# ON THE REPRESENTATIONS OF OPERATOR ALGEBRAS, II

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**1. Introduction.** When we intend to represent a $B^*$-algebra as an operator algebra on a Hilbert space $H$, we must construct the Hilbert space $H$ at first. In the previous paper [9], we investigated the general method for the construction of $H$ and especially pursued the space on which a given $B^*$-algebra is represented as a $W^*$-algebra. In this paper we clarify the relations between $W^*$-algebras which are generated from $C^*$-representations of a given $B^*$-algebra $A$ on various underlying spaces (a $C^*$-representation means a faithful representation as a uniformly closed operator algebra on a certain Hilbert space). *A distinguished state of $A$ with respect to a $C^*$-representation* $\{A^\#, H\}$[1] is a state $\rho$ which permits an expression

$$\rho(x) = \sum_{i=1}^{\infty} < x^\# \varphi_i, \varphi_i > \tag{1}$$

where $x^\#$ is the representative operator for $x \in A$ and $\varphi_i (i = 1, 2, \ldots.)$ are elements in the underlying space $H$ which satisfy the condition

$$\sum_{i=1}^{\infty} \| \varphi_i \|^2 = 1. \tag{2}$$

Let $S(A^{\#_1}, H_1)$ and $S(A^{\#_2}, H_2)$ be the sets of all distinguished states with respect to the $C^*$-representations of $A$ on $H_1$ and on $H_2$ respectively and $M_1, M_2$ be the $W^*$-algebras generated from these $C^*$-representations. Then the main assertion in this paper is following: *If $S(A^{\#_1}, H_1) \supset S(A^{\#_2}, H_2)$, there exists a normal homomorphism of $M_1$ onto $M_2$ and if $S(A^{\#_1}, H_1) = S(A^{\#_2}, H_2)$, $M_1$ is algebraically $*$-isomorphic to $M_2$.* As an application, an alternative proof of Y. Misonou's theorem [4] is given in the last section of this paper, which shows the space-free character of direct product of $W^*$-algebras.

**2. Weak closures of operator algebras.** For every state $\rho$ of a $B^*$-algebra $A$, we can construct a representation $A^\#_\rho$ of $A$ on a Hilbert space $H_\rho$ by the well known method. By $a^\#_\rho$ we denote the representative operator on $H_\rho$ for $a \in A$. Let $\rho$, $\sigma$ be two states of $A$ and $\{A^\#_\rho, H_\rho\}$, $\{A^\#_\sigma, H_\sigma\}$ be representations of $A$ on Hilbert spaces $H_\rho$ and $H_\sigma$ constructed by $\rho$ and $\sigma$ respectively. If there exists an invariant subspace in $H_\rho$ on which the restriction of $A^\#_\rho$ is unitarily equivalent to the representation $\{A^\#_\sigma, H_\sigma\}$, we define an order for $\rho$ and $\sigma$ by $\rho > \sigma$. Then the set of all distinguished states $S(A^\#, H)$ with respect to a $C^*$-representation $\{A^\#, H\}$ of $A$ has the following properties [9, Theorem 1]:

(i) $S(A^\#, H)$ is weakly dense in the state space $\Omega$ of $A$,

1) The author called it a strongest continuous state with respect to a $C^*$-representation in the previous [9].

(ii) $S(A^{\#}, H)$ is closed by the norm topology of $\Omega$.

(iii) $S(A^{\#}, H)$ is convex,

(iv) if $\rho \in S(A^{\#}, H)$ and $\rho > \sigma$, then $\sigma \in S(A^{\#}, H)$.

