ON RIEMANNIAN MANIFOLDS WITH HOMOGENEOUS HOLONOMY GROUP $Sp(n)$

HIDEKIYO WAKAKUWA

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According to the results of M. Berger (M. Berger, [1], [2], [3]), it is known that the restricted homogeneous holonomy group of a non-symmetric, irreducible N-dimensional Riemannian manifold V_N is one of the followings: $SO(N)$ (full rotation group), $U(m)$ (unitary group; $N=2m$), $SU(m)$ (special unitary group; $N=2m$), $Sp(n)$ (unitary symplectic group; $N=4n$), $Sp(n)\otimes T^1$, $Sp(n)\otimes SU(2)$ or some other exceptions. The Riemannian manifold with restricted homogeneous holonomy group $U(m)$ or $SU(m)$ is characterized by the fact that it is pseudo-kaehlerian or pseudo-kaehlerian with Ricci tensor zero (Iwamoto, [1]; Lichnerowicz, [8]). The purpose of this paper is to study the $4n$-dimensional Riemannian manifold whose restricted homogeneous holonomy group is the real representation of the unitary symplectic group $Sp(n)$ or one of its subgroups. Since the group $Sp(n)$ is a subgroup of the special unitary group $SU(2n)$; our manifolds in consideration are special pseudo-kaehlerian manifolds. In Part I, we treat local properties and in Part II the theory of harmonic forms and the cohomology theory.

PART I

In this Part I, unless otherwise stated, the summation convention will be used and the indices run over the following ranges:

$$\begin{aligned}
&i, j, k, \dots\dots &&= 1, 2, \dots, \dots, \dots, \dots\ 4n;\\
&a, b, c, \dots\dots &&= 1, 2, \dots, n;\\
&\alpha, \beta, \gamma, \dots\dots &&= 1, 2, \dots, \dots, 2n;\\
&\bar\alpha, \bar\beta, \bar\gamma \dots &&= 1+2n,\ 2+2n, \dots, 4n.
\end{aligned}$$

1. Preliminary remarks. Let C_{2n} be a $2n$-dimensional complex Cartesian space. Unitary symplectic group $Sp(n)$ operating on C_{2n} is a subgroup of unitary group $U(2n)$ which leaves bilinear form

$$z^a \wedge w^{a+n} = (z^a w^{a+n} - z^{a+n} w^a)/2 \qquad ((z^\alpha),\ (w^\alpha) \in C_{2n})$$

invariant and it is necessarily special unitary. Hence, the necessary and sufficient conditions that a linear endomorphism of C_{2n}

$$z^{*\alpha} = U^\alpha_\beta z^\beta \qquad ((U^\alpha_\beta)\text{: complex matrix of order } 2n) \tag{1.1}$$

be unitary symplectic are as follows:

(i) $U = (U^\alpha_\beta)$ be unitary, that is, ${}^t\bar{U}U = E_{2n}$ (E_{2n}: unit matrix of order $2n$), where the bar over U denotes the complex conjugate of U and ${}^tU$ the transpose of U.

(ii) U leaves the matrix $\begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$ invariant, where E_n denotes the unit matrix of order n.

Such a matrix U is called unitary symplectic. The condition (ii) is equivlent to the fact that U be of the form

$$U=\begin{pmatrix}\Sigma & -\bar{\Theta}\\ \Theta & \bar{\Sigma}\end{pmatrix}, \tag{1.2}$$

where Σ, Θ denote complex matrices of order n. If we put

$$\Sigma = P+Ri,\quad \Theta = Q+Si\quad (i=\sqrt{-1})$$

where P, Q, R, S denote real matrices of order n, we have a real representation of (1.2):

$$T=\begin{pmatrix}P & -Q & -R & -S\\ Q & P & -S & R\\ R & S & P & -Q\\ S & -R & Q & P\end{pmatrix} \tag{1.3}$$

The condition (i) implies that this T be an orthogonal matrix. Therefore, with respect to an orthogonal base $[e_i]$, a transformation of $Sp(n)$ is expressed by

$$e^*_j = T^i_j e_i, \tag{1.4}$$

where $T=(T^i_j)$ is an orthogonal matrix of the form (1.3). With respect to a new base $[e'_i]$ which is obtained from $[e_i]$ by an imaginary transformation

$$e'_\alpha = \frac{1}{\sqrt{2}}(e_\alpha - ie_{\bar{\alpha}}),\quad e'_{\bar{\alpha}} = \frac{1}{\sqrt{2}}(e_\alpha + ie_{\bar{\alpha}}) \tag{1.5}$$

the transformation (1.4) takes the form

$$e'^*_j = T'^i_j e'_i,$$

where

$$(T'^j_i)=\begin{pmatrix}P+Ri, & -Q+Si & & \\ Q+Si, & P-Ri & & \\ & & P-Ri, & -Q-Si\\ & & Q-Si, & P+Ri\end{pmatrix}=\begin{pmatrix}U & 0\\ 0 & \bar{U}\end{pmatrix}. \tag{1.6}$$

By an orthogonal matrix of the form (1.3), the three matrices

$$I=\begin{pmatrix}0 & E_n & 0 & 0\\ -E_n & 0 & 0 & 0\\ 0 & 0 & 0 & -E_n\\ 0 & 0 & E_n & 0\end{pmatrix},\quad J=\begin{pmatrix}0 & 0 & E_n & 0\\ 0 & 0 & 0 & E_n\\ -E_n & 0 & 0 & 0\\ 0 & -E_n & 0 & 0\end{pmatrix},\quad K=\begin{pmatrix}0 & 0 & 0 & E_n\\ 0 & 0 & -E_n & 0\\ 0 & E_n & 0 & 0\\ -E_n & 0 & 0 & 0\end{pmatrix} \tag{1.7}$$

are left invariant, that is, ${}^tTIT=I$, etc. Among these I, J, K there are following relations:

$$\begin{cases}\text{(I)} & I^2=J^2=K^2=-E_{4n}\\ \text{(II)} & {}^tII={}^tJJ={}^tKK=E_{4n}\\ \text{(III)} & IJ=-JI=K,\ JK=-KJ=I,\ KI=-IK=J.\end{cases} \tag{1.8}$$

The necessary and sufficient condition that an orthogonal matrix be unitary symplectic is that it is conjugate to a matrix which leaves the three matrices (1.7) invariant.

2. Characterization of V_{4n}. Let V_{4n} be a $4n$-dimensional Riemannian manifold of class $C^r(r \geqq 2)$ whose restricted homogeneous holonomy group h^0 is the real representation of $Sp(n)$. With respect to a suitable orthogonal frame of reference, there exist three covariant constant tensor fields I, J, K of the form (1.7). Let $\overset{(1)}{F}$ (components $\overset{(1)}{F}{}^i{}_j$), $\overset{(2)}{F}$ (components $\overset{(2)}{F}{}^i{}_j$), $\overset{(3)}{F}$ (components $\overset{(3)}{F}{}^i{}_j$ be the three tensor fields I, J, K with respect to the natural frame of reference of V_{4n} respectively, then the relations (1.8) assert that

$$(2.1)\quad \begin{cases} \text{(I)} & \overset{(1)}{F}{}^2 = \overset{(2)}{F}{}^2 = \overset{(3)}{F}{}^2 = -E_{4n}, \\ \text{(II)} & {}^t\overset{(1)}{F}G\overset{(1)}{F} = {}^t\overset{(2)}{F}G\overset{(2)}{F} = {}^t\overset{(3)}{F}G\overset{(3)}{F} = G, \\ \text{(III)} & \overset{(1)}{F}\overset{(2)}{F} = -\overset{(2)}{F}\overset{(1)}{F} = \overset{(3)}{F},\ \overset{(2)}{F}\overset{(3)}{F} = -\overset{(3)}{F}\overset{(2)}{F} = \overset{(1)}{F},\ \overset{(3)}{F}\overset{(1)}{F} = -\overset{(1)}{F}\overset{(3)}{F} = \overset{(2)}{F}, \end{cases}$$

where G means the matrix of (g_{ij}) of the fundamental metric tensor of V_{4n}.

It is remarked that using the relations (I), (II) and one of (III), the other two relations of (III) can be proved.

If we use the components of $\overset{(1)}{F}$, $\overset{(2)}{F}$ and $\overset{(3)}{F}$, (2.1) is also written in the following forms:

$$(2.1')\quad \begin{cases} \text{(I)} & \overset{(1)}{F}{}^i{}_k\overset{(1)}{F}{}^k{}_j = \overset{(2)}{F}{}^i{}_k\overset{(2)}{F}{}^k{}_j = \overset{(3)}{F}{}^i{}_k\overset{(3)}{F}{}^k{}_j = -\delta^i{}_j, \\ \text{(II)} & g_{ij}\overset{(1)}{F}{}^i{}_k\overset{(1)}{F}{}^j{}_h = g_{ij}\overset{(2)}{F}{}^i{}_k\overset{(2)}{F}{}^j{}_h = g_{ij}\overset{(3)}{F}{}^i{}_k\overset{(3)}{F}{}^j{}_h = g_{kh}, \\ \text{(III)} & \overset{(1)}{F}{}^i{}_k\overset{(2)}{F}{}^k{}_j = -\overset{(2)}{F}{}^i{}_k\overset{(1)}{F}{}^k{}_j = \overset{(3)}{F}{}^i{}_j,\ \overset{(2)}{F}{}^i{}_k\overset{(3)}{F}{}^k{}_j = -\overset{(3)}{F}{}^i{}_k\overset{(2)}{F}{}^k{}_j = \overset{(1)}{F}{}^i{}_j,\ \overset{(3)}{F}{}^i{}_k\overset{(1)}{F}{}^k{}_j = -\overset{(1)}{F}{}^i{}_k\overset{(3)}{F}{}^k{}_j = \overset{(2)}{F}{}^i{}_j. \end{cases}$$

If we put

$$(2.2)\qquad g_{ik}\overset{(1)}{F}{}^k{}_j = \overset{(1)}{F}_{ij},\quad g_{ik}\overset{(2)}{F}{}^k{}_j = \overset{(2)}{F}_{ij},\quad g_{ik}\overset{(3)}{F}{}^k{}_j = \overset{(3)}{F}_{ij},$$

then $\overset{(1)}{F}_{ij}$, $\overset{(2)}{F}_{ij}$, $\overset{(3)}{F}_{ij}$ are anti-symmetric tensor fields. This fact is easily verified from (I) and (II) of (2.1′).

Now we have seen that if the restricted homogeneous holonomy group of V_{4n} is the real representation of $Sp(n)$, then there exist three covariant constant tensor fields $\overset{(1)}{F} = (\overset{(1)}{F}{}^i_j)$, $\overset{(2)}{F} = (\overset{(2)}{F}{}^i{}_j)$ and $\overset{(3)}{F} = (\overset{(3)}{F}{}^i{}_j)$ over V_{4n} satisfying (2.1) or (2.1′) in each coordinate neighborhood.

We shall prove, conversely, that if there exist three covariant constant tensor fields over V_{4n} satisfying (2.1) or (2.1′) in a $4n$-dimensional Riemannian manifold V_{4n}, then the restricted homogeneous holonomy group of V_{4n} is the real representation of $Sp(n)$ or one of its subgroups.

LEMMA 2.1. *Let u^i be an arbitrary non-zero vector field. Then u^i, $\overset{(1)}{F}{}^i{}_ju^j$, $\overset{(2)}{F}{}^i{}_ju^j$ and $\overset{(3)}{F}{}^i{}_ju^j$ are mutually orthogonal. If u^i is a unit vector, then the other three are also unit vectors.*

PROOF. The orthogonality of u^i to the other three is evident from (2.2). The orthogonality of $\overset{(1)}{F}{}^i{}_ju^j$ to $\overset{(2)}{F}{}^i{}_ju^j$, for example, is verified as follows:

$$g_{ij}(\overset{(1)}{F}{}^i{}_ku^k)(\overset{(2)}{F}{}^j{}_hu^h) = \overset{(1)}{F}_{jk}\overset{(2)}{F}{}^j{}_h u^k u^h = -g_{ik}\overset{(1)}{F}{}^i{}_j\overset{(2)}{F}{}^j{}_h u^k u^h$$

$$= -g_{ik}\overset{(3)}{F}{}^{i}{}_{h}u^{k}u^{h} = -\overset{(3)}{F}_{kh}u^{k}u^{h} = 0.$$

If u^i is a unit vector, then the other three are also unit vectors by vitue of (II) of (2.1).

LEMMA 2.2. *Let u^i be an arbitrary non-zero vector field and v^i be a vector field which is orthogonal to all of four vectors u^i, $\overset{(1)}{F}{}^{i}{}_{j}u^j$, $\overset{(2)}{F}{}^{i}{}_{j}u^j$ and $\overset{(3)}{F}{}^{i}{}_{j}u^j$. Then, $\overset{(1)}{F}{}^{i}{}_{j}v^j$, $\overset{(2)}{F}{}^{i}{}_{j}v^j$, $\overset{(3)}{F}{}^{i}{}_{j}v^j$ are mutually orthogonal and orthogonal to all the other five vectors.*

PROOF. For example, the orthogonality of $\overset{(1)}{F}{}^{i}{}_{j}v^j$ to u^i, $\overset{(1)}{F}{}^{i}{}_{j}u^j$ $\overset{(2)}{F}{}^{i}{}_{j}u^j$, $\overset{(3)}{F}{}^{i}{}_{j}u^j$ is verified as follows. By assamption, v^j is orthogonal to all of u^i, $\overset{(1)}{F}{}^{i}{}_{j}u^j$, $\overset{(2)}{F}{}^{i}{}_{j}u^j$ and $\overset{(3)}{F}{}^{i}{}_{j}u^j$, we have

$$g_{ij}u^i v^j = 0,$$

$g_{ij}(\overset{(1)}{F}{}^{i}{}_{k}u^k)v^j = 0$ or

$$\overset{(1)}{F}_{ij}u^i v^j = 0$$

and similarly

$$\overset{(2)}{F}_{ij}u^i v^j = 0,\ \ \overset{(3)}{F}_{ij}u^i v^i = 0.$$

Hence we see that

$$g_{ij}u^i(\overset{(1)}{F}{}^{j}{}_{k}v^k) = \overset{(1)}{F}_{ik}u^i v^k = 0,\ \ g_{ij}(\overset{(1)}{F}{}^{i}{}_{k}u^k)(\overset{(1)}{F}{}^{j}{}_{h}v^h) = g_{kh}u^k v^h = 0,$$

$$g_{ij}(\overset{(2)}{F}{}^{j}{}_{k}u^h)(\overset{(1)}{F}{}^{i}{}_{h}v^h) = \overset{(3)}{F}_{kh}u^k v^h = 0,\ \ g_{ij}(\overset{(3)}{F}{}^{i}{}_{k}u^k)(\overset{(1)}{F}{}^{j}{}_{h}v^h) = -\overset{(2)}{F}_{kh}u^k v^h = 0.$$

The others can be proved similarly. q.e.d.

