RIEMANN-CESÀRO METHODS OF SUMMABILITY II*)

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1. Introduction. In the previous paper [3], we defined Riemann-Cesàro method of summability which includes well-known Riemann methods of summability (R,p) and (R_p). In this paper, we shall consider some Tauberian theorems for this summability.

In terms of standard notations used by Zygmund [10; p.42] and others, Cesàro transform of order α of $\sum a_n$ is defined by $\sigma_n^\alpha = s_n^\alpha / A_n^\alpha$ where s_n^α and A_n^α are given by the relations

$$\sum_{n=0}^{\infty} A_n^\alpha x^n = (1-x)^{-\alpha-1} \text{ and } \sum_{n=0}^{\infty} s_n^\alpha x^n = \frac{\sum_{n=0}^{\infty} s_n x^n}{(1-x)^\alpha} = \frac{\sum_{n=0}^{\infty} a_n x^n}{(1-x)^{\alpha+1}}.$$

It is well-known that $A_n^\alpha \sim n^\alpha/\Gamma(\alpha+1)$, $\alpha \neq -1, -2, \dots,$ as $n \to \infty$. A series is said to be evaluable (C,α) to s if $\sigma_n^\alpha \to s$ as $n \to \infty$. In the following, let α be a real number, not necessarily an integer, for which $\alpha \geqq -1$ and let p be a positive integer. A series $\sum_{n=1}^{\infty} a_n$ is said to be evaluable to zero by Riemann-Cesàro method of order p and index α, or briefly, to be evaluable (R,p,α) to zero, if the series

$$t^{\alpha+1} \sum_{n=1}^{\infty} s_n^\alpha \left(\frac{\sin nt}{nt}\right)^p$$

converges in some interval $0 < t < t_0$ and its sum tends to zero as $t \to 0$. Under these definitions, summability $(R,p,-1)$ and $(R,p,0)$ is reduced to summability (R,p) and (R_p), respectively. It is known [3] that summability (R,p,α), when $-1 \leqq \alpha < p-1$ and $p \geqq 2$, is regular, or more precisely, summability $(C, p-1-\delta)$, $0<\delta<1$, implies summability (R,p,α) when $-1 \leqq \alpha < p-1-\delta$, while summability $(R,1,\alpha)$ is not regular when $-1 \leqq \alpha \leqq 0$.

Concerning summability (R,p), Kanno [5] proved the following

THEOREM K. *Let p be a positive integer. Suppose that*

$$s_n^\beta = o(n^\gamma), \tag{1.1}$$

when $0 < \gamma < \beta$, *and*

$$\sum_{\nu=n}^{\infty} \frac{|a_\nu|}{\nu} = O(n^{-(1-\delta)}), \tag{1.2}$$

*) This paper is a continuation of the previous paper [3]. Cf. R. P. Agnew, Properties of generalized definitions of limit, Bull. Amer. Math. Soc., 45 (1939), 689-730.

when $0<\delta<1$ *and* $\delta=p(\beta-\gamma)/(\beta+1-p)$. *Then, the series* $\sum_{n=1}^{\infty}a_n$ *is evaluable* (R,p) *to zero.*

Concerning summability (R,p,α), we shall prove the following

THEOREM 1. *Let* p *be a positive odd integer. If the conditions* (1.1) *and* (1.2) *are satisfied, then the series* $\sum_{n=1}^{\infty}a_n$ *is evaluable* (R,p,α) *to zero when* $-1\leqq\alpha\leqq 0$.

In this Theorem, we may replace the condition (1.1) by

$$\sum_{\nu=1}^{n}|s_\nu^\beta| = o(n^{\gamma+1}).$$

The proof of this result runs similarly to the one of Theorem 1 in section 3. But a slight modification will be needed therein. Now, using Lemma 3 in section 2, we see that we may replace, in Theorem 1, the condition (1.2) by

$$\sum_{\nu=n}^{2n}(|a_\nu|-a_\nu)=O(n^\delta). \tag{1.3}$$

Thus, Theorem 1 in this form is a generalized form of Sunouchi's theorem [8; Corollaries 1 and 2].

In our Theorem 1, if we put $\alpha=0$, then we have

COROLLARY 1. *If* p *is a positive odd integer, then the series* $\sum_{n=1}^{\infty}a_n$ *is evaluable* (R_p) *to zero under the conditions* (1.1) *and* (1.2).

This Corollary when $p=1$ was proved by Hirokawa and Sunouchi [4]. On the other hand, this was already proved by K. Kanno when he completed the proof of Theorem K, but it was not published.

In Theorem 1, we have only the case $\beta>p-1$. If $\beta<p-1$, then the series $\sum_{n=1}^{\infty}a_n$ is evaluable (C,β) under the condition (1.1); hence it is also evaluable (R,p,α). Therefore, we need a consideration for the critical case $\beta=p-1$. In this case, we get the following

THEOREM 2. *Let* p *be a positive odd integer. Suppose that*

$$\sum_{\nu=1}^{n}|s_\nu^{p-1}| = o(n^p/\log n) \tag{1.4}$$

and

$$\sum_{\nu=n}^{2n}(|a_\nu|-a_\nu)=O(n^{p-\delta}), \tag{1.5}$$

when $0 < \delta < 1$. *Then the series* $\sum_{n=1}^{\infty} a_n$ *is evaluable* (R, p, α) *when* $-1 \leqq \alpha \leqq 0$.