Conversely, if a subset $S$ in the state space $\Omega$ satisfies the condition (i)-(iv), we can construct a $C^*$-representation of $A$ for which the set of all distinguished states of $A$ coincides with $S$. Let $R$ be a collection of states of $A$ and $\{A^{\#}_\rho, H_\rho\}$ be the representation of $A$ by $\rho \in R$. The *representation* $\{A^{\#}_R, H_R\}$ of $A$ by $R$ is the representation on the direct sum $H_R$ of $H_\rho$ ($\rho \in R$) which coincides with $\{A^{\#}_\rho, H_\rho\}$ on each component space $H_\rho$. That is, for $a^{\#}_R \in A^{\#}_R$ and $\varphi_R = (\ldots, \varphi_\rho, \ldots, \varphi_{\rho'}, \ldots) \in H_R$, $(\ldots, \rho, \ldots, \rho', \ldots \in R)$

$$a^{\#}_R \varphi_R = (\ldots, a^{\#}_\rho \varphi_\rho, \ldots a^{\#}_{\rho'} \varphi_{\rho'} \ldots) \in H_R. \tag{3}$$

Especially, when $R$ is all states of $A$, we describe the above representation by $\{A^{\#}_\Omega, H_\Omega\}$.

THEOREM 1.[2)] *Let $\{A^{\#_1}, H_1\}$, $\{A^{\#_2}, H_2\}$ be two $C^*$-representations of a $B^*$-algebra $A$ on Hilbert spaces $H_1$ and $H_2$ respectively and $M_1$, $M_2$ be the weak closures of these operator representations, furthermore, $S(A^{\#_1}, H_1)$, $S(A^{\#_2}, H_2)$ be the sets of all distinguished states with respect to these $C^*$-representations. Then there is an algebraical*-isomorphism $\eta$ from $M_1$ onto $M_2$ such that $\eta(a^{\#_1}) = a^{\#_2}$ if and only if $S(A^{\#_1}, H_1) = S(A^{\#_2}, H_2)$.*

PROOF. Let $S_1, S_2$ be the sets of all normale states of $M_1$ and $M_2$ respectively. Then, since $A^{\#_1}$ is weakly dense in $M_1$ every state in $S(A^{\#_1}, H_1)$ can be uniquely extended to a state in $S_1$, hence $S(A^{\#_1}, H_1)$ can be identified with $S_1$. Similarly $S(A^{\#_2}, H_2)$ is identified with $S_2$. Since the normality of a state is purely algebraic property, if $M_1$ is algebraically $*$-isomorphic to $M_2$ satisfying $\eta(a^{\#_1}) = a_{\#_2}$, $S_1 = S_2$. Hence, in this case, $S(A^{\#_1}, H_1) = S(A^{\#_2}, H_2)$.

Next, we construct the representation $\{M^{\#}_{1S_1}, H_{S_1}\}$ of $M_1$ by the set of states $S_1$. As $S_1$ is weakly dense in the state space of $M_1$, this representation is a $C^*$-representation of $M_1$. Moreover, as each state $\sigma$ in $S_1$ is normal, the representation $\{M^{\#}_{1\sigma}, H_\sigma\}$ of $M_1$ by the state $\sigma$ is weakly closed. Hence $M^{\#}_{1S_1}$ on $H_{S_1}$ is weakly closed. For if a directed set $m^{\#}_{\alpha S_1}$ ($\alpha \in I$) in $M^{\#}_{1S_1}$ converges weakly to $m_0$ and $m^{\#}_{\alpha S_1} \leqq m_0$, a subfamily $m_{\alpha'}(\alpha' \in I')$ in $m_\alpha$ ($\alpha \in I$) converges weakly to $m$ in $M_1$ as $M_1$ is weakly closed. Put $m^{\#}_{S_1}$ be the image of $m$ in $M^{\#}_{1S_1}$ then clearly $m^{\#}_{S_1} = m_0$. We notice the representation of $A$ is weakly dense in $M^{\#}_{1S_1}$. Similarly we construct the representation $\{M^{\#}_{2S_2}, H_{S_2}\}$ of $M_2$ by $S_2$, which is weakly closed and contains the representation of $A$ as a weakly dense subalgebra. If $S(A^{\#_1}, H_1) = S(A^{\#_2}, H_2)$, the two representation of $A$ is unitarily equivalent, hence the weak closure of these representations of $A$ must be unitarily equivalent each other. That is, $M^{\#}_{1S_1}$ is unitarily equivalent to $M^{\#}_{2S_2}$. This assures the algebraic (normal) isomorphism between $M_1$ and $M_2$. q. e. d.