By the aid of above two Lemmas, we prove that the restricted homogeneous holonomy group h^0 of our V_{4n} is the real representation of $Sp(n)$ or one of its subgroups by showing that $\overset{(1)}{F}$, $\overset{(2)}{F}$, $\overset{(3)}{F}$ can be taken in the form (1.7) by choosing a suitable orthogonal frame of reference $[e_1, e_2, \ldots, e_{4n}]$.

At first, choose an arbitrary unit vector as e_1, then its components are $\delta^i{}_1$. The three vectors (components $\overset{(1)}{F}{}^{i}{}_{1}$, $\overset{(3)}{F}{}^{i}{}_{1}$, $\overset{(2)}{F}{}^{i}{}_{1}$) obtained from e_1 by performing collineations given by $\overset{(1)}{F}$, $\overset{(2)}{F}$, $\overset{(3)}{F}$ respectively, are mutually orthogonal by Lemma 2.1. If we choose these vectors as $-e_{n+1}$, $-e_{2n+1}$, $-e_{3n+1}$, then with respect to such frame of reference, we have

$$\overset{(1)}{F}{}^{n+1}{}_{1} = -1,\ \ \overset{(2)}{F}{}^{2n+1}{}_{1} = -1,\ \ \overset{(3)}{F}{}^{3n+1}{}_{1} = -1$$

and the other $\overset{(1)}{F}{}^{i}{}_{1}$, $\overset{(2)}{F}{}^{i}{}_{1}$, $\overset{(3)}{F}{}^{i}{}_{1}$ are all zero.

Next, choose a vector which is orthogonal to all of the above e_1, e_{n+1}, e_{2n+1} and e_{3n+1} as e_2. Then the components of the last vector are $\delta^i{}_2$. The three vectors (components $\overset{(1)}{F}{}^{i}{}_{2}$, $\overset{(3)}{F}{}^{i}{}_{2}$, $\overset{(3)}{F}{}^{i}{}_{2}$) obtained from e_2 by collineations $\overset{(1)}{F}$, $\overset{(2)}{F}$, $\overset{(3)}{F}$ respectively are mutualy orthogonal and orthogonal to e_1, e_{n+1} e_{2n+1}, e_{3n+1} by

Lemma 2.1 and 2.2. If we choose these three vectors as $-e_{n+2}$, $-e_{2n+2}$, $-e_{3n+2}$, then with respect to such a frame of reference

$$\overset{(1)}{F}{}^{n+1}{}_2=-1,\ \overset{(2)}{F}{}^{2n+1}{}_2=-1,\ \overset{(3)}{F}{}^{3n+1}{}_2=-1$$

and the other $\overset{(1)}{F}{}^i{}_2$, $\overset{(2)}{F}{}^i{}_2$, $\overset{(3)}{F}{}^i{}_2$ are all zero.

Repeating similar process n times, we get an orthogonal frame of reference. Taking account of the fact that with respect to this orthogonal frame of reference, $\overset{(1)}{F}{}^i{}_j$, $\overset{(2)}{F}{}^i{}_j$ and $\overset{(3)}{F}{}^i{}_j$ are anti-symmetric with respect to the upper and lower indices, we see that $\overset{(1)}{F}=(\overset{(1)}{F}{}^i{}_j)$, $\overset{(2)}{F}=(\overset{(2)}{F}{}^i{}_j)$ and $\overset{(3)}{F}=(\overset{(3)}{F}{}^i{}_j)$ are of the forms

$$\overset{(1)}{F}=\begin{pmatrix} 0 & E_n & 0 & 0\\ -E^n & X_1 & X_2 & X_3\\ 0 & X_1' & X_2' & X_3'\\ 0 & X_1'' & X_2'' & X_3''\end{pmatrix},\quad \overset{(2)}{F}=\begin{pmatrix} 0 & 0 & E_n & 0\\ 0 & Y_1 & Y_2 & Y_3\\ -E_n & Y_1' & Y_2' & Y_3'\\ 0 & Y_1'' & Y_2'' & Y_3''\end{pmatrix},\quad \overset{(3)}{F}=\begin{pmatrix} 0 & 0 & 0 & E_n\\ 0 & Z_1 & Z_2 & Z_3\\ 0 & Z_1' & Z_2' & Z_3'\\ -E_n & Z_1'' & Z_2'' & Z_3''\end{pmatrix},$$

respectively, where $X_1, X_2, \ldots.;\ Y_1, Y_2, \ldots.;\ Z_1, Z_2\ldots.$ denote real matrices of order n. From (I) of 2.1, we have

$$\overset{(1)}{F}{}^2=\begin{pmatrix} -E_n & X_1 & X_2 & X_3\\ -X_1 & & & \\ -X_1 & & * & \\ -X_1'' & & & \end{pmatrix}=-E_{4n}$$

hence

$$X_1=X_2=X_3=X_1'=X_1''=0.$$

Similarly, from $\overset{(2)}{F}{}^2=\overset{(3)}{F}{}^2=-E_{4n}$ we get

$$Y_2=Y_1'=Y_2'=Y_3'=Y_2''=0,$$

$$Z_3=Z_3'=Z_1''=Z_2''=Z_3''=0.$$

So, $\overset{(1)}{F}$, $\overset{(2)}{F}$ and $\overset{(3)}{F}$ have the following forms:

$$\overset{(1)}{F}=\begin{pmatrix} 0 & E_n & 0 & 0\\ -E_n & 0 & 0 & 0\\ 0 & 0 & X_2' & X_3'\\ 0 & 0 & X_2'' & X_3''\end{pmatrix},\quad \overset{(2)}{F}=\begin{pmatrix} 0 & 0 & E_n & 0\\ 0 & Y_1 & 0 & Y_3\\ -E_n & 0 & 0 & 0\\ 0 & Y_1'' & 0 & Y_3''\end{pmatrix},\quad \overset{(3)}{F}=\begin{pmatrix} 0 & 0 & 0 & E_n\\ 0 & Z_1 & Z_2 & 0\\ 0 & Z_1' & Z_2' & 0\\ -E_n & 0 & 0 & 0\end{pmatrix},$$

By virtue of (III) of (2.1), that is, $\overset{(1)}{F}\overset{(2)}{F}=\overset{(3)}{F}$, we have

$$\overset{(1)(2)}{FF}=\begin{pmatrix} 0 & Y_1 & 0 & Y_3\\ 0 & 0 & -E_n & 0\\ -X_2' & X_3'Y_1' & 0 & X_3'Y_3''\\ -X_2'' & X_3'Y_1'' & 0 & X_3''Y_3''\end{pmatrix}=\overset{(3)}{F}=\begin{pmatrix} 0 & 0 & 0 & E_n\\ 0 & Z_1 & Z_2 & 0\\ 0 & Z_1' & Z_2' & 0\\ -E_n & 0 & 0 & 0\end{pmatrix},$$

hence we get

$$X_2'=0,\ X_2''=E_n,\ Y_1=0,\ Y_3=E_n,\ Z_1=0,\ Z_2=-E_n,\ Z_2'=0.$$

Since $\overset{(1)}{F}$, $\overset{(2)}{F}$ and $\overset{(3)}{F}$ are anti-symmetric, we find

$$X_3' = -E_n, \ Y_1'' = -E_n, \ Z_1' = E_n.$$

Hence, from $X_3'' Y_1'' = 0$ and $X_3' Y_3'' = 0$, we get

$$X_3'' = 0, \ Y_3'' = 0$$

respectively.

Consequently, we find finally that $\overset{(1)}{F}$, $\overset{(2)}{F}$, $\overset{(3)}{F}$ are of the form

$$\overset{(1)}{F} = \begin{pmatrix} 0 & E_n & 0 & 0 \\ -E_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -E_n \\ 0 & 0 & E_n & 0 \end{pmatrix}, \ \overset{(2)}{F} = \begin{pmatrix} 0 & 0 & E_n & 0 \\ 0 & 0 & 0 & E_n \\ -E_n & 0 & 0 & 0 \\ 0 & -E_n & 0 & 0 \end{pmatrix}, \ \overset{(3)}{F} = \begin{pmatrix} 0 & 0 & 0 & E_n \\ 0 & 0 & E_n & 0 \\ 0 & E_n & 0 & 0 \\ -E_n & 0 & 0 & 0 \end{pmatrix}$$

These three tensors being covariant constant, hence left invariant by the restricted homogeneous holonomy group h^0, which means that h^0 is $Sp(n)$ or one of its subgroups as mentioned in §1.

THEOREM 2.1. *If the restricted homogeneous holonomy group of V_{4n} is the real representation of $Sp(n)$ or one of its subgroups, then there exist covariant constant tensor fields $\overset{(1)}{F}{}^i{}_j$, $\overset{(2)}{F}{}^i{}_j$ and $\overset{(3)}{F}{}^i{}_j$ over V_{4n} satisfying* (I), (II) *and* (III) *of* (2.1′). *The converse is also true.*

3. An example of 4-dimensional case. We shall show an example of 4-dimensional Riemannian manifold V_4 with homogeneous holonomy group $Sp(1)$, following to Prof. T. Ôtsuki's method.[1].

At first, we shall investigate the necessary condition for such a V_4. Introduce in V_4 an orthogonal frame of reference $[P, e_i]$ $(i = 1,2,3,4)$, then the connection of V_4 is given by

$$dP = \omega^i e_i, \ de_j = \omega^i{}_j e_i\,, \tag{3.1}$$

where ω^i, $\omega^i{}_j$ are Pfaffian forms with respect to the coordinate neighborhood (x^1, x^2, x^3, x^4) of V_4. The structural equations are given by

$$d\omega^i = \omega^j \wedge \omega^i{}_j, \ d\omega^i{}_j = \omega^a{}_j \wedge \omega^i{}_a + \Omega^i{}_j \qquad (i, j, k, a = 1,2,3,4). \tag{3.2}$$

We can easily see from the remark of §1 that

$$\omega^1{}_2 = \omega^3{}_4, \ \omega^1{}_3 = -\omega^2{}_4, \ \omega^1{}_4 = \omega^2{}_3,$$

since the homogeneous holonomy group is $Sp(1)$. If we put

$$\omega^1{}_2 = \omega^3{}_4 = \theta_2, \ \omega^1{}_3 = -\omega_4{}^2 = \theta_3, \ \omega^1{}_4 = \omega^2{}_3 = \theta_4,$$

then the structural equation can be written as

$$\begin{cases} d\omega^1 = \omega^2 \wedge \theta_2 + \omega^3 \wedge \theta_3 + \omega^4 \wedge \theta_4 \\ d\omega^2 = -\omega^1 \wedge \theta_2 + \omega^3 \wedge \theta_4 - \omega^4 \wedge \theta_3 \\ d\omega^3 = -\omega^1 \wedge \theta_3 - \omega^2 \wedge \theta_4 + \omega^4 \wedge \theta_2 \\ d\omega^4 = -\omega^1 \wedge \theta_4 + \omega^2 \wedge \theta_3 - \omega^3 \wedge \theta_2, \end{cases} \tag{3.3}$$

1) Prof. T. Ôtsuki set forth some examples of fundamental forms of 4-dimensional Riemannian manifolds with homog. holonomy group $Sp(1)$ (Ôtsuki, [6]), but it seems to contain some errors. The details of his method should be referred to his paper.

and

$$(3.4)\qquad \begin{cases} d\theta_2 = 2\theta_3 \wedge \theta_4 + \Omega^1{}_2 \\ d\theta_3 = 2\theta_4 \wedge \theta_2 + \Omega^1{}_3 \\ d\theta_4 = 2\theta_2 \wedge \theta_3 + \Omega^1{}_4 \end{cases} .$$

Let i, j, k be the imaginary units of quaternions and put

$$\omega = \omega^1 + i\omega^2 + j\omega^3 + k\omega^4,\ \ \Gamma = i\theta_2 + j\theta_3 + k\theta_4 \ .$$

If we define formally $d\omega$, $\Gamma \wedge \omega$, then (3.3) can be represented by

$$(3.5)\qquad d\omega = \Gamma \wedge \omega \ .$$

We can see that ω is reducible to the form

$$\omega = a\{dx^1 + i\,dx^2 + \Pi(dx^3 + i\,dx^4)\}$$

where Π is a quaternic function and a is a real function. Substituting ω, Γ in (3.5) and eliminating θ_2, θ_3 and θ_4, we have a differential equation for Π:

$$(3.6)\qquad \frac{\partial\Pi}{\partial x^1}\overline{\Pi} + \frac{\partial\Pi}{\partial x^2}\overline{\Pi}i - \frac{\partial\Pi}{\partial x^3} - \frac{\partial\Pi}{\partial x^4}\,i = 0,$$

where $\overline{\Pi}$ is the quaternic conjugate of Π.

