)/z_1^m| = b > 0$. Now let $z_1 = z_3$ for $z = z_1e_3 + z_3e_3$ in $P$. Then $\lim_{z_3 \to 0} |g(z_3)/z_3^m| = b > 0$, and there exists a $\delta > 0$ such that for $|z_3| < \delta$, $|g(z_3)/z_3^m| \geqq b/2$. Since

$$\sqrt{|g(z_3)h(z_3)|} \leqq M[(1/\sqrt{2})\sqrt{|z_3|^2 + |z_3|^2}\,]^n = M|z_3|^n,$$

then $$|h(z_3)| \leqq \{M^2/[\,|g(z_3)/z_3^m|\,]\}\,|z_3|^{2n-m} \leqq (2M^2/b)\,|z_3|^{2n-m}$$

for $|z_3| < \delta$. Thus $h(z_3)$ has a zero of order at least $2n - m$ at $z_1 = 0$. In any case the sum of the orders is at least $2n$.

The conclusion is best possible, for if $f(z) = z_1^m e_1 + z_3^{2n-m} e_3$, then $|f(z)| = \sqrt{|z_1|^m |z_3|^{2n-m}} \leqq \sqrt{[\,\|z\|\sqrt{2}\,]^m [\,\|z\|\sqrt{2}\,]^{2n-m}} = [\sqrt{2}\,\|z\|\,]^n$, since $|z_1| \leqq \|z\|\sqrt{2}$ and $|z_3| \leqq \|z\|\sqrt{2}$.

**31.** REMARK. A function $f(z)$ which has a zero at the origin has the further interesting property that for $z$ in the intersection of the first nil-plane and the region of analyticity of $f(z)$, the function $f(z)$ has a value which is a first nil-factor, and correspondingly for the second nil-plane. For if $f(0)=0$, then $g(0)=h(0)=0$ and for $z$ in the first nil-plane $z_3=0$, so that $f(z)=g(z_1)e_1$.

## IV. INTEGRATION

**32.** Let $C$ be a rectifiable curve connecting the two distinct points $a$ and $b$ of the bicomplex space. Let $C_1$ be the set of values which $z_1$ takes for all $z$ on $C$, and let $C_3$ be the set of values which $z_3$ takes for $z$ on $C$. $C_1$ will be called the projection of $C$ on the $z_1$-plane. If $C$ is contained in a region $T$, then $C_1$ is contained in the component region $T_1$ and $C_3$ in the component region $T_3$.

A straight line in the bicomplex space is defined as the set of points $z=k\alpha+(1-k)\beta$, where $k$ is real and $\alpha$ and $\beta$ are two distinct points. Decomposing the defining formula into its components, one verifies that a straight line projects into a straight line or a point. Therefore if $C$ is a rectifiable curve, $C_1$ and $C_3$ are also rectifiable curves.

Let further $z_i$, $i=0, 1, 2, \cdots, n$ be $n+1$ distinct points on the curve $C$, where $z_0=a$, $z_n=b$ and $z_i$ is situated on $C$ between $z_{i-1}$ and $z_{i+1}$ as $C$ is traced from $a$ to $b$; finally, let $f(z)$ be a function analytic at all points of $C$, including its end points $a$ and $b$.

DEFINITION. Consider the expression

$$S_n = \sum_{i=1}^{n} f(\xi_i)(z_i-z_{i-1})$$

where $\xi_i$ is an arbitrary point on the section of $C$ which connects $z_{i-1}$ and $z_i$ and denote by $\Delta_n$ the $\max\limits_{i=1,\cdots,n} \|z_i-z_{i-1}\|$. Let the number of points $z_i$ on $C$ tend to infinity in such a way that $\Delta_n$ tends to zero. Then $\lim\limits_{n\to\infty} S_n$, (shown below to exist and be the same for all sequences of subdivisions) will be called the *integral of $f(z)$ along $C$ from $a$ to $b$* and denoted by $\int_C f(z)dz$.

Let $\xi_i=\xi_1{}^i e_1+\xi_3{}^i e_3$; $z_i=z_1{}^i e+z_3{}^i e_3$; $\Delta_1{}^n=\max\limits_{i=1,\cdots,n} |z_1{}^i-z_1{}^{i-1}|$; $\Delta_3{}^n=\max\limits_{i=1,\cdots,n} |z_3{}^i-z_3{}^{i-1}|$; $S_n=S_1{}^n e_1+S_3{}^n e_3$. Then

$$S_n=\left[\sum_{i=1}^{n} g(\xi_1^i)(z_1^i-z_1^{i-1})\right]e_1+\left[\sum_{i=1}^{n} h(\xi_3^i)(z_3^i-z_3^{i-1})\right]e_3$$

and

$$\overset{n}{S_1}=\sum_{i=1}^{n} g(\xi_1^i)(z_1^i-z_1^{i-1}),\qquad \overset{n}{S_3}=\sum_{i=1}^{n} h(\xi_3^i)(z_3^i-z_3^{i-1}).$$

$\varDelta_n$ tends to zero if and only if $\varDelta_1^n$ and $\varDelta_3^n$ tend to zero. From the theory of functions of a complex variable,

$$\lim_{n\to\infty} \overset{n}{\underset{1}{S}} = \int_{C_1} g(z_1)dz_1; \quad \lim_{n\to\infty} \overset{n}{\underset{3}{S}} = \int_{C_3} h(z_3)dz_3.$$

Since these limits exist, $\lim\limits_{n\to\infty} S_n$ exists and

$$\int_C f(z)dz = \left[ \int_{C_1} g(z_1)dz_1 \right] e_1 + \left[ \int_{C_3} h(z_3)dz_3 \right] d_3.$$

A number of elementary properties of the integral may be verified from the definition. Among them that if $C$ is divided at a point of $C$ into two curves $C'$ and $C''$, then $\int_C f(z)dz = \int_{C'} f(z)dz + \int_{C''} f(z)dz$. Then the integral over a closed rectifiable curve in a given direction may be defined by taking two distinct points on $C$ and combining the integrals over the separate parts of $C$ in the given direction. If the closed curve is traced in the opposite direction, the value of the integral will of course be the negative of the previous.