2) This theorem can be considered as an extension of the WECKEN-PLESSNER-ROKHLIN Theorem for non-commutative operator algebras. C. f. M. NAKAMURA and Z. TAKEDA. Normal states of commutative operator algebras, this journal Vol. 5 (1953) p. 116 Theorem 5 and Proposition 7.

LEMMA 1. *Let $M$ be the weak closure of a $C^*$-representation $A^\#$ of a $B^*$-algebra $A$ on a Hilbert space $H$ and $\sigma_\varphi$ be a state of $A$ defined by an element $\varphi$ of $H$ with norm unity. Then the restriction of $M$ on $[M\varphi]$, the invariant subspace spanned by $m\varphi\,(m \in M)$, is unitarily equivalent to the weak closure of the representation $\{A^\#_{\sigma_\varphi}, H_{\sigma_\varphi}\}$ of $A$ by the state $\sigma_\varphi$.*

PROOF. As $A^\#$ is dense in $M$ by the strong topology, for any $m \in M$ and $\varepsilon > 0$, there exists $a^\# \in A^\#$ such as $\|(m - a^\#)\varphi\| < \varepsilon$, hence $[A^\#\varphi] = [M\varphi]$. $A^\#$ is weakly dense in $M$ on the space $[M\varphi]$. As the restriction of $A^\#$ on $[A^\#\varphi]$ and the representation of $A$ by the state $\sigma_\varphi$ are unitarily equivalent, the weak closures of these algebras are unitarily equivalent each other. Hence the lemma is proved.

THEOREM 2. *Let $\{A^{\#_1}, H_1\}$, $\{A^{\#_2}, H_2\}$, $M_1, M_2$, $S(A^{\#_1}, H_1)$ and $S(A^{\#_2}, H_2)$ be same as in Theorem 1. Then if $S(A^{\#_1}, H_1) \supset S(A^{\#_2}, H_2)$, there exists a normal homomorphism $h$ from $M_1$ onto $M_2$ such as $h(a^{\#_1}) = a^{\#_2}$ for all $a \in A$.*

PPOOF. Put $S_1, S_2$ be the sets of all normal states of $M_1$ and $M_2$ respectively. Then by Theorem 1, it is sufficient to prove only for the representations $(M^\#_{1S_1}, H_{S_1})$ of $M_1$ and $(M^\#_{2S_2}, H_{S_2})$ of $M_2$ constructed by $S_1$ and $S_2$ respectively. As $S(A^{\#_1}, H_1) \supset S(A^{\#_2}, H_2)$, by Lemma 1 and the constructions of $H_1$ and $H_2$, there exists an invariant subspace in $H_{S_1}$ on which the restriction of the representation $(A^\#_{S_1}, H_{S_1})$ of $A$ is unitarily equivalent to the representation $(A^\#_{S_2}, H_{S_2})$. Since $M^\#_{1S_1}$, $M^\#_{2S_2}$ are weak closures of $A^\#_{S_1}$ and $A^\#_{S_2}$ on $H_{S_1}$ and $H_{S_2}$ respectively, $M^\#_{2S_2}$ on $H_{S_2}$ is unitarily equivalent to the restriction of $M^\#_{1S_1}$ on the invariant subspace in $H_{S_1}$. Thus we get the desired conclusion.

COROLLARY. *Let $W$ be the weak closure of the representation $A^\#_\Omega$ of a $B^*$-algebra $A$ on the Hilbert space $H_\Omega$ and $M$ be the weak closure of a $C^*$-representation $A^\#$ of $A$ on a Hilbert space $K$. Then there exists a normal homomorphic mapping of $W$ onto $M$.*

Thus $W$ is a $W^*$-algebra having a character to be named *the universal weak closure* of $A$. We notice here that the above stated $W$ has been used in the proof of Sherman's theorem in [8]. $W$, considered as a Banach space, is isomorphic to the double conjugate space of $A$.