Put $\Pi = b_1 + ib_2 + jb_3 + kb_4$, then the fundamental form of V_4 becomes

$$ds^2 = a^2[(dx^1)^2 + (dx^2)^2 + \sum_{r=1}^{4} b_r^2\{(dx^3)^2 + (dx^4)^2\}$$
$$+ 2b^1(dx^1\,dx^3 + dx^2\,dx^4) - 2b_2(dx^1\,dx^4 - dx^2\,dx^3)],$$

we may put $b_4 = 0$ and consider the special case where $b_2 = 0$. Then the dfferential equation (3.6) for b_1 and b_3 becomes

$$\begin{cases} R\dfrac{\partial R}{\partial x^1} = \dfrac{\partial b_1}{\partial x^3}, \qquad R\dfrac{\partial R}{\partial x^2} = \dfrac{\partial b_1}{\partial x^4}, \\ R\dfrac{\partial R}{\partial x^3} = -\dfrac{\partial b_1}{\partial x^1}R^2 + 2b_1\dfrac{\partial b_1}{\partial x^3}, \\ R\dfrac{\partial R}{\partial x^4} = -\dfrac{\partial b_1}{\partial x^2}R^2 + 2b_1\dfrac{\partial b_1}{\partial x^4}, \end{cases}$$

where $R^2 = b_1^2 + b_3^2$. These are satisfied for example by

$$b_1 = c\,x^3 + c'x^4,\ \ b_3 = \{2(cx^1 + c'x^2) + (cx^3 + c'x^4)\}^{\frac{1}{2}},$$

where c and c' are non-zero constants and we have

$$(3.7)\qquad \begin{cases} \omega^1 = adx^1 + ab_1\,dx^3 , \\ \omega^2 = adx^2 + ab_1\,dx^4, \\ \omega^2 = ab_3\,dx^3, \\ \omega^4 = - ab_3\,dx^4 \end{cases} .$$

Putting $\theta_2 = p_i\,dx^i$, $\theta_3 = q_i\,dx^i$, $\theta_4 = r_i\,dx^i$ and substituting these and (3.7) in (3.3), we get after long but straightforward calculations,

$$(3.8)\qquad p_1 = \frac{\partial \log a}{\partial x^2},\ p_2 = -\frac{\partial \log a}{\partial x^1},\ p_3 = \frac{\partial \log a}{\partial x^4},\ p_4 = -\frac{\partial \log a}{\partial x^3};$$

$$(3.9)\quad \begin{cases} q_1 = r_2 = \dfrac{1}{b_3}\left(b_1 \dfrac{\partial \log a}{\partial x^1} - \dfrac{\partial \log a}{\partial x^3}\right) = 0, \\ q_2 = r_1 = \dfrac{1}{b_3}\left(b_1 \dfrac{\partial \log a}{\partial x^2} - \dfrac{\partial \log a}{\partial x^4}\right) = 0; \end{cases}$$

$$(3.10)\quad \begin{cases} q_3 = -\dfrac{c}{b_3} - b_3 \dfrac{\partial \log a}{\partial x^1} = -r_4 = b_3 \dfrac{\partial \log a}{\partial x^1}, \\ q_4 = b_3 \dfrac{\partial \log a}{\partial x^2} = r_3 = -\dfrac{c'}{b_3} - b_3 \dfrac{\partial \log a}{\partial x^2}. \end{cases}$$

From (3.9), we see that $\log a$ must be a solution of differential equations

$$b_1 \frac{\partial f}{\partial x^1} - \frac{\partial f}{\partial x^3} = 0, \quad b_1 \frac{\partial f}{\partial x^2} - \frac{\partial f}{\partial x^4} = 0.$$

Solving these we find

$$\log a = -\frac{1}{2} \log b_3$$

as one of the solutions. This satisfies (3.10) and some other relations imposed to p_i, q_i and r_i by (3.3). Hence we find finally

$$\begin{cases} p_1 = -\dfrac{c'}{2b_3^{\frac{3}{2}}}, \ p_2 = \dfrac{c}{2b_3^{\frac{3}{2}}}, \ p_3 = -\dfrac{c'b_1}{2b_3^{\frac{3}{2}}}, \ p_4 = \dfrac{cb_1}{2b_3^{\frac{3}{2}}}, \\ q_1 = q_2 = r_1 = r_2 = 0, \\ q_3 = -r_4 = -\dfrac{c}{2b_3}, \ q_4 = r_3 = -\dfrac{c'}{2b_3}. \end{cases}$$

Consequently, the structural equations (3.3) are fulfilled by (3.7) and

$$\begin{cases} \theta_2 = -\dfrac{1}{2b_3^{\frac{3}{2}}}(c'\,dx^1 - c\,dx^2 + c'\,b_1\,dx^3 - cb_1\,dx^4), \\ \theta_3 = -\dfrac{1}{2b_3}(c dx^3 + c'\,dx^4), \\ \theta_4 = -\dfrac{1}{2b_3}(c'\,dx^3 - c\,dx^4), \end{cases}$$

where

$$a = b_3^{-\frac{1}{2}} = \{2(cx^1 + c'x^2) + (cx^3 + c'x^4)^2\}^{-\frac{1}{4}}$$

$$l_1 = cx_3 + c'x^4, \quad b_3 = \{2(cx^1 + c'x^2) + (cx^3 + c'x^4)^2\}^{\frac{1}{2}}.$$

Furthermore, from (3.4) we see that

$$\Omega^1{}_2 \neq 0, \ \Omega^1{}_3 \neq 0, \ \Omega^1{}_4 \neq 0,$$

for non-zero c, c'. Therefore, we consider each domain of the 4-dimensional number space separated by a 3-dmensional cylindrical surface

$$2(cx^1 + c'x^2) + (c\,x^3 + c'\,x^4)^2 = 0.$$

Then

$$ds^2 = a^3[(dx^1)^2 + (dx^2)^2 + 2\{cx^1 + c'x^2 + (cx^3 + c'x^4)^2\}\{(dx^3)^2 + (dx^4)^2\} + 2(cx^3 + c'x^4)(dx^1 dx^3 + dx^2 dx^4)]$$

$$(a = \{2(cx^1 + c'x^2) + (cx^3 + c'x^4)^2\}^{-\frac{1}{4}}, \quad c\cdot c' \neq 0)$$

gives an example of an analytic Riemannian metric which defines a Euclidean connection with homogeneous holonomy group $Sp(1)$ in such domain.

4. Root spaces. The characteristic roots of the equation $|\overset{(1)}{F} - \rho E| = 0$ (E: unit matrix of order $4n$) for $\overset{(1)}{F} = (\overset{(1)}{F^i{}_j})$ being i and $-i$ (multiplicity $2n$) respectively, there exist two $2n$-dimensional imaginary root spaces $L(\overset{(1)}{F})$ and $\overline{L}(\overset{(1)}{F})$ corresponding to the two characteristic roots i and $-i$ respectively. A vector x in the tangent space at a point of V_{4n} belongs to $L(\overset{(1)}{F})$ at the point if and only if

$$(\overset{(1)}{F} - i E)^\nu x = 0 \qquad (1 \leqq \nu \leqq 2n) \quad ,$$

but this condition is equivalent to

$$(\overset{(1)}{F} - i E) x = 0$$

by virtue of $\overset{(1)}{F^2} = -E$.

There exist also root spaces $L(\overset{(2)}{F})$, $\overline{L}(\overset{(2)}{F})$ $(\overset{(2)}{F} = (\overset{(2)}{F^i{}_j}))$; $L(\overset{(3)}{F})$, $\overline{L}(\overset{(3)}{F})$ $(\overset{(3)}{F} = (\overset{(3)}{F}{}^i{}_j))$; $L(\overset{(2)}{F})$, $L(\overset{(3)}{F})$ corresponding to characteristic roots i and $\overline{L}(\overset{(2)}{F})$, $\overline{L}(\overset{(3)}{F})$ to $-i$.

These root spaces form (imaginary) parallel fields of $2n$-dimensional planes respectively which is easily verified from the fact that $\overset{(1)}{F}$, $\overset{(2)}{F}$, $\overset{(3)}{F}$ are covariant constant and from the above remark.

These $2n$-planes have no intersections in common except the origin, for, if, for example, $L(\overset{(1)}{F})$ and $L(\overset{(2)}{F})$ contain a vector x in common, we have

$$\overset{(1)}{F}x = \overset{(2)}{F}x,$$

from $\overset{(1)}{F}x = ix$, $\overset{(2)}{F}x = ix$. Operating $\overset{(1)}{F}$ to the above equation from the left and taking account of $\overset{(1)}{F^2} = -E$, $\overset{(1)}{F}\overset{(2)}{F} = \overset{(3)}{F}$, we get

$$-x = \overset{(3)}{F}x.$$

This means that $\overset{(3)}{F}$ have a characteristic root -1, which is a contradiction.

Next, consider a vector $x \in L(\overset{(1)}{F})$ and operating $\overset{(2)}{F}$ to $\overset{(1)}{F}x = ix$ from the left we have

$$-\overset{(3)}{F}x = i\overset{(2)}{F}x \quad \text{or} \quad \overset{(2)}{F}x = i\overset{(3)}{F}x \quad .$$

From this and from $\overset{(1)}{F}\overset{(2)}{F} = \overset{(3)}{F}$, we see that

$$\overset{(1)}{F}(\overset{(2)}{F}x) = \overset{(3)}{F}x = -i\overset{(2)}{F}x \quad ,$$

that is, for a vector $x \in L(\overset{(1)}{F})$, $\overset{(2)}{F}x$ is a vector in $\overline{L}(\overset{(1)}{F})$. This means that

$$\overset{(2)}{F}(L(\overset{(1)}{F})) = \overline{L}(\overset{(1)}{F}).$$

We can see analogously that $\overset{(3)}{F}(L(\overset{(1)}{F})) = \overline{L}(\overset{(1)}{F})$ and so on. Accordingly, we get the following

THEOREM 4.1. *Let* $L(\overset{(1)}{F})$, $\overline{L}(\overset{(1)}{F})$; $L(\overset{(2)}{F})$, $\overline{L}(\overset{(2)}{F})$; $L(\overset{(3)}{F})$, $\overline{L}(\overset{(3)}{F})$ *be* $2n$*-dimensional root spaces determined by* $\overset{(1)}{F}$, $\overset{(2)}{F}$, $\overset{(3)}{F}$; $L(\overset{(1)}{F})$, $L(\overset{(2)}{F})$, $L(\overset{(3)}{F})$ *corresponding to the characteristic root* i *and* $\overline{L}(\overset{(1)}{F})$, $\overline{L}(\overset{(2)}{L})$, $\overline{L}(\overset{(3)}{F})$ *to* $-i$. *These are imaginary parallel fields of* $2n$*-planes which have no point in common except the origin and the following relations hold good*:

$$\left\{\begin{aligned} &\overset{(2)}{F}(L(\overset{(1)}{F})) = \overset{(3)}{F}(L(\overset{(1)}{F})) = \overline{L}(\overset{(1)}{F}), \\ &(\overset{(3)}{F}(L(\overset{(2)}{F})) = \overset{(1)}{F}(L(\overset{(2)}{F})) = \overline{L}(\overset{(2)}{F}), \\ &\overset{(1)}{F}(L(\overset{(3)}{F})) = \overset{(2)}{F}(L(\overset{(3)}{F})) = \overline{L}(\overset{(3)}{F}), \end{aligned}\right. \qquad \left\{\begin{aligned} &\overset{(2)}{F}(\overline{L}(\overset{(1)}{F})) = \overset{(3)}{F}(\overline{L}(\overset{(1)}{F})) = L(\overset{(1)}{F}), \\ &\overset{(3)}{F}(\overline{L}(\overset{(2)}{F})) = \overset{(1)}{F}(\overline{L}(\overset{(2)}{F}) = L(\overset{(2)}{F}), \\ &\overset{(1)}{F}(\overline{L}(\overset{(3)}{F})) = \overset{(2)}{F}(\overline{L}(\overset{(3)}{F})) = \overline{L}(\overset{(3)}{F}), \end{aligned}\right.$$

where $\overset{(2)}{F}(L(\overset{(1)}{F}))$ *designates the* $2n$*-plane obtained from* $L(\overset{(1)}{F})$ *by operating the collineation* $\overset{(2)}{F} = (\overset{(2)}{F}{}_j)$, *etc.*

5. **Connection in complex form.** For each point of our V_{4n}, associate an orthogonal frame of reference $[e_i]$, then the connection in V_{4n} is given by

$$dP = \omega^i e_i, \quad de_j = \omega^i{}_j e_i, \qquad (\omega^i{}_j = -\omega^j{}_i) \tag{5.1}$$

where the matrix $(\omega^i{}_j)(= -\omega^j{}_i)$ is of the form

$$(\omega^i{}_j) = \begin{pmatrix} \omega & -\omega^* & -\widetilde{\omega} & -\widetilde{\omega}^* \\ \omega^* & \omega & -\widetilde{\omega}^* & \widetilde{\omega} \\ \widetilde{\omega} & \widetilde{\omega}^* & \omega & -\omega^* \\ \widetilde{\omega}^* & -\widetilde{\omega} & \omega^* & \omega \end{pmatrix}, \tag{5.2}$$

ω, ω^*, $\widetilde{\omega}$, $\widetilde{\omega}^*$ being matrices of order n. Hence, of course, we see that

$$\omega^\alpha{}_\beta = \omega^{\bar\alpha}{}_{\bar\beta}, \quad \omega^{\bar\alpha}{}_\beta = -\omega^\alpha{}_{\bar\beta} \tag{5.3}$$

If we perform an imaginary transformation for the base $[e_i]$:

$$e'_\alpha = (e_\alpha - ie_{\bar\alpha})/\sqrt{2}, \quad e'_{\bar\alpha} = (e_\alpha + ie_{\bar\alpha})/\sqrt{2},$$

and we write again $[e_i]$ instead of $[e'_i]$, then (5.1) can be written as

$$dP = \pi^\alpha e_\alpha + \pi^{\bar\alpha} e_{\bar\alpha}, \quad de_j = \pi^i{}_j e_i, \qquad \text{(and compl. conj.)} \tag{5.4}$$

where we have put

$$\left\{\begin{aligned} &\pi^\alpha = (\omega^\alpha + i\,\omega^{\bar\alpha})/\sqrt{2} = \overline{\pi^{\bar\alpha}}, \\ &\pi^\alpha{}_\beta = \omega^\alpha{}_\beta + i\,\omega^{\bar\alpha}{}_\beta = \omega^{\bar\alpha}{}_{\bar\beta} - i\,\omega^\alpha{}_{\bar\beta} = \overline{\pi^{\bar\alpha}{}_{\bar\beta}}, \\ &\pi^{\bar\alpha}{}_\beta = \pi^\alpha{}_{\bar\beta} = 0. \end{aligned}\right.$$

From (1.2) of §1, the matrix $(\pi^\alpha{}_\beta)$ have the form

$$(\pi^\alpha{}_\beta) = \begin{pmatrix} \pi & -\overline{\widetilde{\pi}} \\ \widetilde{\pi} & \overline{\pi} \end{pmatrix} = (\overline{\pi^{\bar\alpha}{}_{\bar\beta}}), \tag{5.5}$$

where π, $\widetilde{\pi}$ denote matrices of order n: $\pi = (\pi^a{}_b)$, $\widetilde{\pi} = (\widetilde{\pi}^a{}_b)$ and $(\pi^\alpha{}_\beta)$ being unitary, we have

$$\pi^{a}{}_{b} + \overline{\pi}^{b}{}_{a} = 0, \quad \widetilde{\pi}^{a}{}_{b} - \overline{\widetilde{\pi}}^{b}{}_{a} = 0. \tag{5.6}$$

