ON THE ASPHERICITY OF THE HIGHER DIMENSIONAL COMPLEX

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1. W. H. Cockcroft [1] discussed the non-asphericity of the two-dimensional complex K, which is composed of a given non-aspherical two-dimensional complex L, and two-dimensional cells attached to L. According to his consequences, if $\pi_1(L)$ is (i) Abelian or (ii) a finite group or (iii) a free group, or if L contains only one two-dimensional cell, then K is non-aspherical.

In the present note, we consider the n-dimensional complex K $(n \geqq 3)$, which is composed of a given non-aspherical n-dimensional complex L, and n-dimensional cells attached to it. In this case, we shall prove the asphericity of K in the complete form; namely, K is aspherical, if and only if $\pi_r(L) = 0$ for $1 < r < n-1$ when $n \geqq 4$, $\pi_{n-1}(L)$ is a non-zero free $\pi_1(L)$-module and $H_n(\widetilde{L}) = 0$, where $\widetilde{L}$ is the universal covering complex L of. Then, it is shown that $\widetilde{L}$ is of the same homotopy type as a set of $(n-1)$-spheres having a point in common.

2. Let L be a connected, n-dimensional CW-complex [3] $(n \geqq 3)$. We shall say, following Hurewicz [2], that L is aspherical, if and only if its homotopy groups satisfy the conditions

$$\pi_r(L) = 0 \qquad (r > 1). \tag{2.1}$$

LEMMA (2.2). *L is aspherical, if and only if*

$$\pi_r(L) = 0 \qquad (1 < r \leqq n). \tag{2.3}$$

In fact, we need only to show the sufficiency. From (2.3), we obtain, using the Hurewicz' theorem,

$$\pi_{n+1}(L) \approx \pi_{n+1}(\widetilde{L}) \approx H_{n+1}(\widetilde{L}) = 0,$$

where $\widetilde{L}$ is the universal covering complex of L, and $H_{n+1}(\widetilde{L})$ is its integral homology group. Using the same arguments as above, we get inductively (2.1) for every $r > n$.

Next, let K be a complex such that

$$K = L \cup \{e_i^n\} \tag{2.4}$$

where $\{e_i^n\}$ is a set of n-cells attached to the $(n-1)$-skeleton of L.

LEMMA (2.5). *If L is a non-aspherical complex such that $n \geqq 4$, and if*

$$\pi_r(L) \neq 0 \qquad (1 < r < n-1),$$

for at least one r, then K is non-aspherical.

In fact, we can easily see the non-asphericity of K from a part of the exact homotopy sequence

$$0=\pi_{r+1}(K,L)\rightarrow\pi_r(L)\rightarrow\pi_r(K).$$

LEMMA (2.6). *If L is non-aspherical, and if*

$$\pi_{n-1}(L)=0 \qquad (n\geqq 3),$$

then K is non-aspherical.

In fact, we assume that K is aspherical. Then, from a part of the exact homotopy sequence

$$0=\pi_n(K)\rightarrow\pi_n(K,L)\rightarrow\pi_{n-1}(L)=0,$$

we see that $\pi_n(K,L)=0$. This holds only when $K=L$. As L is non-aspherical, this contradicts to the hypothesis that K is aspherical.

Now, under the construction of (2.4), let us take a set of elements $\{\alpha_i\}$ of $\pi_{n-1}(L)$ as follows: Let the characteristic map for e_i^n be $f_i: E^n\rightarrow \bar{e}_i^n$, and let $x_0\in\dot{E}^n$ be a fixed point. Let us take a fixed path ρ_i from $f_i(x_0)$ to a fixed point y_0 of L, for every i. Then ρ_i can be considered as a homotopy of $f_i(x_0)$, which can be extended to a homotopy from $\dot{E}^n$ into L. The terminal map of this homotopy represents an element of $\pi_{n-1}(L)$ with reference points x_0 and y_0, which we shall call α_i.

LEMMA (2.7). *Let K of* (2.4) *be aspherical. Then $\pi_{n-1}(L)$ is a free $\pi_1(L)$-module with the basis $\{\alpha_i\}$.*