This Theorem is a generalized form of Theorem 5 in [3], which is a generalization of Szász's Theorem [9; Theorem 7′]. On the other hand, even if p is a positive even integer, we may prove, by the methods analogous to the one of the proof of Theorem 2 in section 4, that

(i) *the seires* $\sum_{n=1}^{\infty} a_n$ *is evaluable* (R, p) *to zero under the conditions* (1.4) *and* (1.5),
and

(ii) *the seires* $\sum_{n=1}^{\infty} a_n$ *is evaluable* (R_p) *to zero provided that the condition* (1.4) *holds and* $\sum_{\nu=n}^{2n} (|s_\nu| - s_\nu) = O(n^{p-\delta})$.

THEOREM 3. *Let p be a positive integer. If*

$$s_n^{p-1} = o(n^{p-1}/(\log n)^{1+\delta}), \tag{1.6}$$

when $\delta > 0$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is evaluable* (R, p, α) *when* $-1 \leqq \alpha < p-1$ *or* $\alpha = 0, p = 1$.

In this Theorem, we may replace the condition (1.6) by

$$s_n^{p-1} = o(n^{p-1}\lambda_n),$$

where $\lambda_n > 0$ and $\Sigma\lambda_n/n$ converges. The proof of this result runs similarly to the one of Theorem 3 in section 5.

THEOREM 4. *Let p be a positive integer and let α be an integer such that* $-1 \leqq \alpha < p-1$ *or* $\alpha = 0$, $p = 1$. *Then, there exists a series* $\sum_{n=1}^{\infty} a_n$ *which is not evaluable* (R, p, α), *but it satisfies the condition*

$$s_n^{p-1} = o(n^{p-1}/\log n). \tag{1.7}$$

Since the condition (1.7) implies summability $(C, p-1)$ of $\Sigma\, a_n$, we have

COROLLARY 2. *Let p be a positive integer and let α be an integer such that* $-1 \leqq \alpha < p-1$ *or* $\alpha = 0$, $p = 1$. *Then, summability* $(C, p-1)$ *does not necessarily imply summability* (R, p, α).

Paticular case in Corollary 2 shows that summability $(C, p-1)$ does not necessarily imply summability (R, p) and (R_p), respectively.

On the other hand, Rajchman and Zygmund [7] proved that summability $(C, p-1)$ implies approximate summability (R, p) and (R_p), respectively. As Rajchman and Zygmund defined approximate summability (R, p) in relation to summability (R, p), so we may define *approximate summability* (R, p, α) in relation to summability (R, p, α). We shall say that a series $\sum_{n=1}^{\infty} a_n$ is approximately evaluable (R, p, α) to zero if there is a set E of points having unit density at the origin such that the series

$$t^{\alpha+1} \sum_{n=1}^{\infty} s_n^{\alpha} \left(\frac{\sin nt}{nt}\right)^p$$

converges at all points of E, and such that its sum tends to zero as $t \to 0$ through point of E. Let p be a posive integer and let α be an integer such that $-1 \leqq \alpha < p-1$ or $\alpha = 0, p = 1$. Then, by the method analogous to one of the proof of Rajchman and Zygmund's theorem, we see that summability $(C, p-1)$ implies approximate summability (R, p, α).

2. Preliminary Lemmas.

LEMMA 1. *Let p be a positive odd integer. Then, we have*

$$\sum_{\nu=1}^{n} (\sin \nu t)^p = O(t^{-1}) \tag{2.1}$$

and

$$\sum_{\nu=n}^{\infty} (\sin \nu t)^p / \nu^p = O(n^{-p} t^{-1}). \tag{2.2}$$

Proof is obvious from the identity

$$(-1)^{\frac{p-1}{2}} 2^{p-1} (\sin t)^p = \sin pt - \binom{p}{1} \sin (p-2) t + \dots + (-1)^{\frac{p-1}{2}} \binom{p}{(p-1)/2} \sin t.$$

LEMMA 2. *Let p be a positive integer and let $\Delta^m \varphi(nt)$ denote the m-th difference of $\varphi(nt)$ with respect to n. Then, we have*

$$\Delta^m \varphi(nt) = O(n^{-p} t^{m-p}),$$

where m is a non-negative integer and $\varphi(t) = (\sin t/t)^p$.

This Lemma is due to Obreschkoff [6].

LEMMA 3. *Let $p \geqq 1$ and let $s_n^{\beta} = o(n^{\beta})$ when $\beta > 0$. Then*

$$\sum_{\nu=n}^{2n} (|a_\nu| - a_\nu) = O(n^{p-\delta}), \tag{2.3}$$

when $0 < \delta < 1$, implies

$$\sum_{\nu=n}^{\infty} |a_\nu| / \nu^p = O(n^{-\delta}) \tag{2.4}$$

and conversely.

