ON EXTENSIONS OF PURE STATES OF AN ABELIAN OPERATOR ALGEBRA

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1. Combining results due to J. Dixmier [2] and I. E. Segal [9], a pure state of an abelian C^*-algebra acting on a separable Hilbert space will be expressed as a wave function or a pure state of ideal type. The difference of these two expressions comes from the spectral property of the algebra. The main purpose of the present note is to discuss this (§2).

In last two sections (§§3, 4) we shall give two applications of our results. The first one is an alternative proof for Rosenberg's theorem*) and the second is a remark on a recent theorem due to Ogasawara [6]; finally we shall characterize the space (m) as a special W^*-algebra.

A state σ of the algebra $B(H)$ of all (bounded) linear operators on a separable Hilbert space H will be called a *wave function* (*or of wave type*) if there exists an element φ of H such that

$$\sigma(x) = <\varphi x, \varphi> \tag{1}$$

for all operators; and it will be called a state *of ideal type* if it vanishes whenever x is completely continuous. It is established by Dixmier that a pure state of the algebra $B(H)$ is either of wave type or of ideal type.

Since a pure state π of C^*-algebra A acting on H can be extended to a pure state of $B(H)$, it can be expressible as a pure state of wave type or a pure state of ideal type. According to this expression we shall say even for a pure state of A it is of wave type or of ideal type.

2. At first, we shall discuss the case of pure states of wave type.

PROPOSITION 1. *If a pure state π of an abelian C^*-algebra A acting on a separable space H is of wave type* (1), *then φ is the common proper vector of all hermitean operators belonging to A.*

PROOF. If a is an hermitean element of A, and if $\pi(a) = \lambda$, then

$$<\varphi(a-\lambda), \varphi x^*> = <\varphi(a-\lambda)x, \varphi> = <\varphi(a-\lambda), \varphi><\varphi x, \varphi> = 0$$

for all $x \in A$ since a pure state of an abelian C^*-algebra is multiplicative, whence $\varphi a = \lambda\varphi$.

THEOREM 2. *If all pure states of a C^*-algebra are of wave type, then each hermitean element of the algebra has only point spectrum.*

PROOF. Let A be a subalgebra generated by an hermitean element a of

*) A mistake of our previous paper [4] has been pointed out by a letter of A. Rosenberg. The authors express here their hearty thanks to Dr. A. Rosenberg for his kindness.

the given algebra B. A is clearly abelian and any pure state π of A is of wave type by the hypothesis. Since the value $\pi(a)$ determines the spectrum of a and conversely, Proposition 1 implies our statement.

THEOREM 3. *A commutative C^*-algebra acting on a separable space whose pure states are of wave type has the spectrum with countable points.*

Next, we shall concern with the case of pure states of ideal type. The following proposition, which is basic for us, is a version of a theorem of J. W. Calkin [1]:

PROPOSITION 4. *If a pure state π of an abelian C^*-algebra A is of ideal type, then the value $\pi(a)$ is a point of condensed spectrum of a, where a is an hermitean element of A.* **)

PROOF. Let $\{\varphi_i\}$ be an orthonormal set of proper vectors of a, and let F be the span of it. If $\pi(a)=\lambda$ does not belong to the continuous spectrum of a, then the inverse of $a-\lambda$ exists on $F^{\perp}$.

For a certain positive ε and for all n if it is true

$$|<\varphi_n(a-\lambda), \varphi_n>| \geqq \varepsilon, \tag{2}$$

and if $\varphi=\sum_{i=1}^{\infty}\alpha_i\varphi_i$ we have then

$$\|\varphi(a-\lambda)\|^2=\|\sum_{i=1}^{\infty}\alpha_i\varphi_i(a-\lambda)\|^2=\sum_{i=1}^{\infty}|\alpha_i|^2|<\varphi_i a, \varphi_i>-\lambda|^2$$
$$\geq \varepsilon^2\sum_{i=1}^{\infty}|\alpha_i|^2=\varepsilon^2\|\varphi\|^2,$$

whence $a-\lambda$ has an inverse of F, and so λ can not belong to the spectrum of a.

If (2) is true up to finite number of n, say $n=1,2,\ldots,m$; let E be the span of $\varphi_1,\ldots,\varphi_m$. Considering π as a pure state of $B(H)$, it vanishes at ea where e is the projection belonging to E. On the other hand, the inverse of $a-\lambda$ exists on $E^{\perp}$, a pure state of ideal type can not take as its value on a, which is a contradiction.

Therefore, there exists at least infinitely many n's which do not satisfy (2), that is (i) $<\varphi_n a, \varphi_n>=\lambda$ for infinitely many n or (ii) λ is a limit point of $\{<\varphi_n a, \varphi_n>\}$. In both cases, λ belongs to the condensed spectrum of a.