As well known [1], [6], the ring of all bounded operators on a Hilbert space is isometrically isomorphic as a Banach space to the double conjugate space of the $C^*$-algebras composed of all completely continuous operators on that space. Then, does there exist for every $W^*$-algebra a $C^*$-algebra whose double conjugate space should be isomorphic to the $W^*$-algebra considered as a Banach space? The answer is negative even for factors as shown in the following:

PROPOSITION 1. *If a factor considered as a Banach space is isometrically isomorphic to the double conjugate space of a $C^*$-algebra, the factor is of type I.*

PROOF. Let $\Pi$ be the set of all pure states of a $C^*$-algebra $C$ and $M$ be the weak closure of the representation $(C^\#_\Pi, H_\Pi)$ of $C$ by $\Pi$. Then by the definition of the representation $(C^\#_\Pi, H_\Pi)$, the representation $(C^\#_\pi, H_\pi)$ by a state $\pi$ in $\Pi$ can be considered as a restriction of $C^\#_\Pi$ on an invariant subspace $H_\pi$ in $H_\Pi$ and the restriction of $M$ on $H_\pi$ is the weak closure of $C^\#_\pi$. As $\pi$ is a pure state, $C^\#_\pi$ is irreducible on $H_\pi$, hence the restriction of $M$ on $H_\pi$ is of type $I$. Put $e_\pi$ the projection to the manifold $H_\pi$ and $e^0_\pi$ the smallest projection in the centre of $M$ such as $e_\pi \leqq e^0_\pi$. Then $e_\pi \in M'$ and, as well known [5], $Me_\pi$ is algebraically $*$-isomorphic to $Me^0_\pi$. Thus $Me^0_\pi$ is of type I. As the supremum of $e^0_\pi(\pi \in \Pi)$ is 1, $M$ is of type $I$.

If a factor $A$ as a Banach space is isometrically isomorphic to the double conjugate space of a $C^*$-algebra $C$, $A$ is isomorphic or anti-isomorphic to the weak closure $W$ of $A^\#_\Omega$ on $H_\Omega$ [3; Theorem 14]. But as the type of factor is invariant for anti-isomorphism, we can assume $A$ is isomorphic to $W$. Then by Theorem 2 there exists a normal homomorphism of $A$ onto $M$. But there is no normal homomorphism except an isomorphism for a factor since every factor has no non-trivial weakly closed two-sided ideal. Therefore, if exists such a $C^*$-algebra, $A$ must be of type $I$. q. e. d.

3. **Direct product of operator algebras.** T. Turumaru has defined the direct product $A_1 \times A_2$ of two $C^*$-algebras $A_1, A_2$ and has shown the uniqueness of the product [10]. This means that the algebraical structure of the product does not depend on the choice of the underlying spaces on which the component algebras act as operators. For two $W^*$-algebras $A_1$ and $A_2$ on Hilbert spaces $H_1$ and $H_2$ respectively, the $C^*$-direct product of $A_1$ and $A_2$ in the sense of Turumaru can be seen as a $C^*$-algebra on the Hilbert space $H_1 \times H_2$. Hence its weak closure on $H_1 \times H_2$ is naturally considered as a direct product of two $W^*$-algebras $A_1$ and $A_2$. In the followings we denote this product by $A_1 \otimes A_2$. Recently Y. Misonou has proved that the algebrical structure of $A_1 \otimes A_2$ does not depend on the underlying spaces $H_1$ and $H_2$ similarly as the $C^*$-algebra case [4]. That is, if $A_1, A_2$ are represented as $W^*$-algebras on another Hilbert spaces $K_1, K_2$ respectively, the direct product $A_1 \otimes A_2$ on $H_1 \times H_2$ is algebriacally $*$-isomorphic to the product on $K_1 \times K_2$. As an application of Theorem 1, we give here an alternative proof of this theorem. T. Turumaru has given an another proof of Misonou's theorem depending on the cross-space theory [10].