This is a generalization of Szász's Lemma [9; Lemma 1].

PROOF. Let $0 < l < n$ and let (2.3) hold. Then, we have

$$s_{n+l} - s_n = \sum_{\nu=n+1}^{n+l} a_\nu \geqq - \sum_{\nu=n+1}^{n+l} (|a_\nu| - a_\nu) \geqq - \sum_{\nu=n}^{2n} (|a_\nu| - a_\nu) \geqq -Cn^{p-\delta}.$$

Then, by the method analogous to one of the proof of Bosanquet's convexity theorem [2; Theorem 6], we have $s_n = O(n^{p-\delta})$*); hence, by (2.3),

$$\sum_{\nu=n}^{2n} |a_\nu| = \sum_{\nu=n}^{2n} (|a_\nu| - a_\nu) + s_{2n} - s_{n-1} = O(n^{p-\delta}). \tag{2.5}$$

Consequently

$$\begin{aligned}\sum_{\nu=n}^{\infty} |a_\nu|/\nu^p &= \sum_{k=0}^{\infty} \sum_{\nu=2^k n}^{2^{k+1}n-1} |a_\nu|/\nu^p \\ &\leqq n^{-p} \sum_{k=0}^{\infty} 2^{-pk} \sum_{\nu=2^k n}^{2^{k+1}n} |a_\nu| \\ &= O\Big(n^{-p} \sum_{k=0}^{\infty} 2^{-pk}(2^k n)^{p-\delta}\Big) \\ &= O\Big(n^{-\delta} \sum_{k=0}^{\infty} 2^{-k\delta}\Big) \\ &= O(n^{-\delta}),\end{aligned}$$

which is the desired inequality (2.4). The converse part is obvious. Let (2.4) hold. Then we have (2.5) and further (2.3).

LEMMA 4. *If p is a positive integer and $-1 \leqq \alpha \leqq 0$, then we have*

$$H_\nu(t) \equiv t^{\alpha+1} \sum_{n=\nu}^{\infty} A_{n-\nu}^{\alpha-1} \Big(\frac{\sin nt}{nt}\Big)^p = O(\nu^{-p} t^{-p+1}).$$

Proof of Lemma is analogous to the proof of Lemma 7 in [3], which asserts that Lemma 4 is true when $p = 1$.

LEMMA 5. *Let p be a positive integer. Then we have*

$$\Delta^m H_\nu(t) = O(\nu^{-p} t^{m-p+1})$$

where $-1 \leqq \alpha \leqq 0$ and $\Delta^m H_\nu(t)$ is the m-th difference of $H_\nu(t)$ with respect to ν.

Proof is analogous to the proof of Lemma 8 in [3].

LEMMA 6. *Let p be a positive odd integer and let $-1 \leqq \alpha \leqq 0$. Then*

$$\eta_m(t) \equiv \sum_{\nu=m}^{\infty} H_\nu(t) = O(m^{-p} t^{-p}).$$

Proof is also similar to the proof of Lemma 9 in [3].

*) The Author learned this result from Mr. Kenji Yano, whom the Author expresses his hearty thanks.

LEMMA 7. *Let p be a positive integer and let $-1 \leqq \alpha \leqq 0$. If $s_n^p = o(n^p)$, then we have*

$$t^{\alpha+1} \sum_{n=1}^{\infty} s_n^{\alpha} \left(\frac{\sin nt}{nt}\right)^p = \sum_{n=1}^{\infty} s_n H_n(t).$$

PROOF. If $\alpha = -1$ or $\alpha = 0$, then Lemma is obvious. Hence we shall consider the case $-1 < \alpha < 0$. Since

$$s_n^{\alpha} = \sum_{k=1}^{n} A_{n-k}^{\alpha-1} s_k,$$

we have

$$t^{\alpha+1} \sum_{n=1}^{\infty} s_n^{\alpha} \left(\frac{\sin nt}{nt}\right)^p = \sum_{k=1}^{\infty} s_k \left\{ t^{\alpha+1} \sum_{n=k}^{\infty} A_{n-k}^{\alpha-1} \left(\frac{\sin nt}{nt}\right)^p \right\}$$

$$= \sum_{k=1}^{\infty} s_k H_k(t).$$

Here, we shall prove that this rearrangement is permissible. For this purpose, it is sufficient to prove that, for fixed $t > 0$,

$$t^{\alpha+1} \sum_{k=1}^{N} s_k \sum_{n=N+1}^{\infty} A_{n-k}^{\alpha-1} \left(\frac{\sin nt}{nt}\right)^p = o(1) \qquad \text{as } N \to \infty.$$