**33.** CAUCHY'S INTEGRAL THEOREM. *Let $C$ be a closed rectifiable curve in a simply connected region $T$, and let $f(z)$ be analytic in $T$. Then*

$$\int_C f(z)dz = 0.$$

PROOF. Let $C_1$ and $C_3$ be the projections of $C$ on the $z_1$-and $z_3$-planes, respectively. Since $T$ is simply connected the component regions $T_1$ and $T_3$ are simply connected. Now

$$\int_C f(z)dz = \left[ \int_{C_1} g(z_1)\, dz_1 \right] e_1 + \left[ \int_{C_3} h(z_3)dz_3 \right] e_3.$$

The quantities in brackets are zero. (See Bieberbach [11], page 108, for an argument showing that Cauchy's theorem holds for rectifiable curves which, as $C_1$ and $C_3$, may intersect themselves.)

**34.** DEFINITION. Let $C$ be a rectifiable curve in the bicomplex space whose projections $C_1$ and $C_3$ in the $z_1$- and $z_3$-planes, respectively, are simple closed curves, and such that $C_1$ and $C_3$ are traced once as $C$ is traced once. A curve $C$ in this class will be called a *P-curve.*

Since $C_1$ and $C_3$ are simple closed curves, a $P$-curve is a simple closed curve.

DEFINITION.[3] Let $C$ be a $P$-curve and let $z = z_1e_1 + z_3e_3$ be a point such that $z_1$ is in the interior of $C_1$ and $z_3$ is in the interior of $C_3$. Define the *interior I* of the $P$-curve $C$ as the totality of all points $z$ satisfying this condition.

3) This definition has been given by Prof. Price.

DEFINITION. Let $C$ be a $P$-curve. As $C_3$ is traced in the positive direction in the $z_3$-plane, which is an ordinary complex plane, $C$ will be traced in a certain direction. Designate this direction as the *principal direction on C.*

DEFINITION. Let $C$ be a $P$-curve. As $C$ is traced in the principal direction, $C_1$ will be traced in a certain direction in the $z_1$-plane. If this direction is positive, designate $C$ as a *P-plus curve.* If it is negative, designate $C$ as a *P-minus curve.*

**35.** The proof of the next theorem requires the following

LEMMA. $ijz=z_1e_1-z_3e_3$.

PROOF. $ijz=ij(x+jy) = -iy+ixj=[-iy-i(ix)]e_1+[-iy+i(ix)]e_3$
$=(x-iy)e_1-(x+iy)e_3=z_1e_1-z_3e_3$.

Then, of course, $ijf(z)=g(z_1)e_1-h(z_3)e_3$.

THEOREM. (*Cauchy's Integral Formula*). *Let C be a P-curve and let $f(z)$ be analytic in its interior I and continuous on the closure of I. Let z be any point in the interior I of C. Then*

*Case* 1. *If C is a P-plus curve,*

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)dw}{w-z},$$

*where C is traced in the principal direction.*

*Case* 2. *If C is a P-minus curve*

$$f(z) = \frac{1}{2\pi j}\int_C \frac{f(w)dw}{w-z},$$

*where C is traced in the principal direction.*

PROOF. $f(z)=g(z_1)e_1+h(z_3)e_3$ in $I$. By Cauchy's integral formula in the complex case

$$g(z_1) = \frac{1}{2\pi i}\int_{C_1^+}\frac{g(w_1)dw_1}{w_1-z_1}; \qquad h(z_3) = \frac{1}{2\pi i}\int_{C_3^+}\frac{h(w_3)dw_3}{w_3-z_3}.$$

Case 1.

$$\frac{1}{2\pi i}\int_C \frac{f(w)dw}{w-z} = \left[\frac{1}{2\pi i}\int_{C_1^+}\frac{g(w_1)dw_1}{w_1-z_1}\right]e_1 + \left[\frac{1}{2\pi i}\int_{C_3^+}\frac{h(w_3)dw_3}{w_3-z_3}\right]e_3$$
$$=g(z_1)e_1+h(z_3)e_3=f(z).$$

Case 2.

$$\frac{1}{2\pi j}\int_C \frac{f(w)dw}{w-z} = \left[\frac{1}{2\pi j}\int_{C_1^-}\frac{g(w_1)dw_1}{w_1-z_1}\right]e_1 + \left[\frac{1}{2\pi j}\int_{C_3^+}\frac{h(w_3)dw_3}{w_3-z_3}\right]e_3$$
$$= \left[\frac{-1}{2\pi j}\int_{C_1^+}\frac{g(w_1)dw_1}{w_1-z_1}\right]e_1 + \left[\frac{1}{2\pi j}\int_{C_3^+}\frac{h(w_3)dw_3}{w_3-z_3}\right]e_3$$
$$= -\frac{i}{j}\left[\frac{1}{2\pi i}\int_{C_1^+}\frac{g(w_1)dw_1}{w_1-z_1}\right]e_1 + \frac{i}{j}\left[\frac{1}{2\pi i}\int_{C_3^+}\frac{h(w_3)dw_3}{w_3-z_3}\right]e_3$$
$$= -\frac{i}{j}g(z_1)e_1 + \frac{i}{j}h(z_3)e_3 = -\frac{i}{j}[g(z_1)e_1-h(z_3)e_3] = -\frac{i}{j}ijf(z)=f(z).$$

**36.** In order to apply Cauchy's integral formula, a criterion is needed to determine if the curve is a $P$-plus curve or a $P$-minus curve. The remainder of this section will be devoted to establishing such a criterion for plane curves. A plane is defined as the set of points of the form $z=m\alpha + n\beta + (1-m)(1-n)\gamma$ where $m$ and $n$ are real and $\alpha$, $\beta$, and $\gamma$ are three distinct points not on a line. For convenience the point $z$ about which the integration is performed will be translated to the origin. Then the question asked becomes the following:

Let $P$ be plane through the origin, and let $C$ be a simple closed rectifiable curve in $P$. For which planes $P$ will the curve be a $P$-curve, and in which of those will the curve be a $P$-plus curve and in which will the curve be a $P$-minus curve?

