ON THE DIRECT PRODUCT OF FINITE FACTORS

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The notion of the direct product of W^*-algebras, originally introduced by F. J. Murray and J. von Neumann [3] for factors of type (I), is recently generalized for arbitrary W^*-algebras by Y. Misonou in his preceding [2]. He proved, among others, the definition of the direct product is space-free, whereas his definition heavily depends on the direct product of the underlying Hilbert spaces on which the given algebras act. From a point of view, to study the algebraical structure of W^*-algebras apart from the spatial structure, the above cited situation is somewhat unpleasant, and a purely algebraical definition is desirable[1].

On the other hand, in the theory of classical algebras, the direct product can be introduced, roughly speeking, as follows: A normally simple algebra A is the direct product of normally simple algebras B and C if A contains B and C as subalgebras[2] which satisfy

PROPERTY I. *B and C commute in elementwise;*

PROPERTY II. *B and C generate A.*

This definition of classical algebra can not adapt for the direct product of factors (i. e., normally simple W^*-algebras in some sense) in the sense of Misonou without further restrictions, since a pair of commutable factors B and $C = B'$ in the full operator algebra A on a Hilbert space H satisfy the above Properties I and II, and since $A = B \otimes C$, (in the sense of Misonou) can not be true if B is of a factor of type (II_1) by theorems of Murray-von Neumann [3; Chap. XI] and Misonou [2; Thm. 3].

However, for finite factors the above definition of the direct product of classical algebra coincides with that of Y. Misonou. The purpose of the present note is to show this in the following[3][4]

THEOREM. *If A, B and C are finite factors, then A is the direct product of B and C in the sense of Misonou if and only if they satisfy the above Properties I and II.*

Since "only if" part is obvious by Misonou's results, it suffices to prove "if" part. The proof will be divided into the following sequence of lemmas. Throughout the remainder we shall assume Properties I and II.

LEMMA 1. *If $b \in B$ and $c \in C$, then*

1) It seems to the author that an algebraical definition is given by T. Turumaru [8].

2) Usually, the existence of subalgebras, which are isomorphic to B and C, is required. However, we shall identify them for notational convenience.

3) Compare with Murray-von Neumann [3; Lemma 11. 1. 1].

4) In the theorem, we shall understand that the term "generate" means the generation as W^*-algebra.

$$\tau(bc) = \tau(b)\tau(c) \tag{1}$$

where τ denotes the normalized Trace of A.

PROOF. Without loss of the generality we can assume $c \geqq 0$. Put $\sigma(x) = \tau(xc)$ for $x \in B$. Then σ is a positive linear functional on B and

$$\sigma(ab) = \tau(abc) = \tau(acb) = \tau(bac) = \sigma(ba)$$

for any pair of a and b in B, whence σ is a trace on B by a theorem of Murray-von Neumann [4; Chap. II]. Hence by the unicity of the trace of finite factors, there exists a constant $\gamma \geqq 0$ such that $\sigma(x) = \gamma\tau(x)$. Therefore $\gamma = \tau(c)$ implies (1) which is desired.[5]

LEMMA 2. *If D is the set of all elements of A such that*

$$x = \sum_{i=1}^{n} b_i c_i \qquad \text{for } b_i \in B,\ c_i \in C, \tag{2}$$

then D is algebraically isomorphic to $B \odot C$. Moreover, the isomorphism preserves the value of the traces:

$$\tau\left(\sum_{i=1}^{n} b_i c_i\right) = \tau\left(\sum_{i=1}^{n} b_i \times c_i\right) \tag{3}$$

where $b_i \in B$ and $c_i \in C$.

PROOF. It suffices to show that

$$\sum_{i=1}^{n} b_i c_i = 0 \text{ implies } \sum_{i=1}^{n} b_i \times c_i = 0 \tag{4}$$

for the first half. Let $b \in B$ and $c \in C$, and put $\sigma(x) = \tau(xb)$ and $\rho(y) = \tau(yc)$ for $x \in B$ and $y \in C$ respectively. Then

$$\sigma \times \rho\left(\sum_{i=1}^{n} b_i \times c_i\right) = \sum_{i=1}^{n} \sigma(b_i)\rho(c_i) = \sum_{i=1}^{n} \tau(b_i b)\tau(c_i c) = \sum_{i=1}^{n} \tau(b_i c_i bc) = 0$$

by the hypothesis and Lemma 1. Since such σ's and ρ's are fundamental in B_* and C_* respectively which are easily follows from a result of J. Dixmier [1], a theorem of R. Schatten [6; Thm. 1] implies our (4). The remainder is a consequence of Lemma 1 and a theorem of Y. Misonou [2; Thm. 4].

LEMMA 3. *Representing A standardly on H with a separating and generating vector φ, consider the closures E and F of φB and φC respectively. Then B and C are represented standardly on E and F respectively. Moreover, H is the direct product of E and F.*

PROOF. The first half of Lemma is obvious. Since by Lemma 2

$$\|\varphi \sum_{i=1}^{n} b_i c_i\|^2 = \sum_{i,j=1}^{n} \tau(b_i b_j)\tau(c_i c_j) = \|\sum_{i=1}^{n} \varphi b_i \times \varphi c_i\|^2,$$

and since φD is dense in H by the assumption, it remains to shows that $\varphi B \odot \varphi C$ is dense in $E \times F$. This follows from an easy computation.[6]

PROOF OF THE THEOREM. Since A, B and C act standardly on H, E and

5) Lemma 1 shows that B and C of the theorem is *uncorrelated.*

6) Lemma 3 follows also from a Theorem of R. Pallu de la Barrier [5].

F respectively by Lemma 3, and since A and $B \otimes C$ are the weak closure of D and $B \odot C$ on H and $E \times F$ respectively, Misonou's definition implies our statement. This completes the proof.

REMARK. After the material of the present note is read before the Seminar, T. Turumaru pointed out that the present proof is also applicable for *uncorrelate* subalgebras B and C with Properties I and II in a probability W^*-algebra A (i. e., a finite W^*-algebra having a faithful normal trace τ), where B and C are uncorrelate if and only if the statement of Lemma 1 is true.

An alternative definition of the direct product of W^*-algebras can be given in an analogous manner to R. Sikorski [7] as follows: A W^*-algebra A is the *Sikorski product* of W^*-subalgebras B and C if and only if (1) B and C generate A as W^*-algebra, (2) for any normal homomorphisms ϕ_B and ϕ_C of B and C into an arbitrary W^*-algebra D, there exists a normal homomorphism ϕ of A into D such that ϕ is the common extension of ϕ_B and ϕ_C. For finite factors A, B and C, the Sikorski product coincides with our direct product, however, in the present stage, the author can not prove the coincidence for arbitrary W^*-algebras.

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