PROOF. Let $\widetilde{K}$ be the universal covering complex of K, and let $\widetilde{L}$ be its part over L. Then $\widetilde{L}$ is evidently the universal covering complex of L. Let the reference point of $\pi_n(\widetilde{K},\widetilde{L})$, $\pi_{n-1}(\widetilde{L})$ etc. be $\widetilde{y}_0\in p^{-1}y_0$, where p is the projection. Let $\bigcup_q\{\widetilde{e}_{i,q}^n\}$ be n-cells of $\widetilde{K}$ which cover e_i^n, and whose indices i,q are given as follows: The boundary map of $\widetilde{e}_{i,q}^n$ together with a suitable path from $\widetilde{f}_{i,q}(x_0)$ to $\widetilde{y}_0$ is projected to a map representing $\xi_q\cdot\alpha_i$ ($\xi_q\in\pi_1(L)$), where $\widetilde{f}_{i,q}$ is the characteristic map for $\widetilde{e}_{i,q}^n$. Evidently we obtain

$$\widetilde{K}=\widetilde{L}\bigcup_{i,q}\{\widetilde{e}_{i,q}^n\}. \tag{2.8}$$

Therefore $\pi_n(\widetilde{K},\widetilde{L})\approx H_n(\widetilde{K},\widetilde{L})$ is the free Abelian group with generators $\{\alpha_{i,q}\}$ corresponding one to one with $\{\widetilde{e}_{i,q}^n\}$. If K is aspherical, then, from a part of the exact homotopy sequence

$$\begin{array}{c} 0=\pi_n(\widetilde{K})\rightarrow\pi_n(\widetilde{K},\widetilde{L})\xrightarrow{d}\pi_{n-1}(\widetilde{L})\rightarrow\pi_{n-1}(\widetilde{K})=0 \\ \qquad\qquad\qquad p\downarrow \\ \qquad\qquad\qquad \pi_{n-1}(L), \end{array}$$

we obtain

$$pd:\ \pi_n(\widetilde{K},\widetilde{L})\approx\pi_{n-1}(L),$$

where d is the homotopy boundary homomorphism, and p is the isomorphism induced by p itself. From the construction, it is evident that $pd(\alpha_{i,q})=\xi_q\cdot\alpha_i$, which shows that $\{\xi_q\cdot\alpha_i\}$ ($\xi_q\in\pi_1(L)$) constitute a set of free generators of $\pi_{n-1}(L)$. Namely, $\pi_{n-1}(L)$ is a free $\pi_1(L)$-module with the basis $\{\alpha_i\}$ [4].

LEMMA (2.9). *If K of (2.4) is aspherical, then*

$$H_n(\widetilde{L}) = 0.$$

In fact, we obtain the required result from a part of the exact homology sequence

$$0 = H_{n+1}(\widetilde{K}, \widetilde{L}) \to H_n(\widetilde{L}) \to H_n(\widetilde{K}) \approx \pi_n(\widetilde{K}) = 0.$$

3. We shall prove the following main result in this note:

THEOREM (3.1). *Let L be non-aspherical. Then n-cells $\{e_i^n\}$ can be attached to L so that the resulting complex K of (2.4) is aspherical, if and only if*

(i) *when $n \geqq 4$, $\pi_r(L) = 0$ $(1 < r < n-1)$;*

(ii) *$\pi_{n-1}(L)$ is a non-zero free $\pi_1(L)$-module with a basis $\{\alpha_i\}$;*

(iii) $H_n(\widetilde{L}) = 0$.

PROOF. As the necessity is an immediate consequence of Lemmas (2.5), (2.6), (2.7) and (2.9), we shall prove the sufficiency.

Let $f_i: (\dot{E}^n, x_0) \to (L, y_0)$ be a representing map of α_i, and let $\widetilde{f}_{i,q}: (\dot{E}^n, x_0) \to (\widetilde{L}, p^{-1}y_0)$ be its covering map, where q is an index such that every path ρ_q from $\widetilde{y}_0$ to $\widetilde{f}_{i,q}(x_0)$ is projected by p to an element $\xi_q \in \pi_1(L)$. Let $\beta_{i,q}$ be an element of $\pi_{n-1}(\widetilde{L})$ represented by $\widetilde{f}_{i,q}$ together with ρ_q^{-1}. It is evident that $\beta_{i,q}$ is projected to $\xi_q \cdot \alpha_i \in \pi_{n-1}(L)$ isomorphically by p. Let e_i^n be an n-cell attached to L by f_i, and let $\widetilde{e}_{q,i}^n$ be attached to $\widetilde{L}$ by $\widetilde{f}_{i,q}$. Then $\widetilde{e}_{i,q}^n$ represents a generator $\alpha_{i,q}$ of the free Abelian group $\pi_n(\widetilde{K}, \widetilde{L}) \approx H_n(\widetilde{K}, \widetilde{L})$, where $\widetilde{K}$ is given as in (2.8). As $\pi_n(\widetilde{K}, \widetilde{L})$ and $\pi_{n-1}(\widetilde{L}) \approx \pi_{n-1}(L)$ are free Abelian groups, whose generators satisfy the condition

$$d\alpha_{i,q} = \beta_{i,q},$$

d is an isomorphism onto. Therefore from a part of the exact homotopy sequence

$$\pi_n(\widetilde{K}, \widetilde{L}) \xrightarrow{d} \pi_{n-1}(\widetilde{L}) \to \pi_{n-1}(\widetilde{K}) \to \pi_{n-1}(\widetilde{K}, \widetilde{L}) = 0,$$