Let $\rho$ and $\sigma$ be states of $C^*$-algebras $A$ and $B$ respectively then $\rho \times \sigma$ is a state on $A \times B$ such as

$$(\rho \times \sigma)\left(\sum_{k=1}^{n} a_k \times b_k\right) = \sum_{k=1}^{n} \rho(a_k)\sigma(b_k)$$

for elements of the form $\sum_{k=1}^{n} a_k \times b_k$ in $A \times B$ [10].

LEMMA 2. *If $A$, $B$ are $C^*$-algebras with* 1 *and acting on Hilbert spaces $H, K$ respectively, $\rho, \sigma$ are distinguished states of $A$ on $H$ and of $B$ on $K$*

*respectively, then $\rho \times \sigma$ is a distinguished state of $A \times B$ acting on $H \times K$.*

PROOF. By the definition of the distinguished state, there exist two sequences $\{\varphi_i\}$ and $\{\psi_j\}$ of elements in $H$ and $K$ respectively satisfying

$$\rho(a) = \sum_{i=1}^{\infty} < a\varphi_i, \varphi_i > \qquad \text{for} \quad a \in A \ \text{ and} \sum_{i=1}^{\infty} \|\varphi_i\|^2 = 1,$$

$$\sigma(b) = \sum_{j=1}^{\infty} < b\psi_j, \psi_j > \qquad \text{for} \quad b \in B \ \text{ and} \sum_{j=1}^{\infty} \|\psi_i\|^2 = 1.$$

Then

$$(\rho \times \sigma)\left(\sum_{k=1}^{n} a_k \times b_k\right) = \sum_{k=1} \rho(a_k)\sigma(b_k)$$

$$= \sum_{k=1}^{n}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} < a_k\varphi_i, \varphi_i > < b_k\psi_j, \psi_j >$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{n} < a_k\varphi_i \times b_k\psi_j, \ \varphi_i \times \psi_j >$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} < \left(\sum_{k=1}^{n} a_k \times b_k\right)\varphi_i \times \psi_i. \varphi_i \times \psi_i >$$

As $\rho \times \sigma$ is a state of $A \times B$

$$(\rho \times \sigma)(1 \times 1) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} < \varphi_i \times \varphi_j, \varphi_i \times \varphi_j > = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \|\varphi_i \times \psi_i\|^2 = 1.$$

$A \odot B$, the set of all elements of the form $\sum_{k=1}^{m} a_k \times b_k$, is uniformly dense in $A \times B$. Hence for every $c$ in $A \times B$,

$$(\rho \times \sigma)(c) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} < c(\varphi_i \times \psi_j), \ \varphi_i \times \psi_j > .$$

That is, $\rho \times \sigma$ is a distinguished state of $A \times B$ acting on the Hilbert space $H \times K$.

LEMMA 3. *Let $\rho$ and $\sigma$ be states of $C^*$-algebras $A$ and $B$ and $\{A^\#_\rho, H_\rho\}$, $\{B^\#_\sigma, H_\sigma\}$ be the representations of $A$ and $B$ by states $\rho$ and $\sigma$ respectively. Then the representation of $A \times B$ by the state $\rho \times \sigma$ is unitarily equivalent to the $C^*$-direct product $A^\#_\rho \times B^\#_\sigma$ on $H_\rho \times H_\sigma$.*

When we construct a representation of $A$ by a state $\sigma$, there exists a linear mapping from $A$ into the representative space $H_\sigma$. By $a^\theta_\sigma$ denote the image of $a \in A$ by this mapping.

PROOF. For elements $\sum_{i=1}^{m} a^\theta_{i\rho} \times b^\theta_{i\sigma}$, $\sum_{j=1}^{n} a^\theta_{j\rho} \times b^\theta_{j\sigma}$ in $H_\rho \times H_\sigma$,

$$< \sum_{i=1}^{m} a^\theta_{i\rho} \times b^\theta_{i\sigma}, \ \sum_{j=1}^{n} a^\theta_{j\rho} \times b^\theta_{j\sigma} > = \sum_{i=1}^{m}\sum_{j=1}^{n} < a^\theta_{i\rho} \times b^\theta_{i\sigma}, \ a^\theta_{j\rho} \times b^\theta_{j\sigma} >$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} < a_{i\rho}^{\theta}, a_{j\rho}^{\theta} > \ < b_{i\sigma}^{\theta}, b_{j\sigma}^{\theta} > = \sum_{i=1}^{m}\sum_{j=1}^{n} \rho(a_j^* a_i)\sigma(b_j^* b_i).$$