THEOREM 5. *If all pure states of a C^*-algebra acting on a separable space are of ideal type, then each hermitean element of the algebra has only condensed spectrum.*

3. In this section we shall give a sketch of an another proof of Rosenberg's Theorem to correct our previous proof [4].

Let A be a C^*-algebra acting irreducibly on a separable Hilbert space H, and let C be the algebra of all completely continuous operators. By Lemma 1 of [4], A is simple and either $A=C$ or $A\cap C=0$. If $A\cap C=0$,

**) A real number λ in the spectrum of an hermitean element a will be called a point of the condensed spectrum provided that (1) λ is the point spectrum of infinite multiplicities or (2) λ is an accumulate point of the spectrum of a.

the generated algebra B by A and C is the direct sum of A and C as Banach space (by a theorem of I. Kaplansky [3] and a method employed in the second half of the proof of Lemma 2 of [4]), whence for every pure state π of A, $\Pi(b) = \pi(a)$ $(b \in B, b = a + c, a \in A,\ c \in C)$ is a pure state of B by the assumption on A, and clearly of ideal type. Therefore all pure states of A are of ideal type. (This is the point that the authors commited the fault in [4], this does not directly mean that the pure states of A are *not* of wave type). Hence the spectrum of an hermitean operator of A is condensed.

On the other hand, all pure states of A are of wave type since A has unique irreducible representation, whence each hermitean element of A has only point spectrum. Therefore, combining these, each hermitean element of A has the spectrum which consists of countable proper values of infinite multiplicities. Hence it is sufficient to show that A contains an hermitean element whose proper value has finite multiplicity. A primitive projection of A which is indicated in a lemma of Rosenberg [7; Lemma 7] is required, since it is one-dimensional by the following***)

PROPOSITION 6. *A maximal abelian C^*-subalgebra of an irreducible C^*-algebra acting on a space has the simple spectrum.*

PROOF. If A is maximal abelian in B which is irreducible on H, we have $A' \cap B = A$, where $A' = A''$ since B' consists of scalars. Thus A'' is maximal abelian.

4. Recently, T. Ogasawara [6] proved that a linear operator a on a Hilbert space H is completely continuous if and only if it is weakly completely continuous as an operator on $B(H)$. He based on the following two facts: (1) $B(H)$ is the second conjugate space of the Banach space $C(H)$ of all completely continuous operators (Theorem of Schatten-von Neumann [8] and J. Dixmier [2]) and (2) $C(H)$ is an ideal of $B(H)$. Therefore, the following theorem can be considered as a direct generalization of Ogasawara's theorem:

THEOREM 7. *If a Banach algebra A having the principal unit is the second conjugate of an ideal I, then I coincides with the ideal of all weakly completely continuous elements.*

PROOF. This is a direct consequence of a theorem in [5] which is the nonseparable extension of a theorem of V. Gantmacher, i. e., a linear operator a on a Banach space E is weakly completely continuous if and only if one of the following conditions is satisfied: (1) a^* is weakly completely continuous, (2) a^{**} maps the second conjugate E^{**} into E, where a^* and a^{**} denote the conjugate and second conjugate operator of a respectively. For, $x \to xa$ on A coincides with $x \to xa^{**}$, and so $x \in I$ if and only if a is weakly completely continuous by the preceding facts. This completes the proof of the theorem.

***) An abelian C^*-algebra A acting on H is called to have the simple spectrum if its weak closure A'' is maximal abelian in $B(H)$.

If A satisfies the property described in Theorem 7, then A will be called having Property O. By Ogasawara's theorem, $B(H)$ has Property O. An another example of algebras having Property O is the space (m), the algebra of all bounded sequence of complex numbers, in which (c_0), the algebra of all sequences converging to 0, is the ideal described in Theorem 7.

As an application of the preceding sections, we shall characterize the space (m) as follows:

THEOREM 8. *A W^*-algebra A acting on a separable Hilbert space H is isomorphic to (m) if and only if A is abelian and has Property O.*

PROOF. By a theorem of R. Pallu de la Barrière (cf. e. g. Takeda [10]), the conjugate I^* of the ideal I described in Theorem 7 contains only the normal states of wave type. Therefore, by Theorem 3, the spectrum Ω of I contains countable points. Hence Ω contains isolated points by Rosenberg-Kaplansky's lemma. We shall show at first that the spectrum of hermitean element h of I can not accumulate except 0.

If w_i is a sequence of isolated points on which $h(w_i)$ converges to $\lambda \neq 0$, then $\{x_i h\}$ can contains no weakly converging subsequence where $x_m = \sum_{k=1}^{m} e_k$ and e_n is the characteristic function on the one point set (w_i). Since $\|x_m\| \leqq 1$, this contradicts to the hypothesis that A has Property O by Theorem 7. This shows that each hermitean and consequently each element of I has discrete spectrum which has no accumulate point except the point at infinity, that is, each element of I corresponds to an element of (c_0). This proves the Theorem.

REMARK. Using a similar method, we can prove more general result: If A is a W^*-algebra acting on a separable Hilbert space having a separating and generating vector, and if A has Property O, then the ideal I consists of completely continuous operators in H, whence I is dual and consequently I is the B_0^*-sum of enumerable numbers of dual simple algebras. For a case of factors, this result is less general than a recent theorem of Z. Tadaka [11] which states that if a factor is isomorphic to the second conjugate of a C^*-algebra, then it is of type (I).

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