The fundamental form is given by

$$ds^2 = \varepsilon_{ij}\pi^i\pi^j = 2\pi^\alpha \pi^{\bar\alpha} \quad ,$$

where
$$(\varepsilon_{ij}) = \begin{pmatrix} 0 & E_{2n} \\ E_{2n} & 0 \end{pmatrix} .$$

Now, if we put

$$d\pi^i{}_j = \pi^k{}_j \wedge \pi^i{}_k - \Omega^i{}_j$$

then $\Omega^i{}_j$ satisfies the following relations similar to (5.5):

$$\begin{cases} \Omega^{\bar\alpha}{}_\beta = \Omega^\alpha{}_{\bar\beta} = 0, \\ (\Omega^\alpha{}_\beta) = \begin{pmatrix} \Omega & -\overline{\widetilde{\Omega}} \\ \widetilde{\Omega} & \overline{\Omega} \end{pmatrix} = (\overline{\Omega}^{\bar\alpha}{}_{\bar\beta}), \quad (\Omega = (\Omega^a{}_b), \ \widetilde{\Omega} = (\widetilde{\Omega}^a{}_b)) \\ \Omega^a{}_b + \overline{\Omega}^b{}_a = 0, \ \widetilde{\Omega}^a{}_b - \overline{\widetilde{\Omega}}^b{}_a = 0. \end{cases} \tag{5.7}$$

A manifold with pseudo-kaehlerian connection (5.4) have $Sp(n)$ as its restricted homogeneous holonomy group if and only if $(\pi^\alpha{}_\beta)$ be of the form (5.5) with (5.6). Then the curvature form $\Omega^i{}_j$ satisfies (5.7). We have especially

$$\Omega^\alpha{}_\alpha = \Omega^a{}_a + \overline{\Omega}^a{}_a = 0 \tag{5.8}$$

and the structural equation becomes

$$\begin{cases} d\pi^\alpha = \pi^\beta \wedge \pi^\alpha{}_\beta \\ d\pi^\alpha{}_\beta = \pi^\gamma{}_\beta \wedge \pi^\alpha{}_\gamma + \Omega^\alpha{}_\beta \end{cases} \qquad \text{(and compl. conj.)} \tag{5.9}$$

under the condition (5.5), (5.6) and (5.7).

If we put

$$\Omega^\alpha{}_\beta = R^\alpha{}_{\beta kh}\pi^k \wedge \pi^h \text{ (conj.)}, \quad R_{kh} = R^l{}_{khl} \quad ,$$

it is easily verified that the non-zero components of $R^\alpha{}_{\beta kh}$ are $R^\alpha{}_{\beta\bar\gamma\delta}(= -R^\alpha{}_{\beta\delta\bar\gamma})$and appearently non-zero components of the Ricci tensor $R_{\beta\bar\gamma}$ are zero by virtue of $R_{\beta\bar\gamma} = R^\alpha{}_{\beta\bar\gamma\alpha} = -R^\alpha{}_{\alpha\beta\bar\gamma} = 0$ and (5.8). So V_{4n} is of Ricci tensor zero, which is also verified from the fact that $Sp(n) \subset SU(2n)$.

6. Sectional curvatures. Return to the real natural frame of reference, then $\overset{(1)}{F}{}^i{}_j$ satisfies the equation $\overset{(1)}{F}{}^i{}_{j,k} = 0$, where the semi-colon denotes the covariant differentiation with respect to the Christoffoel symbols obtained from g_{ij}. From the Ricci's identity, we have $\overset{(1)}{F}{}^i{}_l R^l{}_{jkh} = \overset{(1)}{F}{}^l{}_j R^i{}_{lkh}$ or $\overset{(1)}{F}{}_i{}^l R_{ljkh} = \overset{(1)}{F}{}^l{}_j R_{ilkh}$ and hence

$$\overset{(1)}{F}{}^l{}_i \overset{(1)}{F}{}^m{}_j R_{lmkh} = R_{ijkh} \tag{6.1}$$

(Sasaki, [1]; Yano, K and I. Mogi, [2]). Let x^i, y^i be components of two arbitrary vectors. Then the sectional curvature K with respect to the 2-plane π spanned by x^i and y^i is given by

$$K = -\frac{R_{ijkh}\, x^i y^j x^k y^h}{(g_{ik}g_{jh} - g_{ih}g_{jk})x^i y^j x^k y^h}\,.$$

This quantity being independent from the choice of two vectors in π, we choose especially two orthogonal unit vectors x^i, y^i in π, then K is given by

$$K = -R_{ijkh}\, x^i y^j x^k y^h. \tag{6.2}$$

For two orthogonal unit vectors x^i, y^i, we have again orthogonal unit vectors $\overset{(1)}{F^i}_j x^j$, $\overset{(1)}{F^i}_j y^j$ and the sectional curvature with respect to the plane spanned by $\overset{(1)}{F^i}_j x^j$, $\overset{(1)}{F^i}_j y^j$ is equal to K, which is easily seen from (6.1) and (6.2). Thus we get

LEMMA 6.1. *Let V_{2m} be a pseudo-kaehlerian manifold with pseudo-kahlerian structure $F = (F^i_j)$, then the sectional curvature with respect to an arbitrary 2-plane π is equal to the one with respect to 2-plane $F(\pi)$.*

Now, in our V_{4n}, there exist three covariant constant tensors $\overset{(1)}{F} = (\overset{(1)}{F^i}_j)$, $\overset{(2)}{F} = (\overset{(2)}{F^i}_j)$, $\overset{(3)}{F} = (\overset{(3)}{F^i}_j)$ and hence if a vector x^i is given, we can determine a 4-dimensional linear space $L_4(x)$ spanned by mutually orthogonal four vectors x^i, $\overset{(1)}{F^i}_j x^j$, $\overset{(2)}{F^i}_j x^j$ and $\overset{(3)}{F^i}_j x^j$. An arbitrary vector y^i in $L_4(x)$ being given in the form

$$y^i = \alpha x^i + \beta \overset{(1)}{F^i}_j x^j + \gamma \overset{(2)}{F^i}_j x^j + \delta \overset{(3)}{F^i}_j x^j \qquad (\alpha,\ \beta,\ \gamma,\ \delta: \text{scalar functions}).$$

Hence if we perform a collineation $\overset{(1)}{F}$ to x^i, then we have

$$\overset{(1)}{F^k}_i y^i = \alpha \overset{(1)}{F^k}_i x^i - \beta x^k + \gamma \overset{(3)}{F^k}_i x^i - \delta \overset{(2)}{F^k}_i x^i\ ,$$

by virtue of (III) of (2.1). This means that if a vector $y \in L_4(x)$, then $\overset{(1)}{F}(y) \in L_4(x)$ and we get similar properties for $\overset{(2)}{F}$, $\overset{(3)}{F}$.

THEOREM 6.1. *Let x be an arbitrary vector and $L_4(x)$ be a 4-dimensional linear space spanned by mutually orthogonal four vectors x, $\overset{(1)}{F}(x)$, $\overset{(2)}{F}(x)$, $\overset{(3)}{F}(x)$. If π is an arbitrary 2-planed in $L_4(x)$, then $\overset{(1)}{F}(\pi)$, $\overset{(2)}{F}(\pi)$, $\overset{(3)}{F}(\pi)$ are also in $L_4(x)$, furthermore the sectional curvatures with respect to π, $\overset{(1)}{F}(\pi)$, $\overset{(2)}{F}(\pi)$, and $\overset{(3)}{F}(\pi)$ are all equal.*

Using (III) of (2.1), we can see that if π is a 2-plane spanned by any two of x, $\overset{(1)}{F}(x)$, $\overset{(2)}{F}(x)$ or $\overset{(3)}{F}(x)$ and π' is the one spanned by the other two, then, the 2-plane obtained from π by operating $\overset{(1)}{F}$, $\overset{(2)}{F}$ or $\overset{(3)}{F}$ is π itself or π'.

COROLLARY 6.1. *Let x be an arbitrary vector and π the plane spanned by any two of x, $\overset{(1)}{F}(x)$, $\overset{(2)}{F}(x)$, $\overset{(3)}{F}(x)$ and π' the one spanned by the other two. Then,* *the plane obtained from π by operating $\overset{(1)}{F}$, $\overset{(2)}{F}$ or $\overset{(2)}{F}$ is π itself or π' and the sectional curvatures with respect to π and π' are equal.*

PART II

7. Preliminary Remarks. Let V_N be an N-dimensional Riemannian manifold whose class of differentiability is assumed sufficiently high (so far as the Hodge's theorem concerning the harmonic integrals of Riemannian manifolds be true).

The indices run from 1 to N unless otherwise stated and the summation convention is adopted.

To a p-form

$$\varphi = \frac{1}{p!}\varphi_{i_1\dots i_p} dx^{i_1}\dots dx^{i_p} = \varphi_{k_1\dots k_p} dx^{k_1}\dots dx^{k_p}\text{[2]} \qquad (k_1 < \dots < k_p)$$

of the manifold V_N we introduce the following operators.

d: exterior differentiation.

$$(d\varphi)_{i_1\dots i_{p+1}} = \sum_{\alpha=1}^{p+1} (-1)^{\alpha} \varphi_{i_1\dots \hat{i}_\alpha \dots i_{p+1};i_\alpha}$$

where $(\ )_{i_1\dots i_{p+1}}$ denotes the components of the $(p+1)$-form in the parenthesis and the semi-colon denotes the covarinnt differentiation and Λ the absence of the undermentioned component.

$*$: adjoint operator.

$$(*\varphi)_{j_1\dots j_{n-p}} = \sqrt{g}\ \varepsilon_{i_1\dots i_p j_1\dots j_{n-p}} \varphi^{i_1\dots i_p}$$
$$= \sqrt{g}\ \varepsilon_{i_1\dots i_p j_1\dots j_{n-p}} g^{i_1k_1}\dots g^{i_pk_p} \varphi_{k_1\dots k_p}$$

$(i_1 < \dots < i_p$; not summed with these indices)

where $\varepsilon_{i_1\dots i_p j_1\dots j_{n-p}}$ equals to $+1$ if $i_1\dots i_p\ j_1\dots j_{n-p}$ is an even permutation of $1\dots N$ and equals to -1 if it is an odd permutation and equals to zero if otherwise.

With repect to this $*$-operation, we see that the relation

$$** = (-1)^{(N-p)p}$$

holds true.

$\delta = (-1)^{Np+N+1} *d*$:

$$(\delta\varphi)_{i_1\dots i_{p-1}} = (-1)^p g^{jk} \varphi_{i_1\dots i_{p-1}j;k}$$

$\Delta = d\delta + \delta d$:

$$(\Delta\varphi)_{i_1\dots i_p} = -g^{jk}\varphi_{i_1\dots i_p;j;k} + \sum_{s=1}^{p} R^j{}_{i_s}\varphi_{i_1\dots i_{s-1}j i_{s+1}\dots i_p} \tag{7.1}$$
$$+ \sum_{s<t}^{p} R^{jk}{}_{i_s i_t}\varphi_{i_1\dots i_{s-1} j i_{s+1}\dots i_{t-1} k i_{t+1}\dots i_p},$$

2) In the following the products of differential forms designate the exterior products unless otherwise stated.

where $R^{jk}{}_{i_s i_t} = g^{kh} R^j{}_{h i_s i_t}$ and $R^j{}_{h i_s i_t}$ is the curvature tensor and $R_{ij} = g_{ik} R^k{}_j$ is the Ricci tensor.

If $\Delta\varphi = 0$, the p-form φ is called a harmonic form and the coefficients $\varphi_{i_1 \ldots i_p}$ are called components of a harmonic tensor. If the support of φ is compact, the condition $\Delta\varphi = 0$ is equivalent to the following two conditions:

$$d\varphi = 0, \quad \delta\varphi = 0$$

or

$$\varepsilon^{j_1 \ldots j_{p+1}}_{i_1 \ldots i_{p+1}} \varphi_{j_1 \ldots j_p; j_{p+1}} = 0, \quad g^{jk} \varphi_{i_1 \ldots i_{p-1} j, k} = 0,$$

where $\varepsilon^{j_1 \ldots j_{p+1}}_{i_1 \ldots i_{p+1}}$ equals to $+1$ if $(j_1 \ldots . j_{p+1})$ is an even permutation of $(i_1 \ldots . i_{p+1})$ and equals to -1 if it is an odd permutation and otherwise equals to zero.

If especially V_N is orientable, we can define an inner product (φ^p, ψ^p) of two p-forms φ^p and ψ^p whose supports are compact by

$$(\varphi^p, \psi^p) = \int \varphi^p * \psi^p = \int < \varphi^p, \ \psi^p > dV \tag{7.2}$$

where the integral be extended over the whole manifold and

$$< \varphi^p, \psi^p > = \varphi_{i_1 \ldots i_p} \psi^{i_1 \ldots i_p},$$

$$dV = \sqrt{g}\, dx^{i_1} \ldots . dx^{i_p}.$$

(φ^p, ψ^p) possesses the all properties as an inner product, that is,

$$\begin{cases} (c_1\varphi^p_1 + c_2\varphi^p_2, \psi^p) = c_1(\varphi^p_1, \psi^p) + c_2(\varphi^p_2, \psi^p), \quad (c_1, \ c_2: \text{constants}), \\ (\varphi^p, \ \psi^p) = (\psi^p, \ \varphi^p), \\ (\varphi^p, \varphi^p) \geqq 0, \\ (\varphi^p, \ \varphi^p) = 0 \rightarrow \varphi^p = 0. \end{cases}$$

Furthermore, if $N = 2m$ and V_{2m} is a $2m$-dimentional pseudo-kaehlerian manifold, we can introduce the following important operators where F_{ij} are the components of the pseudo-kaehlerian structure of V_{2m} and

$$F^i{}_j = g^{ik} F_{kj}, \quad F^{ij} = g^{jk} F^i{}_k,$$

the indices runing from 1 to 2m.

L: the exterior multiplication of $\Omega = \frac{1}{2} F_{ij} dx^i dx^j$ to an arbitrary form.

