ON ORDER AND COMMUTATIVITY OF B^*-ALGEBRAS

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(Received June 8, 1954)

The representation theory of (partially) ordered vector spaces has an application to the representation theory of commutative B^*-algebras. Kadison has treated this idea [2]. In this respect, we shall notice that the B^*-algebra with the decomposition property is necessarily commuative, which is a generalization of a commutativity theorem of Sherman [5] and might simplify the argument such as Kadison's when we apply the ordered vector space to the representation theory of B^*-algebras. Incidentally different proofs were obtained, which we shall state in the following. §1 is due to Misonou, §2 to Fukamiya and §3 to Takeda.

1. Theorem and its direct treatment. By a B^*-algebra, we mean a Banach algebra possessing a *-operation such as $\|x^*x\| = \|x\|^2$. It has recently been proved that every B^*-algebra can be represented as a uniformly closed, self-adjoint algebra of bounded operators on a suitable Hilbert space. Let A be a B^*-algebra and H, D be the set of all hermitian elements and positive hermitian elements in A respectively, then H is an archimedian ordered vector space by an order relation $a \leqq b$ in H as $b - a \in D$. We say a B^*-algebra A satisfies the *decomposition property*, originally due to F. Riesz, if for every a such as $0 \leqq a \leqq b + c$ with b and c positive, there exist positive a_1, a_2 such that $a = a_1 + a_2, a_1 \leqq b, a_2 \leqq c$. Then we shall prove

THEOREM 1. *A B^*-algebra A which has an identity e and satisfies the decomposition property is necessarily commutative.*

FIRST PROOF OF THEOREM. As a preparation, we notice that every projection p and hermitian operator a on a Hilbert space such that $0 \leqq a \leqq p$ satisfy $ap = pa$. For, by the assumption, we have $0 \leqq (1 - p)a(1 - p) \leqq 0$, which implies $a^{\frac{1}{2}}(1 - p) = 0$, hence $a(1 - p) = 0$ and $a = ap = pa$.

Since every element of A can be expressed as a linear combination of positive elements of A, it is sufficient to prove that $ab = ba$ for every pair of positive elements $a, b \leqq e$.

Let B be the B^*-subalgebra of A generated by a and e. Then B can be isomorphically represented to a ring $C(\Lambda_a)$ of all continuous function on the spetrum Λ_a of a. We denote by V the weak closure of an operator representation of B on a suitable Hilbert space.

Let $a(t)$ be the function corresponding to a by the function representation of B on Λ_a. Then $a(t)$ can be approximated at each point of Λ_a by a sequence $\{s_n(t)\}$ of step functions. This means there exists a sequence $\{s_n\}$ of linear combinations of projections in V which converges strongly to a. Hence, to prove the theorem it is sufficient to show that b is commutative with each projection p in V which is represented to a characteristic function of a closed interval in Λ_a.

Let $I = [t: \alpha \leqq t \leqq \beta]$ and $I' = \Lambda_a \cap I$ and $p(t)$ be the characteristic function on I'. Then we can find a sequence of positive continuous functions $q_n(t)$ on Λ_a which converges to $p(t)$ at each point satisfying $q_n(t) \leqq p(t)$. Let $r_n(t) = 1 - q_n(t)$ then $r_n(t)$ is continuous on Λ_n. We shall denote the elements of B which are determined by $q_n(t)$ and $r_n(t)$ as q_n, r_n respectively. Then $q_n + r_n = e$. Hence, by the decomposition property of A, there exist positive b_{1n}, b_{2n} such that

$$b = b_{1n} + b_{2n}, \quad b_{1n} \leqq q_n, b_{2n} \leqq r_n.$$

Clearly $b_{1n} \leqq p$, hence $pb_{1n} = b_{1n}$ by the above remark. Since $p(t)b_{2n}(t)$ converges to 0 at each point, pb_{2n} converges to 0 strongly. That is, pb_{1n} converges to pb strongly. Similarly, $b_{1n}p$ converges to bp strongly. This shows $pb = bp$. q. e. d.