It appears at once that a simple closed rectifiable curve in the complex plane is a $P$-plus curve, for there the result in case 1 holds. By analogy it would be expected that the plane determined by the line $z=k$ and the line $z=kj$ ($k$ real) contains $P$-minus curves. This is correct, as can be verified by the promised criterion.

**37.** For the proof of the validity of this criterion, a few elementary results from the geometry of four-dimensional Euclidean space will be needed. They are undoubtedly well-known. For the sake of completeness, these results will be formulated and proved.

For the remainder of this section, advantage will be taken of the isomorphisms $z_1e_1 \longleftrightarrow z_1$ and $z_3e_3 \longleftrightarrow z_3$ to estabish results in the nil-planes by actually performing the computation for the $z_1$- and $z_3$-planes.

The first and second nil-planes will be referrd to as conjugate to each other.

LEMMA 1. *Let $P$ be plane through the origin which intersects one of the nil-planes in a straight line. Then every point of $P$ projects into the same straight line in the conjugate nil-plane.*

PROOF. In the defining formula of a plane, let $\gamma$ be the origin and $\beta$ a point on the line of intersection, thus a nil-factor, say $\beta_1e_1$. Then a point in $P$ is of the form $m\alpha + n\beta_1e_1$. The projection in the $z_3$-plane is of the form $m\alpha_3$. The set of such points lie on a line through the origin of the $z_3$-plane.

LEMMA 2. *Let $P$ be a plane through the origin which has no other point in common with a nil-plane. Let $C$ be a simple closed curve in $P$. Then the projection of $C$ in the conjugate nil-plane is also a simple closed curve.*

PROOF. The projections $C_1$ and $C_3$ of the curve $C$ in the first and second nil-planes, respectively, are clearly closed. Suppose the second nil-plane is the one with which $P$ is assumed to have only the origin in common, and that $C_1$ is not simple. Then for two different points $\alpha$ and $\beta$ on $C$, $\alpha_1=\beta_1$. Therefore

$\alpha_3 \neq \beta_3$. But then $\alpha - \beta$ is in $P$ and is a second nil-factor. This is a contradiction, and the lemma follows.

38. In the remainder of this section there will be a change of notation. A bicomplex point $z$ will sometimes be represented as $x+iy+jz+iju$, where $x, y, z, u$ are real. It will always be clear which interpretation is meant. This will be useful in referring to the geometry of the four-dimensional space.

By separating the defining equation of a plane into its four components in this notation, letting $\gamma$ be the origin, and eliminating $m$ and $n$ in the resulting four equations, it is verified that a plane $P$ through the origin may be represented by two homogeneous independent real linear equations:

$$P: \begin{cases} Ax + By + Cz + Du = 0 \\ ax + by + cz + du = 0. \end{cases}$$

The equation of a line may be represented by three independent real linear equations and a point by four such equations. The intersection of two planes may be a line or a point.

LEMMA 3. *Let $P$ be a plane through the origin, other than a nil-plane. Let $P$ be represented by the system of equations:*

$$P: \begin{cases} Ax + By + Cz + Du = 0 \\ ax + by + cz + du = 0. \end{cases}$$

*Then $P$ intersects the first nil-plane in a line if and only if*

$$\Delta_1(P) = \begin{vmatrix} A+D & B-C \\ a+d & b-c \end{vmatrix} = 0,$$

*$P$ intersects the second nil-plane in a line if and only if*

$$\Delta_2(P) = \begin{vmatrix} A-D & B+C \\ a-d & b+c \end{vmatrix} = 0.$$

PROOF. The equations of the first nil-plane may be taken as

(A) $\quad x - u = 0, \quad y + z = 0.$

The equations of the second nil-plane may be taken as

(B) $\quad x + u = 0, \quad y - z = 0.$

The intersection of $P$ and the first nil-plane is determined by solving the equations (A) with the equations $P$. These equations can have solutions other than $(0,0,0,0)$ if and only if the determinant of the system vanishes. This determinant is

$$\begin{vmatrix} A & B & C & D \\ a & b & c & d \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{vmatrix}$$

and is easily reduced to the determinant $\Delta_1(P)$. Similarly if the system consisting of equations $P$ and (B) are solved simultaneously, the resulting determinant of the system reduces to $\Delta_2(P)$.

LEMMA 4. *Let $P$ be a plane through the origin which does not intersect a nil-plane except at the origin. Let $C$ be a circle in $P$ with center at* 0. *Then the projection of $C$ into the conjugate nil-plane in an ellipse. Here a circle is considered an ellipse, but a straight line segment is not.*

PROOF. Let $P$ be represented by the pair of linear equations:

$$P: \begin{cases} Ax+By+Cz+Du = 0 \\ ax+by+cz+du = 0\,. \end{cases}$$

Then the circle will be the intersection of this plane with the hyper-sphere $x^2+y^2+z^2+u^2=r^2$. The projection of a point $x+iy+jz+iju$ in the $z_1$-plane is $z_1=(x+u)+i(y-z)$ and in the $z_3$-plane the projection is $z_3=(x-u)+i(y+z)$. Then if in the $z_1$- and $z_3$-planes, the usual rectangular cartesian coordinates are denoted here by $X_1$, $Y_1$ and $X_3$, $Y_3$.