Since

$$\sum_{n=N+1}^{\infty} A_{n-k-1}^{\alpha-2} \left(\frac{\sin nt}{nt}\right)^p = O(N^{-p}(N-k+1)^{\alpha-1}),$$

we have, using Abel's transformation and $s_n^p = o(n^p)$,

$$t^{\alpha+1} \sum_{k=1}^{N} s_k \sum_{n=N+1}^{\infty} A_{n-k}^{\alpha-1} \left(\frac{\sin nt}{nt}\right)^p = t^{\alpha+1} \sum_{k=1}^{N-1} s_k^1 \sum_{n=N+1}^{\infty} A_{n-k-1}^{\alpha-2} \left(\frac{\sin nt}{nt}\right)^p$$

$$+ t^{\alpha+1} s_N^1 \sum_{n=N+1}^{\infty} A_{n-N-1}^{\alpha-1} \left(\frac{\sin nt}{nt}\right)^p$$

$$= o\left(\sum_{k=1}^{N} k^p N^{-p} (N-k+1)^{\alpha-1} \right) + o(N^p \cdot N^{-p})$$

$$= o(1).$$

Thus, the rearrangement is permissible and Lemma is proved.

LEMMA 8. *Let us put*

$$\psi(q, n, t) = \left| \sum_{r=0}^{q} (-1)^r \binom{q}{r} \left(\frac{\sin (n+r) t}{(n+r) t}\right)^p \right|$$

where p and q are positive integers such that $q \leqq p$. Then, for all large enough positive integer k, there exists a positive integer m_0 such that

$$\psi(q, mk, 2\pi/k) \geqq (2mk)^{-p},$$

when $m \geqq m_0$.

PROOF. Let us put

$$\varphi(p, q) = \sum_{r=0}^{q} (-1)^r \binom{q}{r} (q-r)^p.$$

Then, we have $\varphi(p, 1) = 1$ and $\varphi(q, q) = q\,!$. If we suppose that $\varphi(p, q) \geqq q\,!$ and $\varphi(p, q+1) \geqq (q+1)\,!$ when $p \geqq q+1$, then we may prove that $\varphi(p+1, q+1) \geqq (q+1)!$. Hence, by the mathematical induction, we see that $\varphi(p, q) \geqq q\,!$ for all p and q such that $p \geqq q \geqq 1$. Thus we get

$$\sum_{r=0}^{q} (-1)^r \binom{q}{r} r^p = (-1)^q \varphi(p, q) \neq 0.$$

Therefore we have

$$\lim_{k\to\infty} \lim_{m\to\infty} (mk)^p \psi(q,\ mk, 2\pi/k) = \left| \sum_{r=0}^{q} (-1)^r \binom{q}{r} r^p \right| > 0.$$

Thus we have the Lemma.

3. **Proof of Theorem 1.** Let us put $r_n = \sum_{\nu=n}^{\infty} |a_\nu| / \nu$, then we have $|a_n| = n(r_n - r_{n+1})$. Hence, by (1.2),

(3.1) $s_n = O\left(\sum_{\nu=1}^{n} |a_\nu|\right) = O\left(\sum_{\nu=1}^{n} r_\nu - n\, r_{n+1}\right) = O\left(\sum_{\nu=1}^{n} \nu^{-1+\delta}\right) + O(n^\delta) = O(n^\delta)$.

Therefore, from (1.1), using Dixson-Ferrer's convexity theorem [1], we have

(3.2) $$s_n^\nu = o(n^{\{\delta(\beta-\nu)+\gamma\nu\}/\beta}), \qquad 0 < \nu \leqq \beta.$$

In particular, if $\beta > p$, then

$$s_n^p = o(n^{\{\delta(\beta-p)+\gamma p\}/\beta}) = o(n^p).$$

On the other hand, if $\beta \leqq p$, then we have easily $s_n^p = o(n^p)$. Hence, by Lemma 7, for the proof it is sufficient to prove that the series

$$\sum_{n=1}^{\infty} s_n H_n(t)$$

is convergent in some interval $0 < t < t_0$ and its sum tends to zero as $t \to 0$.

Let us write $k = [\beta] + 1$ and let

$$\sum_{n=1}^{\infty} s_n H_n(t) = \left(\sum_{n=1}^{M+k} + \sum_{n=M+k+1}^{\infty} \right) = U + V,$$

where $M = [(\varepsilon t)^{-\rho}]$, ε being an arbitrarily fixed positive number, and

$$\rho = \frac{p}{p-\delta} = \frac{\beta+1-p}{\gamma+1-p}.$$

Using Abel's transformation, we have, from Lemma 6 and (1.2),

$$V = \sum_{n=M+k+1}^{\infty} s_n H_n(t)$$

$$= -\sum_{n=M+k+1}^{\infty} a_{n+1}\sum_{\nu=n}^{\infty} H_\nu(t) + s_{M+k+1}\sum_{\nu=M+k}^{\infty} H_\nu(t)$$

$$= O\Big(t^{-p}\sum_{n=M}^{\infty}|a_n|n^{-p}\Big) + O(t^{-p}M^{\delta-p})$$

$$= O(t^{-p}M^{\delta-p}) = O(t^{-p}(\varepsilon t)^{p(p-\delta)}) = O(\varepsilon).$$