2. Second Proof due to direct Generalization of Sherman's Method. In this section, we shall proceed as Krein did and obtain a proof of the theorem by using the method employed by Sherman for the proof of his commutativity theorem. An *order ideal* N in an archimedian ordered vector space E is a linear subspace such that $-a \leqq b \leqq a$ for some $a \in N$ implies $b \in N$; an order ideal is a lattice ideal (normal ideal) when the vector space is a lattice. Every proper order ideal can be extended to a maximal order ideal. For every maximal order ideal M, the quotient space E/M is isomorphic (as a linear and ordered space) to reals. Therefore, the set of all states on the B^*-algebra A (the positive linear normalized functionals on H) is in one-to-one correspondence with the set of all maximal order ideals on H: $f \to N = \{u \in H: f(u) = 0\}$. (See Kadison [2])

At first, we notice that, if $u \geqq w \geqq 0$, $v \geqq w \geqq 0$ and $uv = 0$, then $w = 0$. For, as $u \geqq 0$ is equivalent to $\sigma(u) \geqq 0$ for every state σ, $huh \geqq 0$ for every $h \in H$ along with $u \geqq 0$. Thus $u \geqq w$ means $0 = vuv \geqq vwv \geqq 0$, and as $a^*a = 0$ means $a = 0$, we have $w^{\frac{1}{2}}v = 0$. On the other hand, $0 \leqq w^3 \leqq wvw = (wv)w = 0$ shows $w = 0$.

LEMMA. *If B^*-algebra A satisfies the decomposition property, then the maximal order ideal N_0 corresponding to an extreme state σ_0: $N_0 = \{u \in H: \sigma_0(u) = 0\}$ has the property that, for every $u \in H$ with $u = u_+ - u_-, u_+ \geqq 0$, $u_- \geqq 0, u_+ \cdot u_- = u_- \cdot u_+ = 0$, either u_+ or u_- must belong to N_0. If $u \in N_0$, both u_+ and $u_- \in N_0$.*

This lemma is equivalent to $|\sigma_0(u)| = \sigma_0(|u|)$ for an extreme state σ_0.

PROOF. $N_0 = \{u: \sigma_0(u) = 0, u \in H\}$ is clearly a maximal order-ideal. To show the above statement, assume that a $u \in H$ be such that both u_+ and $u_- \bar{\in} N_0^+$. Put $N_1 = \{v \in H: -(cu_+ + w) \leqq v \leqq cu_+ + w, c \geqq 0, w \in N_0^+\}$. If we have $u_- \in N_1$, then it would follow at once, by the decomposition property, $u_- = v_1 + v_2, v_1 \leqq cu_+, v_2 \leqq w$, so we would have $0 \leqq v_1 \leqq cu_+, \leqq u_-$, and $u_+ \cdot u_- = 0$, thus we have $v_1 = 0$ by the above remark. Hence $u_- = v_2 \in N_0^+$, contrary to the assumption, so that $u_- \overline{\in} N_1 \cdot N_1$ is extended to a maximal order ideal N', for which a state τ corresponds. It is obvious that both $\rho = \sigma_0 \wedge \tau$ and $\kappa = 2\sigma_0 - \rho$ are states and $\sigma_0 = \frac{1}{2}(\rho + \tau)$, which contradicts to the extremity of σ_0. Thus u_+ or $u_- \in N_0$.

SECOND PROOF OF THEOREM 1. From the above lemma we can easily see, as Sherman did, that the set $N_0 = \{x : \sigma_0(x) = 0\}$ for an arbitrary extreme state σ_0 is a two-sided ideal of A, and σ_0 is an homomorphism from A onto the complex number field. As σ_0 is arbitrary, A is commutative.

3. Lattice Property of Conjugate Space.

As shown by Sherman [5], all hermitian elements H of a B^*-algebra A constitute a Banach lattice if and only if A is commutative. Then naturally the conjugate space of H is a complete Banach lattice. On the other hand, as shown in [6], every real-valued functional on H of a non-commutative B^*-algebra A is expressed by a difference of two positive functionals of H— this is easily obtained from the fact that the positive element of H forms a normal convex cone [1] [4]. Thus the conjugate space of H is of like nature as a Banach lattice, but not necessarily a Banach lattice. For any algebra, does this exactly form a Banach lattice? The answer for this question is

THEOREM 2. *The conjugate space of the real Banach space H of all hermitian elements of a B^*-algebra A is a Banach lattice if and only if A is commutative.*

Since Kadison [2] has shown that the conjugate space of H is a Banach lattice (in fact, a complete lattice) for a B^*-algebra with the decomposition property, this theorem gives the third proof of Theorem 1 as a direct corollary.