$$\text{and}\quad \begin{array}{lll} (\bar{A}) & x+u=X_1, & y-z=Y_1\,, \\ (\bar{B}) & x-u=X_3, & y+z=Y_3\,. \end{array}$$

The system of the four equations $(\bar{A})$ and $(P)$ may be solved for $x,y,z,u$ as linear combinations of $X_1,Y_1$ if and only if the determinant of the system is non-zero. By comparing equations (A) with equations (B) of lemma 3, it is seen that this determinant is $\Delta_2(P)$. Then if $P$ does not intersect the second nil-plane in a line, the linear combinations of $X_1$ and $Y_1$ may be substituted in the equation of the hypersphere to obtain the equation of the projection of $C$ in the $z_1$-plane. This eqution is quadratic. The curve is bounded. By lemma 2 it is not degenerate. Therefore it is an ellipes.

Similarly if $P$ does not intersect the first nil-plane in a line the projection of $C$ in the $z_3$-plane is an ellipse.

**39.** THEOREM. *Let a plane $P$ through the origin be represented by the system of equations*:

$$P: \begin{cases} Ax+By+Cz+Du = 0 \\ ax+by+cz+du = 0\,. \end{cases}$$

*Let*

$$\Delta_1(P) = \begin{vmatrix} A+D & B-C \\ a+d & b-c \end{vmatrix},\quad \Delta_2(P) = \begin{vmatrix} A-D & B+C \\ a-d & b+c \end{vmatrix}.$$

*Let $C$ be a simple closed rectifiable curve in $P$ containing the origin in its interior (in the topology of the plane $P$). $C$ is a $P$-curve if and only if $\Delta_1(P)\neq 0$ and $\Delta_3(P)\neq 0$. Further $C$ is a $P$-plus curve if $\Delta_1(P)$ and $\Delta_2(P)$ are of the same sign, and $C$ is a $P$-minus curve if $\Delta_1(P)$ and $\Delta_2(P)$ are of different signs.*

PROOF. If $P$ is a nil-plane, then one of the determinants is zero, and the curve $C$ is not a $P$-curve. Otherwise, by lemmas 2 and 3, $C$ is a $P$-curve if $\Delta_1(P)\neq 0$ and $\Delta_2(P)\neq 0$. By lemmas 1 and 3, $C$ is not a $P$-curve if either

$\Delta_1(P)$ or $\Delta_2(P)$ is zero.

Now let $\Delta_1(P)$ and $\Delta_2(P)$ be different from zero. Since $C$ is a simple closed curve in $P$ containing the origin in its interior (in the topology of the plane $P$), the curve $C$ contains in its interior a circle $\overline{C}$ in $P$ with center at the origin. The projections of $C$ will contain in their respective interiors the respective projections of $\overline{C}$. The curve $C$ will be a $P$-plus curve if and only if $\overline{C}$ is a $P$-plus curve. Thus without loss of generality it may be assumed that $C$ is a circle.

Because the projections $C_1$ and $C_3$ of the circle $C$ are ellipses by lemma 4, it will be found sufficient to consider only two points on $C$ which are distinct and not at opposite ends of a diameter. From the relative positions of these two points and their respective projections, it is possible to determine the directions in which $C_1$ and $C_3$ are traced when $C$ is traced in its principal direction. This will then determine if $C$ is a $P$-plus or a $P$-minus curve.

Let $\alpha: (a_1, b_1, c_1, d_1)$ and $\beta: (a_2, b_2, c_2, d_2)$ be two distinct points on the circle $\overline{C}$ not at opposite ends of a diameter. Then

$$\alpha = a_1 + b_1 i + c_1 j + d_1 ij = [(a_1+d_1)+(b_1-c_1)i]e_1 + [(a_1-d_1)+(b_1+c_1)i]e_3$$

$$\beta = a_2 + b_2 i + c_2 j + d_2 ij = [(a_2+d_2)+(b_2-c_2)i]e_1 + [(a_2-d_2)+(b_2+c_2)i]e_3 .$$

Since $\Delta_1(P)$ and $\Delta_2(P)$ are not zero, $\alpha$ and $\beta$ cannot be nil-factors. Therefore let

$$(a_1+d_1)+(b_1-c_1)i = r_1 e^{i\theta_1}; \ (a_1-d_1)+(b_1+d_1)i = r_3 e^{i\theta_3};$$

$$(a_2+d_2)+(b_2-c_2)i = R_1 e^{i\phi_1}; \ (a_2-d_2)+(b_2+c_2)i = R_3 e^{i\phi_3};$$

then $\alpha_1 = r_1 e^{i\theta_1}$, $\alpha_3 = r_3 e^{i\theta_3}$, $\beta_1 = R_1 e^{i\phi_1}$, $\beta_3 = R_3 e^{i\phi_3}$. Now the direction from $\alpha_1$ to $\beta_1$ on $C_1$, not passing through $-\alpha_1$ or $-\beta_1$, will be positive if

$$\pi < \theta_1 - \phi_1 < 2\pi \quad \text{or} \quad 0 < \phi_1 - \theta_1 < \pi$$

and negative if

$$0 < \theta_1 - \phi_1 < \pi \quad \text{or} \quad \pi < \phi_1 - \theta_1 < 2\pi .$$

(Because of the symmetry of the projections and the assumption that $\alpha$ and $\beta$ are distinct and not at opposite ends of a diameter, $\phi_1 - \theta_1$ cannot be an integral multiple of $\pi$.) Thus the direction will possess the same sign as $\sin(\phi_1 - \theta_1)$. Now

$$\phi_1 - \theta_1 = \arg\frac{\beta_1}{\alpha_1} = \arg\frac{(a_2+d_2)+(b_2-c_2)i}{(a_1+d_1)+(b_1-c_1)i}$$

$$= \arg\frac{[(a_1+d_1)(a_2+d_2)+(b_1-c_1)(b_2-c_2)]+[(a_1+d_1)(b_2-c_2)-(a_2+d_2)(b_1-c_1)]i}{(a_1+d_1)^2+(b_1-c_1)^2} .$$