Next, we shall prove, using the method of the proof of Theorem K, that $U = o(1)$. By repeated use of Abel's transformation k-times, we have

$$U = \sum_{n=1}^{M+k} s_n H_n(t)$$

$$= \sum_{n=1}^{M} s_n^k \Delta^k H_n(t) + \sum_{\nu=1}^{k} s^\nu_{M+k-\nu+1}\Delta^{\nu-1} H_{M+k-\nu+1}(t)$$

$$= U_0 + \sum_{\nu=1}^{k} W_\nu,$$

say, where

$$W_\nu = o(M^{\delta(\beta-\nu)+\gamma\nu\}/\beta}\cdot M^{-p}\,t^{\nu-p})$$

$$= o\Big(t^{\nu-p-\frac{p}{\beta(p-\delta)}\{\delta(\beta-\nu)+\gamma\nu-p\beta\}}\Big)$$

$$= o(1), \qquad \nu = 1, 2, \ldots, k-1,$$

by Lemma 5 and (3.2), and, since $s_n^k = o(n^{k+\gamma-\beta})$,

$$W_k = o(M^{k+\gamma-\beta}\cdot M^{-p}t^{k-p})$$

$$= o(M^{k+\gamma-\beta-p}t^{k-p})$$

$$= o\Big(t^{k-p-\frac{p}{p-\delta}(k+\gamma-\beta-p)}\Big)$$

$$= o(1).$$

Since

$$s_n^k = \sum_{\mu=1}^{n} A_{n-\mu}^{k-\beta-1}s_\mu^\beta,$$

we write U_0 in the form

$$U_0 = \sum_{\mu=1}^{M} s_\mu^\beta \sum_{n=\mu}^{M} A_{n-\mu}^{k-\beta-1}\Delta^k H_n(t)$$

$$= \sum_{\mu=1}^{M}\sum_{n=\mu}^{\mu+N} + \sum_{\mu=1}^{M-N-1}\sum_{n=\mu+N+1}^{M} - \sum_{\mu=M-N+1}^{M}\sum_{n=M+1}^{\mu+N}$$

$$= U_1 + U_2 - U_3,$$

say, where $N = [t^{-1}]$. Since

$$\sum_{n=\mu}^{\mu+N} A_{n-\mu}^{k-\beta-1} = \sum_{n=0}^{N} A_n^{k-\beta-1} = A_N^{k-\beta} \sim \frac{N^{k-\beta}}{\Gamma(k-\beta+1)} = O(t^{\beta-k}),$$

we have, by Lemma 5,

$$U_1 = O\Big(\sum_{\mu=1}^{M} |s_\mu^\beta| \sum_{n=\mu}^{\mu+N} A_{n-\mu}^{k-\beta-1}\, n^{-p}\, t^{k-p+1}\Big) = O\Big(t^{k-p+1}\cdot t^{\beta-k} \sum_{\mu=1}^{M} \mu^{-p} |s_\mu^\beta|\Big)$$
$$= o(t^{\beta-p+1} M^{\gamma-p+1}) = o(t^{(\beta-p+1)-\rho(\gamma-p+1)}) = o(1).$$

On the other hand, using Abel's transformation in the inner sum, we have, since $A_\nu^{k-\beta-1} - A_{\nu-1}^{k-\beta-1} = A_{\nu-1}^{k-\beta-2}$,

$$U_2 = \sum_{\mu=1}^{M-N-1} s_\mu^\beta \sum_{n=\mu+N+1}^{M} A_{n-\mu}^{k-\beta-1}\, \Delta^k H_n(t)$$
$$= \sum_{\mu=1}^{M-N-1} s_\mu^\beta \Big[-\Big\{ \sum_{n=\mu+N+1}^{M-1} A_{n-\mu-1}^{k-\beta-2}\, \Delta^{k-1} H_n(t) + A_N^{k-\beta-1} \Delta^{k-1} H_{N+\mu+1}(t)\Big\}$$
$$+ A_{M-\mu}^{k-\beta-1}\, \Delta^{k-1} H_M(t)\Big] = -U_2' + U_2'',$$

say, where

$$U_2'' = \sum_{\mu=1}^{M-N-1} s_\mu^\beta A_{M-\mu}^{k-\beta-1}\, \Delta^{k-1} H_M(t) = O\Big(\sum_{\mu=1}^{M-N-1} |s_\mu^\beta| (M-\mu)^{k-\beta-1} M^{-p} t^{k-p}\Big)$$
$$= O\Big(t^{k-p} M^{-p}\, N^{k-\beta-1} \sum_{\mu=1}^{M} |s_\mu^\beta|\Big) = o(t^{\beta-p+1}\, M^{\gamma-p+1})$$
$$= o(t^{\beta-p+1-\rho(\gamma-p+1)}) = o(1).$$