Λ: $*^{-1} L * = (-1)^{p(2m-p)} *L* = (-1)^p *L*$, where p is the degree of the operated form. We can see that

$$(\Lambda\varphi^p)_{i_1 \ldots i_{p-2}} = \frac{1}{2} F^{jk} \varphi_{i_1 \ldots i_{p-2} jk} \tag{7.3}$$

for a p-form

$$\varphi^p = \frac{1}{p!} \varphi_{i_1 \ldots i_p} dx^{i_1} dx^{i_p}$$

and the following theorem is known:

THEOREM I. *L and Λ transform harmonic forms into harmonic forms.*

This theorem is showed by the relations

$$L\Delta = \Delta L, \quad \Lambda\Delta = \Delta\Lambda$$

which are proved as follows.[3]

At first, we can easily see that

$$dL = Ld \tag{7.4}$$

by virtue of the property: $d\Omega = 0$. Then if we define an operator $\widetilde{d}$ by

$$\widetilde{d}\varphi^p = \frac{1}{(p+1)!} F^j{}_k\, \varphi_{i_1\ldots i_p;j}\, dx^k dx^{i_1}\ldots dx^{i_p}$$

for a p-form $\varphi^p = \frac{1}{p!}\varphi_{i_1\ldots i_p} dx^{i_1}\ldots dx^{i_p}$, we have

$$\begin{aligned}(d\widetilde{d} + \widetilde{d}d)\varphi^p &= \frac{1}{(p+2)!} F^j{}_k\, \varphi_{i_1\ldots i_p;j;h}\, dx^h dx^k dx^{i_1}\ldots dx^{i_p} \\ &\quad + \frac{1}{(p+2)!} F^j{}_k\, \varphi_{i_1\ldots i_p;h,j}\, dx^k dx^h dx^{i_1}\ldots dx^{i_p} \\ &= \frac{1}{(p+2)!} F^j{}_k (\varphi_{i_1\ldots i_p;j;h} - \varphi_{i_1\ldots i_p;h;j})\, dx^h dx^k dx^{i_1}\ldots dx^{i_p} = 0.\end{aligned} \tag{7.5}$$

Cosider a normal coordinate system with center P_0, we see that

$$(\delta L\varphi^p)_{P_0} = (L\delta\varphi^p - \widetilde{d}\varphi^p)_{P_0}$$

thesefore, at each point of the manifold

$$\delta L = L\delta - \widetilde{d} \tag{7.6}$$

holds good.

By (7.4), (7.5) and (7.6) we can verify the equality

$$L\Delta = \Delta L.$$

The latter equality $\Lambda\Delta = \Delta\Lambda$ is proved by using the former and relations

$$*\Delta = \Delta *, \quad *L = L*, \quad *\Lambda = L*.$$

Let L^r be the iteration of L r times, then we have

$$\Lambda L^r = L^r\Lambda + r(m-p-r+1)L^{r-1}, \qquad (p \leqq m-2r) \tag{7.7}$$

especally if $r = 1$, we have

$$\Lambda L = L\Lambda + (m-p)E, \tag{7.8}$$

where E denotes the identity operation.

A p-form φ^p is called *effective* or *of class* 0 *or primitive* if

$$\Lambda\varphi^p = 0.$$

A p-form $L^h\varphi_0^{p-2h}$ is called of *class h*, where φ_0^{p-2h} is an effective $(p-2h)$-form.

Then, the following decomposition theorems hold good, which are proved by Hodge for Kählerian manifold for the first time and extended by Lichnerowicz to pseudo-kaehlerian manifolds (Hodge, [1]; Lichnerowicz, [3]).

THEOREM II. *An arbitrary p-form φ^p can be decomposed uniquely in the*

3) For example, see Guggenheimer, [3] Anhang.

following form:

$$\varphi^p = \varphi_0^p + L\varphi_0^{p-2} + \dots + L^h \varphi_0^{p-2h} \qquad \left(h \leqq \left[\frac{p}{2}\right]\right)$$

where $\varphi_0^p, \dots, \varphi_0^{p-2h}$ *are effective forms.*

From this theorem, we have

THEOREM III. *ΛL is an isomorphism of the linear vector space Φ^p spanned by all p-forms ($p \leqq m-2$). And therefore L is an isomorphism from Φ^p* into Φ^{p+2} ($p \leqq m-2$).

Cosequently, if $\varphi^p \neq 0$, then $L\varphi^p \neq 0$ ($p \leqq m-2$). Since L and Λ transform harmonic forms into harmonic forms, Theorem II turns into the decomposition theorem of the p-th cohomology group (coefficients real), if V_{2m} is compact and orientable.

THEOREM IV. *If V_{2m} is compact, orientable, the p-th cohomology group H^p ($p \leqq m$) can be decomposed into the form*:

$$H^p = H_0^p + LH_0^{p-2} + \dots + L^h H_0^{p-2h}, \qquad \left(h \leqq \left[\frac{p}{2}\right]\right)$$

where $H_0^p, \dots, H_0^{p-2h}$ are subgroups generated by p-, ..., and *($p-2h$)-th effective cohomology classes respectively.*

The products mean the cup products. From this theorem, we have

THEOREM V. *Let d_0^p be the dimension of the linear vector space spanned by all effective harmonic p-forms and B_p be the p-th Betti number, then*

$$d_0^p = B_p - B_{p-2} \geqq 0 \qquad (p \leqq m).$$

And the odd dimensional Betti numbers are even and the even dimensional Betti numbers are $\geqq 1$.

Using the above theorems, we treat differential forms in our V_{4n}, which is *orintable* but not necessarily compact unless otherwise stated.

8. **Harmonic forms of degree odd.** In this section, the indices $i, j, k, \dots$ run over $1, \dots, 4n$.

Since the three pseudo-kaehlerian structures

$$\overset{(u)}{F} = (\overset{(u)}{F^i}_j) \qquad (u = 1, 2, 3)$$

are covariant constant, the integrability conditions are given by

$$R_{lmkh}\overset{(u)}{F^l}_i \overset{(u)}{F^m}_j = R_{ijkh} \qquad (u = 1, 2, 3; \text{ not summed})$$

or

$$R^{lm}{}_{kh}\overset{(u)}{F^i}_l \overset{(u)}{F^j}_m = R^{ij}{}_{kh} \qquad (u = 1, 2, 3; \text{ not summed}). \tag{8.1}$$

And furthermore

$$R^{ij}{}_{lm}\overset{(u)}{F^l}_k \overset{(u)}{F^m}_h = R^{ij}{}_{kh} \tag{8.2}$$

$$R^a{}_i \overset{(u)}{F^j}_a = R^j{}_a \overset{(u)}{F^a}_i \tag{8.3}$$

$(u = 1, 2, 3; \text{ not summed})$

hold good.

Let $\mathfrak{H}^p$ be the linear vector space spanned by all harmonic p-forms of V_{4n} and put

$$\overset{(u)}{F}{}^{i_1}{}_{j_1}\cdots\overset{(u)}{F}{}^{i_p}{}_{j_p}=\overset{(u)}{F}{}^{i_1\cdots i_p}{}_{j_1\cdots j_p}\qquad(u=1,2,3\,;\text{not summed}).\tag{8.4}$$

For a harmonic p-form $\varphi^p=\frac{1}{p!}\varphi_{i_1\ldots i_p}dx^{i_1}\ldots dx^{i_p}\in\mathfrak{H}^p$, we define a p-tensor

$$\overset{(u)}{\varphi}_{i_1\ldots i_p}=\overset{(u)}{F}{}^{j_1\cdots j_p}{}_{i_1\ldots i_p}\varphi_{j_1\ldots j_p}\qquad(u=1,2,3)\tag{8.5}$$

and consider the transformations

$$\overset{(u)}{\mathfrak{F}}:\varphi_p\rightarrow\overset{(u)}{\varphi}{}^p=\frac{1}{p!}\overset{(u)}{\varphi}_{i_1\ldots i_p}dx^{i_1}\ldots dx^{i_p}\qquad(u=1,2,3).\tag{8.6}$$

LEMMA 8.1. *The transformations $\overset{(u)}{\mathfrak{F}}$ $(u=1,2,3)$ are automorphisms of the linear vebtor space $\mathfrak{H}^p$ spanned by all harmonic p-forms of V_{4n}. That is to say, if $\varphi_{i_1\ldots i_p}$ is a non-zero harmonic p-tensor, then the p-tensors $\overset{(u)}{\varphi}_{i_1\ldots i_p}$ $(u=1,2,3)$ are also non-zero harmonic p-tensors.*

PROOF. Using (7.1), (8.5) and the equation

$$\overset{(u)}{F}{}^{j_1\cdots j_p}{}_{i_1\ldots i_p;k}=0,\qquad(u=1,2,3)$$

we can see that

$$\begin{aligned}(\Delta\overset{(u)}{\varphi}{}^p)_{i_1\ldots i_p}=&-g^{kh}\overset{(u)}{F}{}^{j_1\cdots j_p}{}_{i_1\ldots i_p}\varphi_{j_1\ldots j_p;k;h}\\&+\sum_{s=1}^{p}R^k{}_{i_s}\overset{(u)}{F}{}^{j_1\cdots j_p}{}_{i_1\ldots i_{s-1}k i_{s+1}\ldots i_p}\varphi_{j_1\ldots j_p}\\&+\sum_{s<t}^{p}R^{kh}{}_{i_s i_t}\overset{(u)}{F}{}^{j_1\cdots j_p}{}_{i_1\ldots i_{s-1}k\, i_{s+1}\ldots i_{t-1}h\, i_{t+1}\ldots i_p}\varphi_{j_1\ldots j_p}.\end{aligned}$$

By virtue of (8.1) and (8.2), we have

$$R^{kh}{}_{i_s i_t}\overset{(u)}{F}{}^{j_s}{}_k\overset{(u)}{F}{}^{j_t}{}_h=R^{j_s j_t}{}_{kh}\overset{(u)}{F}{}^k{}_{i_s}\overset{(u)}{F}{}^h{}_{i_t},\qquad(u=1,2,3\,;\text{ not summed})$$

therefore, we get

$$\begin{aligned}&R^{kh}{}_{i_s i_t}\overset{(u)}{F}{}^{j_1\cdots j_p}{}_{i_1\ \,i_{s-1}k\, i_{s+1}\ldots i_{t-1}h\, i_{t+1}\ldots i_p}\varphi_{j_1\ldots j_p}\\&=R^{j_s j_t}{}_{kh}\overset{(u)}{F}{}^{j_1\ldots k\ldots h\ldots j_p}{}_{i_1\ldots i_{s-1}i_s i_{s+1}\ldots i_{t-1}i_t i_{t+1}\ldots i_p}\varphi_{j_1\ldots j_p}\qquad(u=1,2,3)\\&=R^{kh}{}_{j_s j_t}\overset{(u)}{F}{}^{j_1\cdots j_p}{}_{i_1\ldots i_p}\varphi_{j_1\ldots k\ldots h\ldots j_p}\end{aligned}$$

from (8.4). And we also have

$$\begin{aligned}&R^k{}_{i_s}\overset{(u)}{F}{}^{j_1\cdots j_p}{}_{i_1\ldots i_{s-1}k i_{s+1}\ldots i_p}\varphi_{j_1\ldots j_p}\\&=R^{j_s}{}_k\overset{(u)}{F}{}^{j_1\ldots k\ldots j_p}{}_{i_1\ldots i_{s-1}i_s i_{s+1}\ldots i_p}\varphi_{j_1\ldots j_p}\qquad(u=1,2,3)\\&=R^k{}_{j_s}\overset{(u)}{F}{}^{j_1\cdots j_p}{}_{i_1\ldots i_p}\varphi_{j_1\ldots k\ldots j_p}.\end{aligned}$$

Consequently, it becomes that

$$\begin{aligned}
&(\Delta \overset{(u)}{\varphi^p})_{i_1\ldots i_p} \\
&= \overset{(u)}{F}{}^{j_1\ldots j_p}{}_{i_1\ldots i_p}\Big[-g^{kh}\varphi_{j_1\ldots j_p;k;h}+\sum_{s=1}^{p}R^k{}_{j_s}\varphi_{j_1\ldots j_{s-1}kj_{s+1}\ldots j_p} \\
&\qquad\qquad +\sum_{s<t}^{p}R^{kh}{}_{j_sj_t}\varphi_{j_1\ldots j_{s-1}kj_{s+1}\ldots j_{t-1}hj_{t+1}\ldots j_p}\Big] \\
&= \overset{(u)}{F}{}^{j_1\ldots j_p}{}_{i_1\ldots i_p}(\Delta\varphi^p)_{j_1\ldots j_p}, \qquad (u=1,2,3)
\end{aligned}$$

from which we see that

$$\Delta\varphi^p = 0 \rightarrow \Delta\overset{(u)}{\varphi}_p = 0 \qquad (u = 1, 2, 3).$$

The transformation $\overset{(u)}{\mathfrak{F}}$ are non-singular, that is, if $\overset{(u)}{\varphi}{}^p = 0$, then $\varphi^p = 0$ ($u =$ 1, 2, 3), which is easily seen from the definition.

q. e. d.

We consider the case in which p is odd and for the sake of brevity, we put

$$\overset{(u)}{F}{}^{i_1\ldots i_p}{}_{j_1\ldots j_p} = \overset{(u)}{F}{}^{\xi}{}_{\eta} \qquad (u = 1, 2, 3)$$

where $\xi = (i_1\ldots. i_p)$, $\eta = (j_1\ldots. j_p)$. And similarly, we put

$$g_{i_1j_1}\ldots. g_{i_pj_p} = G_{i_1\ldots i_p, j_1\ldots j_p} = G_{\xi,\eta},$$
$$g^{i_1j_1}\ldots. g^{i_pj_p} = G^{i_1\ldots i_p, j_1\ldots j_p} = G^{\xi,\eta},$$

where $\xi = (i_1\ldots. i_p)$, $\eta = (i_1\ldots. j_p)$ as in the above. Then, we can easily see that

$$G_{\xi\eta}G^{\eta\zeta} = \delta^{\zeta}_{\xi}$$

where δ^{ζ}_{ξ} is the Kronecker's delta. Since p is odd, by the definition of $\overset{(u)}{F}{}^{\xi}_{\eta} = F^{i_1\ldots i_p}{}_{j_1\ldots j_p}$ and by (2.1) of §2, we see that

$$(8.7)\begin{cases} \overset{(u)}{F}{}^{\xi}{}_{\eta}\overset{(u)}{F}{}^{\eta}{}_{\zeta} = -\delta^{\xi}_{\zeta}, & (u = 1, 2, 3) \\ G_{\xi\eta}\overset{(u)}{F}{}^{\xi}_{\zeta}\overset{(u)}{F}{}^{\eta}{}_{\kappa} = G_{\zeta\kappa}, & \\ \overset{(u)}{F}{}^{\xi}{}_{\eta}\overset{(v)}{F}{}^{\eta}{}_{\zeta} = \varepsilon_{uvw}\overset{(w)}{F}{}^{\xi}{}_{\zeta}, & (u, v, w = 1, 2, 3;\ \text{any two of them are not equal and not summed in } w) \end{cases}$$

where ε_{uvw} is equal to $+1$ if (uvw) is an even permutation of (123) and -1 if it is an odd permutation.