Thus

$$\operatorname{sign}[\sin(\phi_1-\theta_1)] = \operatorname{sign}[(a_1+d_1)(b_2-c_2)-(a_2+d_2)(b_1-c_1)] = \operatorname{sign}\left[\begin{vmatrix} a_1+d_1 & b_1-c_1 \\ a_2+d_2 & b_2-c_2 \end{vmatrix}\right].$$

Similarly

$$\operatorname{sign}\begin{bmatrix}\text{corresponding} \\ \text{direction on} \\ C_3\end{bmatrix} = \operatorname{sign}[\sin(\phi_3-\theta_3)] = \operatorname{sign}\left[\begin{vmatrix} a_1-d_1 & a_2-d_2 \\ b_1+c_1 & b_2+c_2 \end{vmatrix}\right].$$

Now by multiplication of determinants, using the fact that the points satisfy the equations of $P$

$$\begin{vmatrix} A & B & C & D \\ a & b & c & d \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} a_1 & a_2 & 1 & 0 \\ b_1 & b_2 & 0 & 1 \\ c_1 & c_2 & 0 & 1 \\ d_1 & d_2 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & A-D & B+C \\ 0 & 0 & a-d & b+c \\ a_1-d_1 & a_2-d_2 & 1 & 0 \\ b_1+c_1 & b_2+c_2 & 0 & 1 \end{vmatrix}$$

By reducing the determinants on the left to the second order and making the obvious Laplace expansion by second order minors on the right

$$\begin{vmatrix} A+D & B-C \\ a+d & b-c \end{vmatrix} \cdot \begin{vmatrix} a_1+d_1 & b_1-c_1 \\ a_2+d_2 & b_2-c_2 \end{vmatrix} = \begin{vmatrix} A-D & B+C \\ a-d & b+c \end{vmatrix} \cdot \begin{vmatrix} a_1-d_1 & a_2-d_2 \\ b_1+c_1 & b_2+c_2 \end{vmatrix}$$

or

$$\Delta_1(P) \times \begin{vmatrix} a_1+d_1 & b_1-c_1 \\ a_2+d_2 & b_2-c_2 \end{vmatrix} = \Delta_2(P) \times \begin{vmatrix} a_1-d_1 & a_2-d_2 \\ b_1+c_1 & b_2+c_2 \end{vmatrix}.$$

Thus

$$\text{sign } \Delta_1(P) \times \text{sign} \begin{bmatrix} \text{direction} \\ \text{on } C_1 \end{bmatrix} = \text{sign } \Delta_2(P) \times \text{sign} \begin{bmatrix} \text{direction} \\ \text{on } C_3 \end{bmatrix}.$$

Thus if $\Delta_1(P)$ and $\Delta_2(P)$ have like signs, $C$ is a $P$-plus curve; and if $\Delta_1(P)$ and $\Delta_2(P)$ have unlike signs, $C$ is a $P$-minus curve.

## V. ANALYTIC CONTINUATION

**40.** DEFINITION. Let $f(z)$ be analytic in a region $T$. If there exists a region $T'$ such that $T \subset T'$ and a function $g(z)$ such that $g(z)$ is analytic in $T'$ and $f(z) = g(z)$ for $z$ in $T$, then the function $g(z)$ will be said to *continue the function $f(z)$ analytically into the region $T'$*.

If $f(z)$ is a complex-valued function analytic in a region $S$ of the complex plane (considered as a subset of the bicomplex space), it has previously been pointed out in the introduction, part 6, that the expression $f(z_1)e_1 + f(z_3)e_3$ defines a bicomplex-valued analytic function of a bicomplex variable in the product region $T$ of the component regions of $S$. Any analytic continuation of this function will also be referred to as an analytic continuation of $f(z)$.

It has also been pointed out previously that if $f(z)$ is analytic in $T$ and the component-regions are $T_1$ and $T_3$, then $f(z)$ is automatically continued analytically into the product region of $T_1$ and $T_3$ by the formula $f(z) = g(z_1)e_1 + h(z_3)e_3$ for $z_1$ in $T_1$ and $z_3$ in $T_3$.

The usual method of analyticl continuation by power series is applicable to the bicomplex space (see Futagawa [3]), and theorems similar to those in the complex case can be established.

**41.** The following question has been raised by Futagawa [3]:

Given a complex-valued function $f(z)$ of a complex variable $z$, and a simple closed curve $\Gamma$ in the complex plane. Suppose that $f(z)$ is analytic in the interior of $\Gamma$, but that singularities of $f(z)$ are everywhere dense on $\Gamma$. In other words, $\Gamma$ is a natural boundary of $f(z)$. Is it ever possible to continue $f(z)$ into the bicomplex space along a continuous path which intersects the complex plane outside of $\Gamma$?

Futagawa appears to answer this question in the affirmative (Futagawa [3], pages 80-120). However if $f(z)$ is continued into the bicomplex space, then $f(z)=f(z_1)e_1+f(z_3)e_3$, and since for $z$ complex, $z_1=z_3=z$, the projections $\Gamma_1$ and $\Gamma_3$ of the curve in the $z_1$-and $z_3$-planes, respectively, are curves congruent to $\Gamma$, and are hence the natural boundaries of the component functions $f(z_1)$ and $f(z_3)$, respectively. The projections of a continuous curve $z$ in the bicomplex space joining a point inside $\Gamma$ in the complex plane to a point outside $\Gamma$ in the complex plane are continuous curves joining pairs of points similarly situated with respect to $\Gamma_1$ and $\Gamma_3$, hence crossing $\Gamma_1$ and $\Gamma_3$, respectively. Hence analytic continuation along $C$ is impossible and the above question is answered in the negative.