Since

$$\sum_{n=\mu+N+1}^{M-1} A_{n-\mu-1}^{k-\beta-2}\, \Delta^{k-1} H_n(t) - A_N^{k-\beta-1}\, \Delta^{k-1}\, H_{N+\mu+1}(t)$$
$$= O\Big(t^{k-p} \sum_{n=\mu+N+1}^{M-1} (n-\mu)^{k-\beta-2} n^{-p}\Big) + O(t^{k-p} N^{k-\beta-1} (N+\mu)^{-p})$$
$$= O\Big(t^{k-p} \mu^{-p} \sum_{n=N}^{\infty} n^{k-\beta-2}\Big) + O(t^{\beta-p+1}\, \mu^{-p})$$
$$= O(t^{\beta-p+1}\, \mu^{-p}),$$

we have, by (1.1),

$$U_2' = O\Big(t^{\beta-p+1} \sum_{\mu=1}^{M-N-1} \mu^{-p} |s_\mu^\beta|\Big)$$
$$= o(t^{\beta-p+1}\, M^{\gamma-p+1}) = o(t^{\beta-p+1-\rho(\gamma-p+1)}) = o(1)$$

and then we have $U_2 = o(1)$. Lastly we have

$$U_3 = \sum_{\mu=M-N+1}^{M} s_\mu^\beta \sum_{n=M+1}^{\mu+N} A_{n-\mu}^{k-\beta-1}\, \Delta^k H_n(t)$$
$$= O\Big(\sum_{\mu=M-N+1}^{M} |s_\mu^\beta| \sum_{n=M+1}^{\mu+N} A_{n-\mu}^{k-\beta-1}\, t^{k-p+1} n^{-p}\Big)$$

$$= O\Big(t^{k-p+1}M^{-p} \sum_{\mu=M-N+1}^{M} |s_\mu^\beta| \sum_{n=\mu}^{\mu+N} A_{n-\mu}^{k-\beta-1}\Big)$$

$$= O\Big(t^{k-p+1} M^{-p} \sum_{\mu=M-N+1}^{M} |s_\mu^\beta|\cdot N^{k-\beta}\Big)$$

$$= o(t^{\beta-p+1} M^{-p+\gamma+1}) = o(t^{(\beta-p+1)-\rho(\gamma-p+1)}) = o(1).$$

Thus, summing up the above estimations, we have

$$\limsup_{t\to 0} \Big|\sum_{n=1}^{\infty} s_n H_n(t)\Big| = O(\varepsilon).$$

Since ε is an arbitrary positive number, we have

$$\lim_{t\to 0} \sum_{n=1}^{\infty} s_n H_n(t) = 0,$$

and Theorem is completely proved.

4. **Proof of Theorem 2.** From (1.4), we have $s_n^p = o(n^p)$; hence, by Lemma 3, we may replace (1.5) by (2.4), and, by Lemma 7, for the proof of Theorem it is sufficient to prove that the series $\sum_{n=1}^{\infty} s_n H_n(t)$ is convergent in some interval $0 < t < t_0$ and its sum tend to zero as $t \to 0$. Convergence of the series follows from the estimation of V below. Let us write

$$\sum_{n=1}^{\infty} s_n H_n(t) = \Big(\sum_{n=1}^{M} + \sum_{n=M+1}^{\infty}\Big) = U + V$$

where $M = [t^{-r}]$ and $r\delta > p$. Since $x^p - (x-1)^p = O(x^{p-1})$, we have, using (2.4), $s_n = O(n^{p-\delta})$ by the method analogous to one which we obtained (3.1). Furthermore, by Dixon and Ferrer's convexity theorem [1], using (1.4), we get

(4.1) $$s_n^\nu = o(n^{(p^2-\delta(p-\nu))/p})$$

when $0 < \nu \leqq p$. Then we have, by (2.4) and Lemma 6,

$$V = \sum_{n=M+1}^{\infty} s_n H_n(t)$$

$$= -\sum_{n=M+1}^{\infty} a_{n+1} \sum_{\nu=n}^{\infty} H_\nu(t) + s_{M+1} \sum_{\nu=M+1}^{\infty} H_\nu(t)$$

$$= O\Big(t^{-p} \sum_{n=M}^{\infty} |a_n|/n^p\Big) + O(M^{p-\delta} t^{-p} M^{-p})$$

$$= O(t^{-p}M^{-\delta}) = O(t^{r\delta-p}) = o(1).$$

Now, using Abel's transformation $(p-1)$-times, we have

$$U = \sum_{n=1}^{M} s_n H_n(t)$$

$$= \sum_{n=1}^{M-p+1} s^p \ \Delta^{p-1} H_n(t) + \sum_{\nu=1}^{p-1} s^\nu_{M-\nu+1} \Delta^{\nu-1} H_{M-\nu+1}(t)$$

$$= U_0 + \sum_{\nu=1}^{p-1} U^\nu,$$

say, where, by (4. 1),

$$U_\nu = o\left(M^{p-\delta\left(1-\frac{\nu}{p}\right)} \cdot t^{\nu-p} M^{-p}\right) = o\left(t^{\nu-p} M^{-\delta\left(1-\frac{\nu}{p}\right)}\right)$$

$$= o(t^{r\delta\left(1-\frac{\nu}{p}\right)+\nu-p}) = o(1), \qquad \nu = 1, 2, \ldots, p-1.$$