On the other hand,

$$(\rho \times \sigma)\left(\left(\sum_{j=1}^{n} a_j \times b_j\right)^* \left(\sum_{i=1}^{m} a_i \times b_i\right)\right)$$

$$= (\rho \times \sigma)\left(\left(\sum_{j=1}^{n} a_j^* \times b_j^*\right)\left(\sum_{i=1}^{m} a_i \times b_i\right)\right)$$

$$= (\rho \times \sigma)\left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_j^* a_i \times b_j^* b_i\right)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} \rho(a_j^* a_i)\sigma(b_j^* b_i).$$

Hence the correspondence $\left(\sum_{i=1}^{m} a_i \times b_i\right)_{\rho\times\sigma}^{\theta} \in H_{\rho\times\sigma}$ to $\sum_{i=1}^{n} a_{i\rho}^{\theta} \times b_{i\rho}^{\theta} \in H_\rho \times H_\sigma$ can be extended to an isometric transformation $u$ between $H_{\rho\times\sigma}$ and $H_\rho \times H_\sigma$.

Furtheremore, since

$$< (x_\rho^{\#} \times y_\sigma^{\#}) \sum_{i=1}^{m} a_{i\rho}^{\theta} \times b_{i\sigma}^{\theta}, \ \sum_{j=1}^{n} a_{j\rho}^{\theta} \times b_{j\sigma}^{\theta} >$$

$$= < \sum_{i=1}^{m} (xa_i)_\rho^{\theta} \times (yb_i)_\sigma^{\theta}, \ \sum_{j=1}^{n} a_{j\rho}^{\theta} \times b_{j\sigma}^{\theta} >$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} \rho(a_j^* x a_i)\, \sigma(b_j^* y b_i)$$

$$= (\rho \times \sigma)\left(\sum_{i=1}^{m}\sum_{j=1}^{n} (a_j^* x a_i) \times (b_j^* y b_i)\right)$$

$$= (\rho \times \sigma)\left(\left(\sum_{j=1}^{n} a_j \times b_j\right)^* \left(\sum_{i=1}^{m} x a_i \times y b_i\right)\right)$$

$$= < (x \times y)_{\rho\times\sigma}^{\#}\left(\sum_{i=1}^{m} a_i \times b_i\right)_{\rho\times\sigma}^{\theta}, \ \left(\sum_{j=1}^{m} a_j \times b_j\right)_{\rho\times\sigma}^{\theta} >,$$

$$(x \times y)_{\rho\times\sigma}^{\#} = u^{-1}(x_\rho^{\#} \times y_\sigma^{\#})u.$$

As the elements of the forms $\sum_{i=1}^{m} (x_i \times y_i)_{\rho\times\sigma}^{\#}$, $\sum_{i=1}^{n} x_{i\rho}^{\#} \times y_{i\sigma}^{\#}$ are uniformly dense subalgebra in $(A \times B)_{\rho\times\sigma}^{\#}$, $A_\rho^{\#} \times B_\sigma^{\#}$ respectively, the proof is easily concluded.

LEMMA 4. *If a Hilbert space $H$ is a direct sum of subspaces $H_\alpha$ $(\alpha \in I)$ and a Hilbert space $K$ is a direct sum of subspaces $K_\beta$ $(\beta \in J)$, then the direct product space $H \times K$ is the direct sum of subspaces $H_\alpha \times K_\beta$ $(\alpha \in I, \beta \in J)$.*