## VI. EXTENSION OF VARIOUS THEOREMS TO THE BICOMPLEX SPACE

**42.** Many theorems from the theory of analytic functions of a complex variable can be extended with little or no change to the bicomplex space and proved by the Ringleb decomposition. Some examples chosen rather arbitrarily will be presented here. The maximum-minimum principle will be found to hold for $\|f(z)\|$, $|f(z)|$ and $N[f(z)]$. Schwarz's lemma holds if the norm, $\|z\|$, is used. The condition for equality has a modification, however.

In theorems where the assumptions involve the behavior of an analytic function on a set of points $S$ having a limit point interior to the region $T$ of analyticity, such as Vitali's theorem and the uniqueness theorem for power series, the additional assumption must be made in the bicomplex case that the set of points does not lie in the nil-planes with respect to a finite number of points, and also some assumption such as that the closure of the set of points is in $T$. The purpose of this last assumption is to prevent $S$ from consisting of the sum of two sets $A$ and $B$ such that $A$ is contained in the nil-planes with respect to a finite number of points and yet has a limit point in $T$, while $B$ is not contained in the nil-planes with respect to a finite number of points but has all its limit points on the boundary of $T$.

Periodic functions generalize immediately. If the period is not a nil-factor, then the component functions are both periodic.

The property of conformal mapping in the complex plane can of course not be expected to extend to the bicomplex case, as an example will confirm.

**43.** MAXIMUM-MINIMUM PRINCIPLE. *Let $f(z)$ be analytic in a region $T$. Then none of the expressions $\|f(z)\|$, $N[f(z)]$, or $|f(z)|$ can assume a maximum or a non-zero minimum at a point $z_0$ interior to $T$.*

PROOF. 1. Suppose $\|f(z)\| = (1/\sqrt{2})\sqrt{|g(z_1)|^2 + |h(z_3)|^2}$ assumes a maximum at an interior point $z_0 = z_1^0 e_1 + z_3^0 e_3$ of $T$. Then $|g(z_1)|^2 + |h(z_3^0)|^2$ assumes a maximum at $z_1 = z_1^0$, an interior point of the component region $T_1$. This contradicts the maximum modulus principle in the complex theory. Thus $\|f(z)\|$ cannot assume a maximum at an interior point of $T$.

Suppose now that $\|f(z)\|$ assumes a minimum at $z = z_0$. Then $|g(z_1)|^2 + |h(z_3^0)|^2$, and therefore $|g(z_1)|$, assumes a mimimum at $z_1 = z_1^0$. Thus $|g(z_1^0)| = 0$. Similarly $|h(z_3^0)| = 0$. Therefore $\|f(z)\|$ is zero at $z = z_0$.

2. Let $N[f(z)]$ have a maximum at $z = z_0 = z_1^0 e_1 + z_3^0 e_3$ interior to $T$. Recall that $N[f(z)] = \max[\,|g(z_1)|, |h(z_3)|\,]$. Suppose, for instance, that $|g(z_1^0)| \geqq |h(z_3^0)|$. Then $|g(z_1)|$ has a maximum at $z_1 = z_1^0$. This cannot happen. Therefore $N[f(z)]$ cannot have a maximum at an interor point of $T$. Now suppose $N[f(z)]$ has a minimum other than zero at $z_0 = z_1^0 e_1 + z_3^0 e_3$ in $T$ and $|g(z_1^0)| > |h(z_3^0)|$. Then again $|g(z_1)|$ has a non-zero minimum, which cannot happen. If $|g(z_1^0)| = |h(z_3^0)|$ then either $|g(z_1)|$ or $|h(z_3)|$ has a non-zero minimum, which cannot happen. Thus $N[f(z)]$ cannot have a non-zero minimum.

3. Suppose $|f(z)|$ has a maximum at $z = z_0 = z_1^0 e_1 + z_3^0 e_3$ interior to $T$. Since $|f(z)| = \sqrt{|g(z_1)| \cdot |h(z_3)|}$, then $|g(z_1)| \cdot |h(z_3^0)|$ has a maximum at $z_1 = z_1^0$. This cannot happen. Similarly $|f(z)|$ cannot have a non-zero minimum.

44. SCHWARZ'S LEMMA. *Let $f(z)$ be analytic in the hypersphere $\|z\| < R$. Let $\|f(z)\| < M$ for $\|z\| < R$, and let $f(0) = 0$. Then*

$$\|f(z)\| \leqq \frac{M\|z\|}{R}$$

*for $\|z\| < R$, where equality can hold if and only if $f(z) \equiv (M/R)Kz$, where $K$ is a bicomplex constant such that $\|K\| = |K| = 1$.*

PROOF. Since $f(0) = 0$, $g(0) = h(0) = 0$. Let $|z_1| < R\sqrt{2}$ and $z_3 = 0$. Then $\|z\| < R$, and $\|f(z)\| < M$, by hypothesis. Therefore $|g(z_1)| < M\sqrt{2}$ for $|z_1| < R\sqrt{2}$. By Schwarz's lemma for the complex case

$$|g(z_1)| \leqq \frac{M}{R}|z_1|$$

for $|z_1| < R\sqrt{2}$ and equality holds only if $g(z_1) \equiv (M/R)K_1 z_1$, where $|K_1| = 1$. Similarly

$$|h(z_3)| \leqq \frac{M}{R}|z_3|$$

for $|z_3| < R\sqrt{2}$ and equality holds only if $h(z_3) \equiv (M/R)K_3 z_3$ where $|K_3| = 1$. Then for $\|z\| < R$,

$$\|f(z)\| = (1/\sqrt{2})\sqrt{|g(z_1)|^2 + |h(z_3)|^2} \leqq (1/\sqrt{2})\sqrt{\frac{M^2}{R^2}|z_1|^2 + \frac{M^2}{R^2}|z_3|^2}$$

$$= \frac{M}{R} \cdot \frac{1}{\sqrt{2}}\sqrt{|z_1|^2 + |z_3|^2} = M\|z\|/R,$$

and equality holds only if

$$f(z) = \frac{M}{R} K_1 z_1 e_1 + \frac{M}{R} K_3 z_3 e_3 = \frac{M}{R} Kz,$$

where $K = K_1 e_1 + K_3 e_3$ with $|K_1| = 1$ and $|K_3| = 1$. This is the case if and only if $\|K\| = |K| = 1$.