If we put

$$G_{\xi\eta}\overset{(u)}{F}{}^{\eta}{}_{\zeta} = \overset{(u)}{F}_{\xi\zeta}, \qquad (u = 1, 2, 3)$$

then from the first two equations of (8.7), we see that $F_{\xi\zeta}$ is anti-symmetric with respect to ξ and ζ. We say two differential forms φ^p, ψ^p whose supports are compact to be orthogonal, if

$$(\varphi^p, \psi^p) = \int < \varphi^p,\ \psi^p > dV = 0,$$

where dV is the volume element of the manifold.

It is easily verified that non-zero mutually orthogonal p-forms are linearly independent in real constant coefficients.

LEMMA 8.2. *In V_{4n} (of class C^r, $r \geqq 1$), if φ^p is a differential p-form where p is odd and if the support of φ^p is compact, then φ^p, $\overset{(1)}{\mathfrak{F}}\varphi^p$, $\overset{(2)}{\mathfrak{F}}\varphi^p$, and $\overset{(3)}{\mathfrak{F}}\varphi^p$ are mutually orthogonal.*

PROOF. For brevify, put

$$\varphi_{i_1\ldots i_p} = \varphi_\xi \quad , \qquad \varphi^\xi = G^{\xi\eta}\varphi_\eta,$$

then we have

$$(\overset{(u)}{\mathfrak{F}}\varphi^p)_{i_1\ldots i_p} \equiv (\overset{(u)}{\mathfrak{F}}\varphi^p)_\xi = \overset{(u)}{F}{}^\eta{}_\xi\,\varphi_\eta \qquad (u = 1, 2, 3)$$

where $\xi = (i_1\ldots . i_p)$.

Using (8.7) and in the similar way as the proof of Lemma of §2, we get

$$(\varphi^p, \overset{(u)}{\mathfrak{F}}\varphi^p) = \int < \varphi^p, \overset{(u)}{\mathfrak{F}}\varphi^p > dV = \int (G^{\xi\eta}\varphi_\xi \overset{(u)}{F}{}^\zeta{}_\eta\,\varphi_\zeta) dV = \int (\overset{(u)}{F}_{\xi\eta}\varphi^\xi\varphi^\eta)\,dV = 0,$$

$$(\overset{(u)}{\mathfrak{F}}\varphi^p, \overset{(v)}{\mathfrak{F}}\varphi^p) = \int < \overset{(u)}{\mathfrak{F}}\varphi^p, \overset{(v)}{\mathfrak{F}}\varphi^p > dV = \int (G^{\xi\eta}\overset{(u)}{F}{}^\zeta{}_\xi\varphi_\zeta \overset{(v)}{F}{}^\kappa{}_\eta\varphi_\kappa)\,dV$$

$$= \varepsilon \int (\overset{(w)}{F}_{\xi\eta}\,\varphi^\xi\varphi^\eta)\,dV = 0,$$

$$(u, v, w = 1, 2, 3\,;\ u \neq v \neq w\,;\ \varepsilon = +1 \text{ or } -1)$$

which is to be proved.

LEMMA 8.3. *In V_{4n} (of class C^r, $r \geqq 1$), let φ^p be a non-zero differential p-form with compact support and ψ^p be a non-zero differential p-form with compact support which is orthogonal to four p-forms φ^p, $\overset{(u)}{\mathfrak{F}}\varphi^p$ $(u = 1, 2, 3)$, where p is odd. Then ψ^p, $\overset{(1)}{\mathfrak{F}}\psi^p$, $\overset{(2)}{\mathfrak{F}}\psi^p$ and $\overset{(3)}{\mathfrak{F}}\psi^p$ are mutually orthogonal and orthogonal to the four p-forms φ^p, $\overset{(u)}{\mathfrak{F}}\varphi^p$ $(u = 1, 2, 3)$.*

PROOF. The orthogonality of any two of ψ^p, $\overset{(u)}{\mathfrak{F}}\psi^p$ $(u=1, 2, 3)$ is already proved by Lemma 8.2.

Since ψ^p is orthogonal to φ^p and $\overset{(u)}{\mathfrak{F}}\varphi^p$ $(u = 1, 2, 3)$, we have

$$(\varphi^p, \psi^p) = \int (G_{\xi\eta}\,\varphi\ \ \psi^\eta)\,dV = 0,$$

$$(\overset{(u)}{\mathfrak{F}}\varphi^p, \psi^p) = -(\overset{(u)}{F}_{\xi\eta}\,\varphi^\xi\,\psi^\eta)\,dV = 0.$$

From these relations, we see that

$$(\varphi^p, \overset{(u)}{\mathfrak{F}}\psi^p) = \int (G^{\xi\eta}\varphi_\xi \overset{(u)}{F}{}^\zeta{}_\eta\,\psi_\zeta)\,dV = \int (\overset{(u)}{F}_{\xi\eta}\,\varphi^\xi\,\psi^\eta)\,dV = 0,$$

$$(\overset{(u)}{\mathfrak{F}}\varphi^p, \overset{(v)}{\mathfrak{F}}\psi^p) = \begin{cases} \varepsilon \int (G_{\xi\eta}\, \varphi^\xi\, \psi^\eta)\, dV = 0 \\ \varepsilon' \int (\overset{(w)}{F}_{\xi\eta}\, \varphi^\xi\, \psi^\eta)\, dV = 0 \end{cases}, \qquad (\text{for } u, v, w = 1, 2, 3)$$

$$(\varepsilon, \varepsilon' = +1 \text{ or } -1)$$

which proves the Lemma.

From Lemma 8.2 and Lemma 8.3, we have

THEOREM 8.1. *In our V_{4n} (of class C^r, $r \geqq 4$), if the number of linearly independent (in real coefficients) harmonic forms with compact suppotrs of odd degree is finite, then it is $\equiv 0$ (mod 4).*

PROOF. If there exists a non-zero harmonic form φ^p, then $\overset{(1)}{\mathfrak{F}}\varphi^p$, $\overset{(2)}{\mathfrak{F}}\varphi^p$ and $\overset{(3)}{\mathfrak{F}}\varphi^p$ are also harmonic by Lemma 8.1. And these are mutually orthogonal by Lemma 8.2, and so linearly independent in real coefficients.

If furthermore there exists another harmonic p-form ψ^p linearly independent from the four p-forms mentioned above, we can find a harmonic p-form orthogonal to them. Then we can find 8 mutually orthogonal and hence 8 linearly independent harmonic p-forms by Lemma 8.3. Repeating similar process we get the conclusion of the theorem.

If especially V_{4n} is compact and the class of differentiability is sufficiently high[4], this theorem can be lead to the following Corollary.

COROLLARY 8.1. *Let V_{4n} be compact and the class of differentiability be sufficiently high[4] and let B_{2q+1} be the odd dimensional Betti numbers of V_{4n}, then*

$$B_{2q+1} \equiv 0 \qquad (mod\ 4).$$

For the 1-dimensional Betti number we can study more precisely, if V_{4n} is compact.

The following theorem is known.

THEOREM. *In a compact Riemanian manifold, in order that a harmonic vector φ^i satisfy*

$$R_{jk}\, \varphi^j\, \varphi^k \geqq 0$$

it is necessary and sufficient that φ^i is a parallel vector field, that is φ^i satisfy $\varphi^i{}_{;j} = 0$ (for ex. Yano, [1]).

Since $R_{jk} = 0$ in our V_{4n}, the above theorcm is applicable if V_{4n} is compact, and hence a vector φ^i is harmonic if and only if it is parallel vector field. Then from Corollary 8.1, we get

$$B_1 = 4r \qquad (r \geqq 0)$$

for the 1-dimensional Betti number B_1.

4) So far as the Hodge's theorem concerning the harmonic integrals of Riemannian manifolds be true.

The linear vector space $\mathfrak{H}^1$ of all harmonic 1-forms is spanned by $4r$ linearly independent (in real coefficients) harmonic forms whose coefficients are components of a harmonic vectors. These $4r$ vectors $\varphi_{(1)i}, \ldots, \varphi_{(4r)i}$ are linearly independent with respect to coefficients of scalar functions. For, if otherwise, we can put without any loss of generality,

$$\text{(8.8)} \qquad \begin{cases} \varphi_{(r'+1)i} = \alpha_{(1)}\varphi_{(1)i} + \ldots + \alpha_{(r')}\varphi_{(r')i}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \varphi_{(4r)i} = \rho_{(1)}\varphi_{(1)i} + \ldots + \rho_{(r')}\varphi_{(r')i}, \end{cases} \qquad (r' < 4r)$$

where $\alpha_{(1)}, \ldots, \alpha_{(r')}, \ldots, \rho_{(1)}, \ldots, \rho_{(r')}$ are scalar functions and $\varphi_{(1)i}, \ldots, \varphi_{(r')i}$ are lenearly independent with respect to coefficients of scalar functions. Since $\varphi_{(\ )i}, \ldots, \varphi_{(4r)i}$ are harmonic and hence parallel vector fields, by differentiating (8.8) covariantly, we get

$$\begin{cases} 0 = \alpha_{(1),j}\varphi_{(1)i} + \ldots + \alpha_{(r'),j}\varphi_{(r')i}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 0 = \rho_{(1),j}\varphi_{(1)i} + \ldots + \rho_{(r'),j}\varphi_{(r')} \end{cases} \qquad \left(\alpha_{(1),j} = \frac{\partial\alpha^{(1)}}{\partial x^j}, \rho_{(1),j} = \frac{\partial\rho_{(1)}}{\partial x^j}, \text{ etc.}\right)$$

Multiplying an arbitrary vector v^j and contracting, these become

$$\begin{cases} 0 = \widetilde{\alpha}_{(1)}\varphi_{(1)} + \ldots + \widetilde{\alpha}_{(r')}\varphi_{(r')}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 0 = \widetilde{\rho}_{(1)}\varphi_{(1)i} + \ldots + \widetilde{\rho}_{(r')}\varphi_{(r')i}, \end{cases}$$

where $\widetilde{\alpha}_{(1)} = \alpha_{(1),j}v^j, \ldots, \widetilde{\alpha}_{(r')} = \alpha_{(r'),j}v^j, \ldots \widetilde{\rho}_{(1)} = \rho_{(1),j}v^j, \ldots, \widetilde{\rho}_{(r')} = \rho_{(r'),j} v^j$ are scalar functions. Since $\varphi_{(1)i}\ldots, \varphi_{(r')i}$ are linearly independent in scalar functions, we have

$$\widetilde{\alpha}_{(1)} = 0, \ldots, \widetilde{\alpha} \quad = 0, \ldots, \widetilde{\rho}_{(1)} = 0, \ldots, \widetilde{\rho}_{(r')} = 0,$$

that is

$$\alpha_{(1),j}v^j = 0, \ldots, \alpha_{(r'),j}v^j = 0, \ldots, \rho_{(1),j}v^j = 0, \ldots, \rho_{(r'),j}v^j = 0.$$

As v^i is arbitrary, we get $\alpha_{(1),j} = 0, \ldots, \alpha_{(r'),j} = 0, \ldots, \rho_{(1),j} = 0, \ldots, \rho_{(r'),j} = 0$ and hence $\alpha_{(1)} =$ const., $\ldots, \alpha_{(r')} =$ const., $\ldots, \rho_{(1)} =$ const., $\ldots, \rho_{(r')} =$ const., which contradicts by (8.8) to the fact that $\varphi_{(1)i}, \ldots, \varphi_{(4r)i}$ are linearly independent in constant coefficients.

Consequently, V_{4n} admits $4r$ linearly independent perallel vector fields, hence V_{4n} decomposes locally into the form

$$V_{4n} = E_r \times V_{4(n-r)}$$

where E_{4r} is a $4r$-dimensional compact flat manifol and $V_{4(n-r)}$ is a Riemannian manifold whose resricted homogeneous holonomy group is $Sp(n-r)$ or one of its subgroups which does not fix any directions. If otherwise, $V_{4(n-r)}$ admits a parallel and hence harmonic vector fields, hence there are more than $4r$ harmonic vector fields, contradictorily to the fact that $B_1 = 4r$.

Conversely, if V_{4n} decomposes into the above form locally, then we can easily see that $B_1 = 4r$.

THEOREM 8.2. *Let V_{4n} in consideration be compact and denote the 1-dimen-*

sional Betti number by B_1, *then*

$$B_1 = 4r \qquad (r:\ \textit{non-negative integers}).$$

Furthermore, V_{4n} *decomposes locally into the direct product*:

$$V_{4n} = E \times V_{4(n-r)}$$

where E_{4n} *is a* $4r$*-dimensional compact flat manifold and* $V_{4(n-r)}$ *is a compact Riemannian manifold whose restricted homogeneous holonomy group is* $Sp(n-r)$ *or one of its subgroups which does hot fix any directions. The converse is also true.*

We see therefore that $B_1 \leqq 4n$. And if V_{4n} is irreducible, then $B_1 = 0$.