Let Ω be the state space of a B^*-algebra A. By the usual method we construct a Hilbert space $\mathfrak{H}_\sigma$ for every state σ in Ω and put $\mathfrak{H}$ the direct sum of $\mathfrak{H}_\sigma(\sigma \in \Omega)$. Then A is isomorphically represented to an operator algebra $A^\#$ on $\mathfrak{H}$ [8]. Let $a^\#$ be the representative operator for $a \in A$ and W be the weak closure of $A^\#$. A *canonical state* of $A^\#$ is a state σ given by $\varphi \in \mathfrak{H}$ such as $\sigma(a^\#) = \langle a^\#\varphi, \varphi \rangle$. Considering all finite linear combinations of canonical states of A which define the same linear functional on A as a class, we get a space S constructed by all such classes, which can be regarded as isomorphic to the conjugate space $\overline{A}$ of A. We denote by $\{f\}$ the class in S which corresponds to f in $\overline{A}$. A *canonical linear functional* $\sigma_{\varphi,\psi}$ on $A^\#$ means a linear functional such as $\sigma_{\varphi,\psi}(a^\#) = \langle a^\#\varphi, \psi \rangle$ where φ, ψ are elements of $\mathfrak{H}$. Then every class of V contains a canonical linear functional, for $\sum_{i=1}^{n} \langle a^\#\varphi_i, \psi_i \rangle$ can be written as $\sum_{l=1}^{m} \alpha_l \sigma_l(a^\#)$ where α_l is a complex number and σ_l is a state on A satisfying $\sigma_l \neq \sigma_{l'}$ for $l \neq l'$. Then, by the definition of $\mathfrak{H}$, there exist $\varphi, \psi \in \mathfrak{H}$ such as $\sum_{i=1}^{n} <a^{\#}\varphi_i, \psi_i> = <a^{\#}\varphi, \psi>$. Hence if we bundle up as a class all canonical functionals which belongs to a same class in S, S can be replaced by the totality R of such classes. Thus every element F of the conjugate space $\bar{\bar{A}}$ of $\bar{A}$ defines a linear functional on R and the latter gives a bilinear functional on $\mathfrak{H}$. By Riesz's well known lemma F defines an operator w on $\mathfrak{H}$ such as $F(f) = <w\varphi, \varphi>$ for every $\sigma_{\varphi,\psi} \in \{f\}$. This w is contained in W, because for a canonical linear functional g defined by $g(a^{\#}) = <a^{\#}a'\varphi, \psi>$ where $a' \in A^{\#\prime}$, $F(g) = <wa'\varphi, \psi> = <w\varphi, a'\varphi^*> = <a'w\varphi, \psi>$ since $<a^{\#}a'\varphi, \psi> = <a^{\#}\varphi, a'^*\psi>$. On the contrary every $w \in W$ defines a linear functional on A and this correspondence between $\bar{\bar{A}}$ and W is linear, norm preserving. For

$$\|F\| = \sup_{\varphi,\psi} \frac{|<w\varphi, \psi>|}{\|\sigma_{\varphi,\psi}\|} \geqq \sup_{\varphi,\psi} \frac{|<w\varphi, \psi>|}{\|\varphi\| \cdot \|\psi\|} = \|w\|.$$

Let $a^{\#}_{\alpha}$ be a directed set in the unit sphere of $A^{\#}$ which converges to $w/\|w\|$ weakly. Then, as $\|\sigma_{\varphi,\psi}\| = \sup_{\|a^{\#}\| \leqq 1} |<a^{\#}\varphi, \psi>|$,

$$\|F\| = \sup_{\varphi,\psi} \frac{|<w\varphi, \psi>|}{\|\sigma_{\varphi,\psi}\|} \leqq \sup_{\varphi,\psi} \left[\left| \frac{<w/\|w\| \varphi, \psi>}{<a^{\#}_{\alpha}\varphi, \psi>} \right| \cdot \|w\| \right] \text{ for every } \alpha.$$

Hence $\|F\| \leqq \|w\|$, that is $\|F\| = \|w\|$.

By the definition of w, w is a positive operator if and only if F is a positive functional on A. Thus we get a precise statement of a theorem in [6].

THEOREM 3. *The double conjugate space $\bar{\bar{H}}$ of the space H af all hermitian elements of a B^*-algebra A is isomorphic as an ordered Banach space to the space of all hermitian operators of a W^*-algebra W.*

PROOF OF THEOREM 2. If H is a Banach lattice, $\bar{\bar{H}}$ is a complete Banach lattice, hence by the above theorem, all hermitian operators in W constitute a vector lattice. Then by S. Sherman's theorem, W is commutative, hence A is necessarily commutative.

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