**45.** Definition. Let $f_n(z)$ be an infinite sequence of bicomplex-valued functions, defined on a set $S$. The sequence will be termed *uniformly convergent* if for every $\varepsilon > 0$ there exists an integer $N(\varepsilon) > 0$ such that $\|f_m(z) - f_n(z)\| < \varepsilon$ for every $m, n > N(\varepsilon)$ and for every $z$ in $S$.

If the $f_n(z)$ are analytic for all $n$ it is easily verified that a necessary and sufficient condition for $f_n(z)$ to converge uniformly on $S$ is that $g_n(z_1)$ converge uniformly on $S_1$ and $h_n(z_3)$ converge uniformly on $S_3$.

Vitali's Convergence Theorem. *Let $f_n(z)$ be a sequence of functions, each analytic in a region $T$. Let*

$$\|f_n(z)\| \leqq M$$

*for every $n$ and for every $z$ in $T$, and let $f_n(z)$ tend to a limit as $n \to \infty$ at a set $S$ of points that is not contained in the nil-planes with respect to a finite number of points, and such that the closure of $S$ is in $T$. Then $f_n(z)$ tends uniformly to a limit on any closed subset of $T$, the limit being therefore an analytic function of $z$ in $T$.*

Proof. Since $\|f_n(z)\| \leqq M$, $|g(z_1)| \leqq M\sqrt{2}$ and $|h(z_3)| \leqq M\sqrt{2}$. The projections $S_1$ and $S_3$ of the set $S$ are infinite point sets having a limit point in $T_1$ and $T_3$, respectively, since the closure of $S$ is in $T$, and $S$ is not contained in the nil-planes with respect to a finite number of points. Therefore Vitali's convergence theorem applies to the $g_n(z_1)$ and $h_n(z_3)$, which are uniformly convergent in every closed subset of $T_1$ and $T_3$, respectively. Thus the sequence $f_n(z)$ is uniformly convergent in every closed subset of $T$.

**46.** This section will be concluded by an example to show that even if an analytic function $f(z)$ is assumed to have a derivative different from zero or a nil-factor at a point, the mapping performed by $f(z)$ need not preserve angles in the bicomplex space. The example will be based on the following:

Lemma. *The transformation $w = az + b$, where $a$ is not equal to zero or a nil-factor, takes every straight line into a straight line.*

Proof. Let $\alpha$ and $\beta$ be two distinct points on the line. Then $a\alpha + b \neq a\beta + b$. On the line, $z$ is of the form $k\alpha + (1-k)\beta$, where $k$ is real. Then $az + b = a[k\alpha + (1-k)\beta] + b = k(a\alpha + b) + (1-k)(a\beta + b)$, which is a straight line through the distinct points $a\alpha + b$ and $a\beta + b$.

Example. Consider the transformation $w = (2e_1 + e_3)z$. This function is analytic in the entire space. The derivative has the constant value $2e_1+e_3$, which is not zero or a nil-factor. Now let the bicomplex variable $z$ be $x+iy+jz+iju$ where $x, y, z, u$ are real. In the plane $x=0$, $u=0$, consider the lines $y=0$ and $y=z$. These lines intersect at the origin at an angle of $\pi/4$. By the lemma the transformation takes these lines into lines again. But a point on the line $y=z$ is a nil-factor and the transformation leaves the line fixed; while a point on the line $y=0$ if on the form $kj$, where $k$ is real. Then

$$(2e_1+e_3)kj = -\frac{1}{2}ki+\frac{3}{2}kj$$

which determines the line $x=0$, $u=0$, $3y+z=0$ and this line does not make an angle of $\pi/4$ with the line $y=z$.

## VII. TAKASU'S ALGEBRA

**47.** Takasu [9] has considered the theory of functions of a generalized bicomplex variable

$$z=x_1+jx_2+j'(x_3+jx_4)$$

where $j^2=\mu+\nu j$, $j'^2=\mu'+\nu' j$ and $\mu, \nu, \mu', \nu'$, are real constants and $x_1, x_2, x_3, x_4$ are real variables. The fundamental operations are defined by requiring the usual formal laws of operation to hold. The system of such numbers $z$ is seen to be an associative commutative linear algebra with the modulus $1+0j+j'(0+0j)$, denoted by 1 (see Dickson [1], pages 4-7).

In view of Ringleb's decomposition theorem one might ask: For what values of $\mu, \nu, \mu', \nu'$ is this system reducible? Scheffers has given the following criterion: (See Dickson [1], page 27).

A linear associative algebra $A$ with a modulus is reducible if and only if it contains an element $x\neq 0, 1$ such that $x^2=x$ and $xz=zx$ for every element $z$ of $A$. An equivalent condition is that there exist in $A$ an element $y\neq \pm 1$ such that $y^2=1$ and $yz=zy$ for every $z$ in $A$.

Proof of equivalence: Assume that there exists $x\neq 0,1$ such that $x^2=x$. Then $(2x-1)^2=4x^2-4x+1=4(x^2-x)+1=1$. Since $x\neq 0$, then $2x-1\neq -1$; since $x\neq 1$, then $2x-1\neq 1$.