we obtain

$$\pi_{n-1}(K) \approx \pi_{n-1}(\widetilde{K}) = 0. \tag{3.2}$$

Next, let us consider a diagram

$$\begin{array}{ccc} \pi_n(\widetilde{K}, \widetilde{L}) & \xrightarrow{d} & \pi_{n-1}(\widetilde{L}) \\ T\downarrow & & \downarrow T \\ H_n(\widetilde{K}, \widetilde{L}) & \xrightarrow{\partial} & H_{n-1}(\widetilde{L}), \end{array} \tag{3.3}$$

where ∂ is the homology boundary homomorphism, and T is the Hurewicz isomorphism. As d is an isomorphism onto, and as the commutativity holds in (3.3), ∂ is an isomorphism onto. Therefore, from a part of the exact homology sequence

$$0 = H_n(\widetilde{L}) \to H_n(\widetilde{K}) \to H_n(\widetilde{K}, \widetilde{L}) \overset{\partial}{\to} H_{n-1}(\widetilde{L}),$$

we obtain $H_n(\widetilde{K}) = 0$ using (iii), which shows

(3.4) $$\pi_n(K) \approx \pi_n(\widetilde{K}) \approx H_n(\widetilde{K}) = 0$$

from (3.2) and the Hurewicz' isomorphism. On the other hand we obtain $\pi_r(L) \approx \pi_r(K)$ for $1 < r < n-1$ when $n \geqq 4$. Therefore from (i), (3.2), (3.4) and from Lemma (2.2), we conclude that K is aspherical.

COROLLARY (3.5). *If L is an n-dimensional compact manifold with finite $\pi_1(L)$, any attaching of n-cells to L does not generate an aspherical complex.*

We can see that L is non-aspherical. In fact, if $\pi_r(L) = 0$ for $1 < r < n$, we obtain $\pi_n(\widetilde{L}) \approx H_n(\widetilde{L}) \neq 0$, as $\widetilde{L}$ is compact and orientable. So, from (iii) of Theorem (3.1), we can see the conclusion.

COROLLARY (3.6). *An $(n-1)$-dimensional real projective space P^{n-1} $(n \geqq 3)$ cannot be attached by n-cells so that the resulting complex is aspherical.*

As $\widetilde{P}^{n-1} = S^{n-1}$, the $(n-1)$-sphere, the conditions (i) and (iii) of Theorem (3.1) are satisfied. On the other hand, (ii) is not satisfied. In fact, $\pi_{n-1}(P^{n-1})$ is not a free $\pi_1(P^{n-1})$-module, but a relation $\xi\cdot\alpha + (-1)^n\alpha = 0$ holds good for the generator ξ of $\pi_1(P^{n-1})$ and α of $\pi_{n-1}(P^{n-1})$.

4. In this section, we shall determine the homotopy type of L, which satisfies the conditions of Theorem (3.1).

THEOREM (4.1). *If a connected non-aspherical n-dimensional complex L can be attached by n-cells $\{e_i^n\}$ so that the resulting complex K of (2.4) is aspherical, then $\widetilde{L}$ is of the same homotopy type as the $(n-1)$-spheres having a point in common.*

PROOF. Let us assume that L satisfies (i), (ii) and (iii) of Theorem (3.1). Let Z be the set of $(n-1)$-spheres $\bigcup S_{q,i}^{n-1}$ having a point z_0 in common, where the indices (q, i) correspond one to one with (ξ_q, α_i), $\{\alpha_i\}$ being a basis given by (ii) of Theorem (3.1). Now, we shall define a map

$$g : Z \to \widetilde{L}$$

such that

$$g \mid S_{q,i}^{n-1} : (S_{q,i}^{n-1}, z_0) \to (\widetilde{L}, \widetilde{y}_0)$$

represents an element of $\pi_{n-1}(\widetilde{L})$, which is the inverse image of $\xi_q \cdot \alpha_i$ by the projection p.

Using the Hurewicz' theorem, the only non-trivial group $H_i(\widetilde{L})$ $(i \geqq 1)$ is $H_{n-1}(\widetilde{L})$ from (i), (ii) and (iii); and $H_{n-1}(\widetilde{L})$ is the free Abelian group, whose generators correspond one to one with $\{\xi_q \cdot \alpha_i\}$. Evidently $H_{n-1}(Z)$ is mapped isomorphically onto $H_{n-1}(\widetilde{L})$ by the induced homomorphism by g. Therefore, from [3, Theorem 3], g is a homotopy equivalence.

From Corollary (3.6), we can see that the condition of Theorem (4.1) is not sufficient.

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