LEMMA 5. *Let $R = \{\rho_i, (i \in I)\}$ and $S = \{\sigma_j, (j \in J)\}$ be collections of states of a $C^*$-algebra $A$ on a Hilbert space $H$ and of a $C^*$-algebra $B$ on a Hilbert space $K$ respectively such that the $C^*$-algebra $A$ acting on $H$ is unitarily* *equivalent to the representation* $(A^{\#}_{R}, H_R)$ *by* $R$ [9 ; Theorem 2] *and the* $C^*$-*algebra* $B$ *acting on* $K$ *is unitarily equivalent to the representation* $(B^{\#}_{S}, H_S)$ *by* $S$. *Denote by* $R \odot S$ *the set of states of* $A \times B$ *which can be expressed as* $\rho \times \sigma (\rho \in R, \sigma \in S)$, *then the* $C^*$-*algebra* $A \times B$ *on the Hilbert space* $H \times K$ *is unitarily equivalent to the representation* $\{(A \times B)^{\#}_{R \odot S}, H_{R \odot S}\}$ *of* $A \times B$ *by* $R \odot S$.

PROOF. As the product space of invariant subspaces in $H$ and in $K$ is invariant in $H \times K$ for the product $A \times B$, the lemma is evident from Lemma 3 and Lemma 4.

THEOREM 3. (Misonou [4 ; Theorem 1]). *If* $A_1$ *is a* $W^*$-*algebra on Hilbert spaces* $H_1$ *and* $K_1$ *and if* $A_2$ *is a* $W^*$-*algebra on Hilbert spaces* $H_2$ *and* $K_2$. *Then the direct product* $A_1 \otimes A_2$ *of* $A_1$ *and* $A_2$ *on* $H_1 \times H_2$ *is algebraically* $*$-*isomorphic to the product of* $A_1$ *and* $A_2$ *on* $K_1 \times K_2$.

PROOF. Let $S(A, H_1)$, $S(B, K_1)$ and $S(A \times B, H_1 \times K_1)$ be the set of all distinguished states of $A$ acting on $H$, of $B$ on $K_1$ and that of the $C^*$-direct product $A \times B$ on $H_1 \times K_1$. By Lemma 2 if $\rho \in S(A, H_1)$ and $\sigma \in S(B, K_1)$, then $\rho \times \sigma \in S(A \times B, H_1 \times K_1)$. Hence $S(A \times B, H_1 \times K_1)$ contains $S(A, H_1) \odot S(B, K_1)$. Thus

$$S(A \times B, H_1 \times K_1) \supset [S(A, H_1) \odot S(B, K_1)]$$

where the bracket means the smallest subset in the state space of $A \times B$ which contains $S(A, H_1) \odot S(B, K_1)$ and satisfies the conditions (ii)-(iv) in the introduction.

On the other hand, since a state $\sigma_{\varphi \times \psi}$ on $A \times B$ such as

$$\sigma_{\varphi \times \psi}(x) = \langle x (\varphi \times \psi), \varphi \times \psi \rangle \qquad \text{for } x \in A \text{ and } \|\varphi \times \psi\| = 1$$

(where $\varphi \in H_1, \psi \in K_1$ $\|\varphi\| = \|\psi\| = 1$), is contained in $[S(A, H_1) \odot S(B, K_1)]$ and by Lemma 5 there exists a collection $T$ of states of the from $\sigma_{\varphi \times \psi}$ in the state space of $A \times B$ such as

$$[T] = S(A \times B, H_1 \times K_1), \quad [9 ; \text{Theorem } 2],$$

we get

$$S(A \times B, H_1 \times K_1) \subset [S(A, H_1) \odot S(B, K_1)].$$

Hence

$$S(A \times B, H_1 \times K_1) = [S(A, H_1) \odot S(B, K_1)].$$

Since $S(A, H)$ is nothing but the set of all normal states for the $W^*$-algebra $A$ and the normality of a state of a $W^*$-algebra is purely algebraical [2], $S(A, H_1) \odot S(B, K_1)$ is independent to $H_1$ and $K_1$. Therefore,

$$S(A \times B, H_1 \times K_1) = S(A \times B, H_2 \times K_2),$$

Then by Theorem 1, $A \otimes B$ on $H_1 \times K_1$ is algebraically $*$-isomorphic to $A \otimes B$ on $H_2 \times K_2$. q. e. d.

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