Putting $N = [t^{-1}]$, we write

$$U_0 = \left(\sum_{n=1}^{N} + \sum_{n=N+1}^{M}\right) = U_0' + U_0''.$$

Then we have

$$U_0' = \sum_{n=1}^{N-1} s_n^p \Delta^p H_n(t) + s_N^p \Delta^{p-1} H_N(t)$$

$$= o\left(\sum_{n=2}^{N} \frac{n^p}{\log n} \cdot \frac{t}{n^p}\right) + o\left(\frac{N^p}{\log N} \cdot \frac{1}{N^p}\right)$$

$$= o(Nt) + o(1) = o(1),$$

and putting $s_n' = \sum_{k=1}^{n} |s_k^{p-1}|$,

$$U_0'' = O\left(\sum_{n=N+1}^{M} |s_n^{p-1}| n^{-p}\right)$$

$$= O\left(\sum_{n=N+}^{M-1} s_n'' (n^{-p} - (n+1)^{-p}) + s_M'' M^{-p} - s_N'' (N+1)^{-p}\right)$$

$$= o\left(\sum_{n=N}^{M} \frac{n^p}{\log n} \cdot \frac{1}{n^{p+1}}\right) + o\left(\frac{M^p}{\log M} \cdot M^{-p}\right) + o\left(\frac{N^p}{\log N} \cdot N^{-p}\right)$$

$$= o(\log r) + o(1) = o(1),$$

and the proof of Theorem is complete.

5. **Proof of Theorem 3.** From (1. 6), we have, for $-1 \leqq \alpha \leqq p-1$,

$$s_n^\alpha = o(n^{p-1}/(\log n)^{1+\delta}).$$

Hence the series

$$t^{\alpha+1} \sum_{n=1}^{\infty} s_n^\alpha \left(\frac{\sin nt}{nt}\right)^p$$

converges (absolutely) for $t > 0$. Now, let us write

$$t^{\alpha+1} \sum_{n=1}^{\infty} s_n^\alpha \left(\frac{\sin nt}{nt}\right)^p = \left(\sum_{n=1}^{M} + \sum_{n=M+1}^{\infty}\right) = U + V,$$

where $M=[\exp(t^{-r})]$ and $r\delta>p$. Then we have

$$V=o\left(t^{\alpha-p+1}\sum_{n=M}^{\infty}n^{-1}(\log n)^{-1-\delta}\right)$$
$$=o(t^{\alpha-p+1}(\log M)^{-\delta})=o(t^{r\delta+\alpha-p+1})=o(1).$$

On the other hand, we shall prove that $U=o(1)$. We suppose that α is not an integer. The case that α is an integer may be easily deduced by the following argument. Since

$$s_n^{\alpha}=\sum_{\nu=1}^{n}A_{n-\nu}^{\alpha-p}s_{\nu}^{p-1},$$

we have

$$U=t^{\alpha+1}\sum_{n=1}^{M}\left(\frac{\sin nt}{nt}\right)^p\sum_{\nu=1}^{n}A_{n-\nu}^{\alpha-p}s_{\nu}^{p-1}$$
$$=t^{\alpha+1}\sum_{\nu=1}^{M}s_{\nu}^{p-1}\sum_{n=\nu}^{M}A_{n-\nu}^{\alpha-p}\left(\frac{\sin nt}{nt}\right)^p$$
$$=t^{\alpha+1}\left(\sum_{\nu=1}^{N}+\sum_{\nu=N+1}^{M}\right)s_{\nu}^{p-1}\sum_{n=\nu}^{M}A_{n-\nu}^{\alpha-p}\left(\frac{\sin nt}{nt}\right)^p$$
$$=U_1+U_2,$$

say, where $N=[t^{-1}]$. Now, we see that, by the method analogous to one which we obtained Lemma 7 and 8 in [3],

$$Q_{\nu}(t)=t^{\alpha+1}\sum_{n=\nu}^{M}A_{n-\nu}^{\alpha-p}\left(\frac{\sin nt}{nt}\right)^p=O(\nu^{-p})$$

and

$$Q_{\nu}(t)-Q_{\nu+1}(t)=O(\nu^{-p}t).$$

Then, we have easily $U_1=o(1)$ and

$$U_2=o\left(\sum_{\nu=N+1}^{M}\frac{\nu^{p-1}}{(\log\nu)^{1+\delta}}\cdot\nu^{-p}\right)=o(1).$$

Thus Theorem is completely proved.