Conversely, assume that there exists $y$ such that $y^2=1$, and $y\neq\pm 1$. Then $[(y+1)/2]^2=(y^2+2y+1)/4=(1+2y+1)/4=(2y+2)/4=(y+1)/2$. Since $y\neq +1$, then $(y+1)/2\neq 1$; since $y\neq -1$, then $(y+1)/2\neq 0$.

Clearly $y$ commutes with every element $z$ of $A$ if and only if $x$ commutes with $z$.

Further $[(1-y)/2]^2=(y^2-2y+1)/4=(2-2y)/4=(1-y)/2$. And $[(1+y)/2]\cdot[(1-y)/2]=(1-y^2)/2=0$. Thus $(1+y)/2$ and $(1-y)/2$ are idempotent divisors of zero (nil-factors).

48. To simplify the computation through which it will be determined for what values of $\mu$, $\nu$, $\mu'$, $\nu'$ the system is reducible, the definitions of $j^2$ and $j'^2$ will be transformed in the following way.

Since $j^2=\nu j+\mu$, then $(2j-\nu)^2=4j^2-4\nu j+\nu^2=4\nu j+4\mu-4\nu j+\nu^2=4\mu+\nu^2$. If $\nu^2+4\mu>0$ then $\sqrt{\nu^2+4\mu}$ is real, and

$$\left(\frac{2j-\nu}{\sqrt{\nu^2+4\mu}}\right)^2=1.$$

If $\nu^2+4\mu<0$ then $\sqrt{-(\nu^2+4\mu)}$ is real, and

$$\left(\frac{2j-\nu}{\sqrt{-(\nu^2+4\mu)}}\right)^2=-1.$$

If $\nu^2+4\mu=0$ then

$$(2j-\nu)^2=0.$$

Similarly from the equation defining $j'^2$,

$$\left(\frac{2j'-\nu'}{\sqrt{\nu'^2+4\mu'}}\right)^2=1 \text{ if } \nu'^2+4\mu'>0,$$

$$\left(\frac{2j'-\nu'}{\sqrt{-(\nu'^2+4\mu')}}\right)^2=-1 \text{ if } \nu'^2+4\mu'<0,$$

$$(2j'-\nu')^2=0 \text{ if } \nu'^2+4\mu'=0.$$

These relations divide the algebra into nine cases, which may be reduced to five by isomorphisms. If the relations are represented briefly as $K^2=1$, $K^2=-1$, $K^2=0$, and $K'^2=1$, $K'^2=-1$, $K'^2=0$, then the cases may be tabulated as follows:

| $K'^2$ \ $K^2$ | 1 | −1 | 0 |
|---|---|---|---|
| 1 | A | B | C |
| −1 | D | E | F |
| 0 | G | H | J |

The cases B and D are seen to be isomorphic simply by interchanging $K$ and $K'$. Case E is seen to be isomorphic to cases B and D, since $(KK')^2=1$, and the elements $K$ and $KK'$ of case E can be made to correspond to the elements $K$ and $K'$ of case B. By interchanging $K$ and $K'$ it is also seen that cases C and G are isomorphic and that cases H and F are isomorphic. These facts summarized in tabular form become:

| $K'^2$ \ $K^2$ | 1 | -1 | 0 |
|---|---|---|---|
| 1 | I | II | III |
| -1 | II | II | IV |
| 0 | III | IV | V |

Cases I, II, and III are reducible since they contain an element whose square is unity, and the system is commutative. Case II is, of course, that of the ordinary bicomplex variable discussed in the previous sections of this paper.

Inc ase IV, let $K^2=-1$, and $K'^2=0$. The elements $e_1=1, e_2=K, e_3=K', e_4=KK'$ form a basis with the multiplication table

| $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|
| $e_2$ | $-e_1$ | $e_4$ | $-e_3$ |
| $e_3$ | $e_4$ | 0 | 0 |
| $e_4$ | $-e_3$ | 0 | 0 |

Then if $a,b,c,d$ are real,

$$(ae_1+be_2+ce_3+de_4)^2=(a^2-b^2)e_1+2abe_2+2(ac-bd)e_3+2(ad+bc)e_4 .$$

This is equal to one if and only if

$$\begin{cases} a^2-b^2=1 \\ ab=0 \\ ac-bd=0 \\ ad+bc=0 . \end{cases}$$

This system of equations has only two solutions, $a=\pm 1$, $b=c=d=0$. Thus the algebra in case IV is irreducible.

In case V, $K^2=0$ and $K'^2=0$. The elements $e_1=1$, $e_2=K$, $e_3=K'$, $e_4=KK'$ form a basis with the multiplication table

| $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|
| $e_2$ | 0 | $e_4$ | 0 |
| $e_3$ | $e_4$ | 0 | 0 |
| $e_4$ | 0 | 0 | 0 |

Then if $a$, $b$, $c$, $d$ are real

$$(ae_1+be_2+ce_3+de_4)^2=a^2e_1+2abe_2+2ace_3+2(ad+bc)e_4 .$$

This is equal to one if and only if

$$\begin{cases} a^2=1 \\ ab=0 \\ ac=0 \\ ad+bc=0 . \end{cases}$$

This system of equations has only two solutions, $a=\pm 1$, $b=c=d=0$. Thus the algebra in case V is irreducible.

**49.** Many questions about the function theory in the separate cases can be raised. Case I decomposes into four separate subalgebras, which are each isomorphic to the algebra of real numbers. Then to what extent will the theory of functions in case I resemble that of a complex or ordinary bicomplex variable?

Cases IV and V have nil-potent elements, for which all power series would terminate. Case IV contains the complex number system as a subalgebra. Then is it possible, or again impossible (see section V), to continue a complex-valued analytic function of a comlpex variable in this space beyond its natural boundary in the complex plane?

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NAVAL ORDNANCE LABORATORY, WHITE OAK, MARYLAND, U. S. A.