9. Harmonic forms of degree even. Let R be the Grassmann ring of differential forms of V_{4n}. For a suitably chosen orthogonal frame of reference, we can take

$$(\overset{(1)}{F}_{ij}) = \begin{pmatrix} 0 & E_n & 0 & 0 \\ -E_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -E_n \\ 0 & 0 & E_n & 0 \end{pmatrix}, \quad (\overset{(1)}{F}_{ij}) = \begin{pmatrix} 0 & 0 & E_n & 0 \\ 0 & 0 & 0 & E_n \\ -E_n & 0 & 0 & 0 \\ 0 & -E_n & 0 & 0 \end{pmatrix}, \quad (\overset{(3)}{F}_{ij}) = \begin{pmatrix} 0 & 0 & 0 & E_n \\ 0 & 0 & -E_n & 0 \\ 0 & E_n & 0 & 0 \\ -E_n & 0 & 0 & 0 \end{pmatrix}.$$

In this section the range of indices are set forth as follows:

$$\begin{cases} a, b, c, \ldots\ldots = 1, 2, \ldots\ldots, n; \\ a^*, b^*, c^*, \ldots\ldots = a+n,\ b+n,\ c+n, \ldots\ldots\ (\leqq 2n) \\ \bar{a}, \bar{b}, \bar{c}, \ldots\ldots = a+2n,\ b+2n,\ c+2n, \ldots\ldots\ (\leqq 3n) \\ \bar{a}^*, \bar{b}^*, \bar{c}^*, \ldots\ldots = a+3n,\ b+3n,\ c+3n, \ldots\ldots\ (\leqq 4n) \end{cases}$$

Then, $\overset{(1)}{\Omega} = \dfrac{1}{2}\overset{(1)}{F}_{ij}\omega^i\omega^j$, $\overset{(2)}{\Omega} = \dfrac{1}{2}\overset{(2)}{F}_{ij}\omega^i\omega^j$, $\overset{(3)}{\Omega} = \dfrac{1}{2}\overset{(3)}{F}_{ij}\omega^i\omega^j$ can be written in the following form

$$(9.1) \qquad \begin{cases} \overset{(1)}{\Omega} = \overset{(1)}{F}_{aa^*}\omega^a\omega^{a^*} + \theta_1 = \sum\limits_a \omega^a\omega^{a^*} + \theta_1 \\ \overset{(1)}{\Omega} = \overset{(2)}{F}_{a\bar{a}}\omega^a\omega^{\bar{a}} + \theta_2 = \sum\limits_a \omega^a\omega^{\bar{a}} + \theta_2 \\ \overset{(3)}{\Omega} = \overset{(3)}{F}_{a\bar{a}^*}\omega^a\omega^{\bar{a}^*} + \theta_3 = \sum\limits_a \omega^a\omega^{\bar{a}^*} + \theta_3, \end{cases}$$

where θ_1, θ_2, θ_3 are the sum of the terms which do not contain ω^a ($a = 1, \ldots\ldots, n$).

Consider the $2r$-form of the type

$$(9.2) \qquad \varphi^{2r} = \overset{(1)}{\Omega}{}^{\lambda}\overset{(2)}{\Omega}{}^{\mu}\overset{(3)}{\Omega}{}^{\nu}, \qquad (\lambda+\mu+\nu = r)$$

where $\overset{(u)}{\Omega}{}^{\lambda}$ ($u = 1, 2, 3$) designate the exterior product of $\overset{(u)}{\Omega}$ λ times and $r \leqq n$. There are ${}_3H_r$ different forms of the type (9.2), where ${}_3H_r = \binom{r+2}{r}$. We denote the set of such forms by Φ^{2r}. In φ^{2r} the sum of the terms which contain just r of ω^a ($a = 1, \ldots\ldots, n$) is given by

$$\sum \omega^{a_1}\ldots\omega^{a_\lambda}\omega^{a_1}\ldots\omega^{b_\mu}\omega^{c_1}\ldots\omega^{c_\nu}(\omega^{a_1^*}\ldots\omega^{a^*}\omega^{\bar{b}_1}\ldots\omega^{\bar{b}_\mu}\omega^{\bar{c}_1*}\ldots\omega^{\bar{c}_\nu*})$$

$(a_1\ldots, a_\lambda, b_1, \ldots, b_\mu, \ c_1, \ldots, c_\nu = 1, \ldots, n$; any two of them are not equal).

Next, let

$$\psi^{2r} = \overset{(1)}{\Omega}{}^{\lambda'} \cdot \overset{(2)}{\Omega}{}^{\mu'} \cdot \overset{(3)}{\Omega}{}^{\nu'} \qquad (\lambda' + \mu' + \nu' = r)$$

be a form in Φ^{2r} different from φ^{2r}. In ψ^{2r} the sum of the terms which contain just r of ω^a $(a = 1, \ldots, n)$ is given by

$$\sum \omega^{a_1}\ldots\omega^{a\lambda'}\,\omega^{b_1}\ldots\omega^{b\mu'}\omega^{c1}\ldots\omega^{c\nu'}(\omega^{a_1^*}\ldots\,\omega^{a^*}_{\lambda}\,\omega^{\bar{b}_1}\ldots\omega^{\bar{b}\mu'}\omega^{\bar{c}^*_1}\ldots\omega^{\bar{c}^*_\nu})$$

$(a_1\ldots, a_{\lambda'}, b_1, \ldots, b_{\mu'}, c_1, \ldots, c_{\nu'} = 1, \ldots, n$; any two of them are not equal). Since φ^{2r} and ψ^{2r} are different, at least one of 'the pairs (λ, λ'), (μ, μ'), (ν, ν') is not equal, for example, $\lambda \neq \lambda'$. Therefore, In Φ^{2r}, there are no forms which contain just λ of ω^{a*}, μ of $\omega^{\bar{b}}$, ν of $\omega^{\bar{c}*}$ other than φ^{2r}. In other words, a form in Φ^{2r} contains some bases of R which are not contained in any other forms in Φ^{2r}. Consequently, the forms in Φ^{2r} are linearly independent with respect to constant coefficients.

And all forms in Φ^{2r} are non-zero harmonic by Theorem I and Theorem III of §7. Therefore we have

THEOREM 9.1. *In V_{4n} (of class C^r, $r \geqq 4$), let h_{2r} be the number of linearly independent (in real constant coefficients) harmonic 2r-forms. Then,* $h_{2r} \geqq {}_3H_r = \binom{r+2}{r}$.

COROLLARY 9.1. *If the V_{4n} is compact, orientable and the class of differentiability is sufficiently high, then the 2r-th ($r \leqq n$) Betti number B_{2r} satisfies the inequblity:*

$$B_{2r} \geqq {}_3H_r = \binom{r+2}{r}.$$

10. Decomposition theorem. In the similar way to pseudo-kaehlerian case, we introduce the following operators for a p-form $\varphi^p = \frac{1}{p!}\varphi_{i_1\ldots i_p}\,dx^{i_1}\ldots dx^{i_p}$

$$\begin{cases} L_{(1)}: \text{exterior multiolication by } \overset{(1)}{\Omega} = \frac{1}{2}\overset{(1)}{F}_{ij}\,dx^i\,dx^j \\ L_{(2)}: \text{———} \quad \text{by } \overset{(2)}{\Omega} = \frac{1}{2}\overset{(2)}{F}_{ij}\,dx^i\,dx^j \\ L_{(3)}: \text{———} \quad \text{by } \overset{(3)}{\Omega} = \frac{1}{2}\overset{(3)}{F}_{ij}\,dx^i\,dx^j, \end{cases}$$

$$\begin{cases} \Lambda_{(1)}: *^{-1}L_{(1)}* = (-1)^p * L_{(1)}* \\ \Lambda_{(2)}: *^{-1}L_{(2)}* = (-1)^p * L_{(2)}* \\ \Lambda_{(3)}: *^{-1}L_{(3)}* = (-1)^p * L_{(3)}*, \end{cases}$$

then, we can see that

$$(10.1) \qquad (\Lambda_{(u)}\varphi^p)_{i_1\ldots i_{p-2}} = \frac{1}{2}\overset{(u)}{F}{}^{kh}\varphi_{i_1\ldots i_{p-2}kh},$$

$$(10.2) \qquad \Lambda_{(u)}L^r_{(u)}\varphi^p = L^r_{(u)}\Lambda_{(u)}\varphi^p + r(2n-p-r+1)L^{r-1}_{(u)}\varphi^p, \qquad (u=1,2,3)$$

analogously to the pseudo-kaehlerian case.

And since the linear combination $\alpha \overset{(1)}{F} + \beta \overset{(2)}{F} + \gamma \overset{(3)}{F}$ (α, β, γ: scalar functions; $\alpha^2 + \beta^2 + \gamma^2 = 1$) is also a pseudo-kaehlerian structure, we can introduce the operators

$$
(10.3) \qquad \begin{cases} L: \alpha L_{(1)} + \beta L_{(2)} + \gamma L_{(3)} \\ \Lambda: \alpha \Lambda_{(1)} + \beta \Lambda_{(2)} + \gamma \Lambda_{(3)} \end{cases} \qquad (\alpha^2 + \beta^2 + \gamma^2 = 1).
$$

The operators L, $L_{(u)}$, Λ, $\Lambda_{(u)}$ transform harmonic forms into harmonic forms. We call a p-form φ^p such as

$$\Lambda_{(u)} \varphi^p = 0$$

$\Lambda_{(u)}$-effective and call Λ-effective if $\Lambda \varphi^p = 0$.

An arbitrary p-form φ^p ($p \leqq 2n$) decomposes in the following three manners:

$$
(10.4) \quad \begin{cases} \varphi^p = \psi^p_{(1)} + L_{(1)} \psi^{p-2}_{(1)} + \dots + L^{q_1}_{(1)} \psi^{p-2q_1}_{(1)} & \left(q_1 \leqq \left[\frac{p}{2}\right]\right) \\ \quad\; = \psi^p_{(2)} + L_{(2)} \psi^{p-2}_{(3)} + \dots + L^{q_2}_{(2)} \psi^{p-2q_2}_{(2)} & \left(q_2 \leqq \left[\frac{p}{2}\right]\right) \\ \quad\; = \psi^p_{(3)} + L_{(3)} \psi^{p-2}_{(3)} + \dots + L^{q_3}_{(3)} \psi^{p-2q_3}_{(3)} & \left(q_3 \leqq \left[\frac{p}{2}\right]\right) \end{cases}
$$

where $\psi^{p-2h}_{(1)}$ ($h = 0, \dots, q_1$), $\psi^{p-2h}_{(2)}$ ($h = 0, \dots, q_2$) aud $\psi^{p-2h}_{(3)}$($h = 0, \dots, q_3$) are $\Lambda_{(1)}$-, $\Lambda_{(2)}$-, $\Lambda_{(3)}$-effective ($p-2h$)-forms respectively.

We also have the decomposition with respect to L:

$$
(10.5) \qquad \varphi^p = \psi^p + L\psi^{p-2} + \dots + L^q \psi^{p-2q} \qquad \left(q \leqq \left[\frac{p}{2}\right]\right)
$$

where ψ^{p-2h}($h = 0, 1, \dots, q$) is a Λ-effective ($p - 2h$)-form.

We call such a form as $L^s_{(u)} \psi^r_{(u)}$ where $\psi^r_{(u)}$ is $\Lambda_{(u)}$-effective to be of $L_{(u)}$-class s.

If $\mathfrak{H}^p$ is the linear vector space of all harmonic p-forms, then $\mathfrak{H}^p$ decomposes in following three manners:

$$
(10.6) \quad \begin{cases} \mathfrak{H}^p = \mathfrak{H}^p_{(1)} + L_{(1)} \mathfrak{H}^{p-2}_{(1)} + \dots + L^{q_1}_{(1)} \mathfrak{H}^{p-2q_1}_{(1)} & \left(q_1 \leqq \left[\frac{p}{2}\right]\right) \\ \quad\; = \mathfrak{H}^p_{(2)} + L_{(2)} \mathfrak{H}^{p-2}_{(2)} + \dots + L^{q_2}_{(2)} \mathfrak{H}^{p-2q_2}_{(2)} & \left(q_2 \leqq \left[\frac{p}{2}\right]\right) \\ \quad\; = \mathfrak{H}^p_{(3)} + L_{(3)} \mathfrak{H}^{p-2}_{(3)} + \dots + L^{q_3}_{(3)} \mathfrak{H}^{p-2q_3}_{(3)} & \left(q_3 \leqq \left[\frac{p}{2}\right]\right) \end{cases}
$$

where $L^h_{(u)} \mathfrak{H}^{p-2h}_{(u)}$ ($u = 1, 2, 3$; $h = 1, \dots, q_u$) are linear vector sub-spaces of all harmonic p-forms of $L_{(u)}$-class h.

Now, let

$$\psi_{(u)} = \frac{1}{r!} \psi_{(u) i_1 \dots i_p} dx^{i_1} \dots dx^{i_p} \qquad (u = 1, 2, 3)$$

be a $\Lambda_{(u)}$-effective r-form and consider the operations $\overset{(v)}{\mathfrak{F}}$ ($v = 1, 2, 3$) of §8, that is

$$\overset{(v)}{\mathfrak{F}}\Psi^r_{(u)} = \frac{1}{r.}\overset{(v)}{F}{}^{j_1}{}_{i_1}\dots\overset{(v)}{F}{}^{j_r}{}_{i_r}\Psi_{(u)j_1\dots j_r}dx^{i_1}\dots dx^{i_r} \qquad (u, v = 1, 2, 3)$$

or in tensor forms

$$(\overset{(v)}{\mathfrak{F}}\Psi^r_{(u)})_{i_1\dots i_p} = \overset{(v)}{F}{}^{j_1}{}_{i_1}\dots\overset{(v)}{F}{}^{j_r}{}_{i_r}\Psi_{(u)j_1\dots j_r}. \qquad (u, v = 1, 2, 3)$$

These operations are non-singular and taking account of the fact that $\Psi^r_{(u)}$ is $\Lambda_{(u)}$-effective, we see that

$$(\Lambda_{(u)}\overset{(v)}{\mathfrak{F}}\Psi^r_{(u)})_{i_1\dots i_{p-2}} = \frac{1}{2}\overset{(u)}{F}{}^{kh}(\overset{(v)}{F}{}^{j_1}{}_{i_1}\dots\overset{(v)}{F}{}^{j_{p-2}}{}_{i_{p-2}}\overset{(v)}{F}{}^{j_{p-1}}{}_k\overset{(v)}{F}{}^{j_p}{}_h\Psi_{(u)j_1\dots j_p})$$

$$= \frac{\varepsilon}{2}\overset{(u)}{F}{}^{j_1}{}_{i_1}\dots\overset{(v)}{F}{}^{j_{p-2}}{}_{i_{p-2}}\overset{(u)}{F}{}_{j_{p-1}}{}^{j_p}\Psi_{(u)j_1\dots j_{p-1}j_p} = 0 \qquad (\varepsilon = \pm 1).$$

That is to say, $\overset{(v)}{\mathfrak{F}}$ $(v = 1, 2, 3)$ transforms $\Lambda_{(u)}$-effective forms $(u = 1, 2, 3)$ again into $\Lambda_{(u)}$-effective forms.