6. **Proof of Theorem 4.** By the repeated Abel transformation ($p-\alpha-1$)-times, we have

$$t^{\alpha+1}\sum_{n=1}^{m}s_n^{\alpha}\left(\frac{\sin nt}{nt}\right)^p=t^{\alpha+1}\sum_{n=1}^{m-p+\alpha+1}s_n^{p-1}\Delta^{p-\alpha-1}\left(\frac{\sin nt}{nt}\right)^p$$
$$+t^{\alpha+1}\sum_{\nu=0}^{p-\alpha-2}s_{m-\nu}^{\alpha+\nu+1}\Delta^{\nu}\left(\frac{\sin(m+\nu)t}{(m+\nu)t}\right)^p,$$

where, using Lemma 2 and (1.7), for fixed t,

$$t^{\alpha+1}\sum_{\nu=0}^{p-\alpha-2}s_{m-\nu}^{\alpha+\nu+1}\Delta^{\nu}\left(\frac{\sin(m+\nu)t}{(m+\nu)t}\right)^p=\sum_{\nu=0}^{p-\alpha-2}o\left(\frac{m^{p-1}}{\log(m-\nu)}\cdot\frac{1}{m^p}\right)=o(1),$$

as $m \to \infty$. Therefore the series

$$(6.1) \qquad t^{\alpha+1} \sum_{n=1}^{\infty} s^{\alpha}_{n} \left(\frac{\sin nt}{nt} \right)^{p} \quad \text{and} \quad t^{\alpha+1} \sum_{n=1}^{\infty} s_n^{p-1} \Delta^{p-\alpha-1} \left(\frac{\sin nt}{nt} \right)^{p}$$

are equiconvergent for fixed t. Thus, for the proof, it is sufficient to prove that (1.7) does not necessarily imply the convergence of the second series in (6.1) in some interval $0 < t < t_0$. Let us write

$$(6.2) \qquad t^{\alpha+1} \sum_{n=1}^{\infty} s_n^{p-1} \Delta^{p-\alpha-1} \left(\frac{\sin nt}{nt} \right)^{p} = \sum_{n=1}^{\infty} \varepsilon_n c_n(t),$$

where $\varepsilon_n = s_n^{p-1} \log n / n^{p-1}$ and $c_n(t) = t^{\alpha+1} \dfrac{n^{p-1}}{\log n} \Delta^{p-\alpha-1} \left(\dfrac{\sin nt}{nt} \right)^{p}$ when $n \geqq 2$. Then we have by (1.7), $\varepsilon_n = o(1)$ as $n \to \infty$. In order that the sequence-to-function transformation (6.2) is convergence-preserving, by the Kojima-Schur Theorem[1], $\sum_{n=2}^{\infty} |c_n(t)|$ must be uniformly bounded in $0 < t < t_0$. But, this series is divergent at some point in an arbitrary neighbourhood of origin. Let $t = 2\pi/k$ and let k be an arbitrarily fixed positive integer, but large enough. Then we have, using Lemma 8,

$$\begin{aligned} \sum_{n=2}^{\infty} |c_n(t)| &= t^{\alpha+1} \sum_{n=2}^{\infty} \frac{n^{p-1}}{\log n} |\psi(p-\alpha-1, n, t)| \\ &\geqq t^{\alpha+1} \sum_{\nu=1}^{\infty} \frac{(k\nu)^{p-1}}{\log (k\nu)} |\psi(p-\alpha-1, k\nu, t)| \\ &\geqq t^{\alpha+1} \sum_{\nu=1}^{\infty} \frac{1}{2^{p-\alpha-1}} \cdot \frac{1}{(k\nu)\log (k\nu)} = +\infty. \end{aligned}$$

This shows that $\sum_{n=2}^{\infty} |c_n(t)|$ is divergent at $t = 2\pi/k$ and the transformation (6.2) is not convergence-preserving. Therefore (1.7) does not necessarily imply the convergence of the first series in (6.1). Thus, Theorem is completely proved.

REFERENCES

[1] A. L. DIXSON AND W. L. FERRER, On the Cesàro sums, Journ. London Math. Soc., 7(1932), 87-93.
[2] L. S. BOSANQUET, Note on convexity theorems, Journ. London Math. Soc., 18 (1943), 239-248.
[3] H. HIROKAWA, Riemann-Cesàro methods of summability, Tôhoku Math. Journ., 7 (1955), 279-295.
[4] H. HIROKAWA AND G. SUNOUCHI, Two theorems on the Riemann summability, Tôhoku Math. Jonrn., 5(1953), 261-267.
[5] K. KANNO, On the Riemann summability, Tôhoku Math. Journ., 6(1954), 155-166.
[6] N. OBRESCHKOFF, Über das Riemannsche Summierungsverfahren, Math. Zeits., 48(1942), 441-454.

1) Cf. R. P. Agnew, Properties of generalized definitions of limit, Bull. Amer. Math. Soc, 445(1939), 689-730.

[7] A. RAJCHMAN AND A. ZYGMUND, Sur la relation du procédé de sommation de Cesàro et celui de Riemann, Bull. de l'Acad. Polonaise, (1925), 69–80.
[8] G. SUNOUCHI, Tauberian theorem for Riemann summability, Tôhoku Math. Journ., 5 (1953), 34–42.
[9] O. SZÁSZ, On convergence and summability of trigonometrical series., Amer. Journ. Math., 64 (1942), 575–591.
[10] A. ZYGMUND, Trigonometrical Series, Warsaw, 1935.

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