Next, consider a form of $L_{(u)}$-class s $(u = 1, 2, 3)$:

$$L^s_{(u)}\Psi^r_{(u)} = \frac{1}{(s+r)!}\frac{1}{2^s r.}\varepsilon^{h_1h'_1\dots h_sh'_sk_1\dots k_r}_{i_1i'_1\dots i_si'_sj_1\dots j_r}\overset{(u)}{F}{}_{h_1h'_1}\dots\overset{(u)}{F}{}_{h_sh'_s}\Psi_{(u)k_1\dots k_r}dx^{i_1}\dots dx^{j_r},$$

$$(u = 1, 2, 3)$$

where $\Psi^r_{(u)}$ is $\Lambda_{(u)}$-effective. Then, we see that

$$\overset{(v)}{\mathfrak{F}}L^s_{(u)}\Psi^r_{(u)} = \frac{1}{(s+r)!}\frac{1}{2^s r.}\overset{(v)}{F}{}^{i_1}{}_{l_1}\overset{(v)}{F}{}^{i'_1}{}_{l'_1}\dots\overset{(v)}{F}{}^{i_s}{}_{l_s}\overset{(v)}{F}{}^{i'_s}{}_{l'_s}\overset{(v)}{F}{}^{j_1}{}_{m_1}\dots\overset{(v)}{F}{}^{j_r}{}_{m_r}$$

$$\times(\varepsilon^{h_1h'_1\dots h_sh'_sk_1\dots k_r}_{i_1i'_1\dots i_si'_sj_1\dots j_r}\overset{(u)}{F}{}_{h_1h'_1}\dots\overset{(u)}{F}{}_{h_sh'_s}\Psi_{(u)k_1\dots k_r})dx^{l_1}\dots dx^{m_r}$$

$$= \frac{1}{(s+r)!}\frac{1}{2^s r.}\varepsilon^{i_1i'_1\dots i_si'_sj_1\dots j_r}_{l_1l'_1\dots l_sl'_sm_1\dots m_r}(\overset{(u)}{F}{}_{i_1i'_1}\dots\overset{(u)}{F}{}_{i_si'_s}\overset{(v)}{F}{}^{k_1}{}_{j_1}\dots\overset{(v)}{F}{}^{k_r}{}_{j_r}\Psi_{(u)k_1\dots k_r})dx^{l_1}\dots dx^{m_r}$$

$$= c(L^s_{(u)}\overset{(v)}{\mathfrak{F}}\Psi^r_{(u)}) \qquad (c\text{: non-zero const.}).$$

Since $\Psi^r_{(u)}$ is $\Lambda_{(u)}$-effective, $\overset{(v)}{\mathfrak{F}}\Psi^r_{(u)}$ is also $\Lambda_{(u)}$-effective. From the above, we have

THEOREM 10.1. *The operations $\overset{(v)}{\mathfrak{F}}$ $(v = 1, 2, 3)$ are automorphisms of the linear vector space of all forms of $L_{(u)}$-class s $(s = 0, 1, \dots, 2n;\ u = 1, 2, 3)$.*

Since $\overset{(v)}{\mathfrak{F}}$ transform harmonic forms into harmonic forms, we have

COROLLARY 10.1. *The operations $\overset{(v)}{\mathfrak{F}}$ $(v = 1, 2, 3)$ are automorphisms of the linear vector spaces of all harmonic forms of $L_{(u)}$-class s $(u = 1, 2, 3;\ s = 0, 1, \dots, 2n)$.*

In particular, if p is odd and if the dimension of $\mathfrak{H}^p$ of all harmonic p-forms whose supports are compact is finite, then we have three decompositions of the forms (10.4) for an arbitrary forms $\varphi^p \in \mathfrak{H}^p$. If there exists a non-zero harmonic p-form $L^s_{(u)}\Psi^{p-2s}_{(u)}$ of $L_{(u)}$-class s, then there exist in $\mathfrak{H}^p$ four non-zero harmonic p-forms $L_{(u)}\Psi^{p-2s}_{(u)}$, $\overset{(v)}{\mathfrak{F}}(L_{(u)}\Psi^{p-2s}_{(u)})$ $(v = 1, 2, 3)$ by Corollary

10.1, these being orthogonal with respect to the inner product and hence linearly independent. If there exists another harmonic p-form of $L_{(u)}$-class s independent from the above four, we can find 8 linearly independent forms in $\mathfrak{H}^p$ in the similar way to §8.

THEOREM 10.2. *Let p be odd. If the dimension of $\mathfrak{H}^p$ of all harmonic p-forms with compact supports is finite, then in each decomposition* (10.6) *of $\mathfrak{H}^p$ the dimension of $L^h_{(u)}\mathfrak{H}^{p-2h}_{(u)}$ $(u = 1, 2, 3)$ is $\equiv 0$ (mod. 4).*

If furthermore V_{4n} is compact the decomposition (10.6) of $\mathfrak{H}^p$ turns into the decomsition of the p-th cohomology group H^p:

$$
\begin{aligned}
H^p &= H^p_{(1)} + L_{(1)} H^{p-2}_{(1)} + \ldots. + L^{q_1}_{(1)} H^{p-2q_1}_{(1)} && \left(q_1 \leqq \left[\frac{p}{2}\right]\right) \\
(10.7) \qquad &= H^p_{(2)} + L_{(2)} H^{p-2}_{(2)} + \ldots. + L^{q_2}_{(2)} H^{p-2q_2}_{(2)} && \left(q_2 \leqq \left[\frac{p}{2}\right]\right) \\
&= H^p_{(3)} + L_{(3)} H^{p-2}_{(3)} + \ldots. + L^{q_3}_{(3)} H^{p-2q_3}_{(3)} && \left(q_3 \leqq \left[\frac{p}{2}\right]\right).
\end{aligned}
$$

Let B_r and B_{r-2} $(r \leqq 2n)$ be the r-th and $(r-2)$-th Betti numbers of V_{4n} and let $d^r_{(u)}$ be the dimension of the linear vector space of $\Lambda_{(u)}$-effective harmonic p-forms, then

$$d^r_{(u)} = B_r - B_{r-2}, \qquad (u = 1, 2, 3)$$

from which we see that the rank of the subgroups $L^h_{(1)} H^{p-2h}_{(1)}$, $L^h_{(2)} H^{p-2h}_{(2)}$ and $L^h_{(3)} H^{p-2h}_{(3)}$ are equal for every $h \leqq \left[\frac{p}{2}\right]$ and $\equiv 0$ (mod 4) by the theorem.

COROLLARY 10.2. *Let V_{4n} be compact. Then the p-th cohomology group H^p decomposes in three manners such as* (10.7) *and the rank of each corresponding subgroups $L^h_{(1)} H^{p-2h}_{(1)}$, $L^h_{(2)} H^{p-2h}_{(2)}$ and $L^h_{(3)} H^{p-2h}_{(3)}$ are equal for every $h \leqq \left[\frac{p}{2}\right]$. If p is odd, these ranks are $\equiv 0$ (mod. 4).*

In the next place, let p be even and consider a harmonic p-form φ^p whose support is compact. Then $L^{r_1}_{(1)} L^{r_2}_{(2)} L^{r_3}_{(3)} \varphi^p$ are harmonic 0-forms, that is, constants for all non-negative integers r_1, r_2, and r_3 satisfying $r_1 + r_2 + r_3 = p/2$. Then the ${}_3H_{p/2}$ linear equations

$$
\begin{aligned}
(10.8) \qquad L^{r_1}_{(1)} L^{r_2}_{(2)} L^{r_3}_{(3)} \varphi^p &= \sum_{r'_1+r'_2+r'_3=p/2} (L^{r_1}_{(1)} L^{r_2}_{(2)} L^{r_3}_{(3)} \cdot 1,\ \overset{(1)}{\Omega}{}^{r'_1} \overset{(2)}{\Omega}{}^{r'_2} \overset{(3)}{\Omega}{}^{r'_3}) \sigma_{(r'_1 r'_2 r'_3)} \\
&= \sum_{r'_1+r'_2+r'_3=p/2} (\overset{(1)}{\Omega}{}^{r_1} \overset{(2)}{\Omega}{}^{r_2} \overset{(3)}{\Omega}{}^{r_3},\ \overset{(1)}{\Omega}{}^{r'_1} \overset{(2)}{\Omega}{}^{r'_2} \overset{(3)}{\Omega}{}^{r'_3}) \sigma_{(r'_1 r'_2 r'_3)}
\end{aligned}
$$

have a unique solution for unknown constants $\sigma_{(r'_1 r'_2 r'_3)}$. To show this, for brevity, write the ${}_3H_{p/2}$ forms $\overset{(1)}{\Omega}{}^{r_1} \overset{(2)}{\Omega}{}^{r_2} \overset{(3)}{\Omega}{}^{r_3}$ $(r_1 + r_2 + r_3 = p/2)$ as v_1, v_2, $\ldots. v_q$ $(q = {}_3H_{p/2})$, and $\sigma_{(r'_1 r'_2 r'_3)}$ as $c_\lambda (\lambda = 1, \ldots. q)$. Then (10.8) can be written in the form

$$(10.9)\quad \begin{cases} c_1(v_1, v_1) + c_2(v_2, v_1) + \dots + c_q(v_q, v_1) = d_1 \\ c_1(v_1, v_2) + c_2(v_2, v_2) + \dots + c_q(v_q, v_2) = d_2 \\ \dots\dots\dots\dots\dots\dots \\ c_1(v_1, v_q) + c_2(v_2, a_q) + \dots + c_q(v_q, v_q) = d_q, \end{cases}$$

where $c_1, \dots, c_q$ are unknown constants and $d_1, \dots, d_q$ and (v_λ, v_μ) $(\lambda, \mu = 1, \dots, q)$ are known constants. The determinant

$$\begin{vmatrix} (v_1, v_1) & (v_2, v_1) \dots (v_q, v_1) \\ (v_1, v_2) & (v_2, v_2) \dots (v_q, v^2) \\ \dots\dots & \\ (v_1, v_q) & (v_2, v_q) \dots (v_q, v_q) \end{vmatrix}$$

is not zero, for if otherwise, there exist constants $c'_1, \dots, c'_q$ which are not simultaneously equal to zero and satisfy

$$\begin{cases} c'_1(v_1, v_1) + c'_2(v_2, v_1) + \dots + c'_q(v_q, v_1) = 0, \\ \dots\dots\dots\dots\dots\dots \\ c'_1(v_1, v_q) + c'_2(v_2, v_q) + \dots + c'_q(v_q, v_q) = 0, \end{cases}$$

or

$$\begin{cases} (c'_1 v_1 + c'_2 v_2 + \dots + c'_q v_q, v_1) = 0, \\ \dots\dots\dots\dots\dots\dots \\ (c'_1 v_1 + c'_2 v_2 + \dots + c'_q v_q, v_q) = 0. \end{cases}$$

Since $c'_1 v_1 + \dots + c'_q v_q$ lies in the vector space spanned by linearly independent $v_1, \dots, v_q$, we must have

$$c'_1 v_1 + \dots + c'_q v_q = 0,$$

from which we get

$$c'_1 = c'_2 = \dots = c'_q = 0,$$

by virtue of the linear independence of $v_1, \dots, v_q$. But this is a contradiction.

Consequently, φ^p decomposes uniquely into the following form:

$$(10.10)\qquad \varphi^p = \tau^p + \sum_{r_1+r_2+r_3=p/2} \overset{(1)}{\Omega}{}^{r_1} \overset{(2)}{\Omega}{}^{r_2} \overset{(3)}{\Omega}{}^{r_3} \sigma_{(r_1 r_2 r_3)}, \quad (\sigma_{(r_1 r_2 r_3)}: \text{constants})$$

where τ^p satisfies the equations

$$(\tau_p, \overset{(1)}{\Omega}{}^{r_1} \overset{(2)}{\Omega}{}^{r_2} \overset{(3)}{\Omega}{}^{r_3}) = \int < \tau^p, \overset{(1)}{\Omega}{}^{r_1} \overset{(2)}{\Omega}{}^{r_2} \overset{(3)}{\Omega}{}^{r_3} > dV = 2^{r_1+r_2+r_3} \int \Lambda^{r_1}_{(1)} \Lambda^{r_2}_{(2)} \Lambda^{r_3}_{(3)} \tau^p \, dV = 0$$

for all r_1, r_2 and r_3 satisfying $r_1 + r_2 + r_3 = p/2$. These equations are equivalent to

$$(10.11)\qquad \Lambda^{r_1}_{(1)} \Lambda^{r_2}_{(2)} \Lambda^{r_3}_{(3)} \tau^p = 0 \qquad (r_1 + r_2 + r_3 = p/2),$$

since $\Lambda^{r_1}_{(1)} \Lambda^{r_2}_{(2)} \Lambda^{r_3}_{(3)} \tau^p$ are harmonic 0-forms, that is, constants.

THEOREM 10.3. *Let p be even. If the dimension of the linear vector space $\mathfrak{H}^p$ of all harmonic p-forms with compact supports is finite, then every p-form $\varphi^p \in \mathfrak{H}^p$ decomposes into the form*

$$\varphi^p = \tau^p + \sum_{r_1+r_2+r_3=p/2} \overset{(1)}{\Omega}{}^{r_1} \overset{(2)}{\Omega}{}^{r_2} \overset{(3)}{\Omega}{}^{r_3} \sigma_{(r_1 r_2 r_3)} \qquad (\sigma_{(r_1 r_2 r_3)} : \textit{constants})$$

where τ^p *is a harmonic* p*-form satisfying*

$$\Lambda_{(1)}^{r_1} \Lambda_{(2)}^{r_2} \Lambda_{(3)}^{r_3} \tau^p = 0,$$

where r_1, r_2 *and* r_3 *are non-negative integers satisfying* $r_1 + r_2 + r_3 = p/2$.

COROLLARY 10.3. *Let* V_{4n} *be compact and* p *be even. Then the* p*-th Betti number* B_p *can be given by*

$$B_p = \mathcal{E}_p + {}_3H_{p/2}$$

where $\mathcal{E}_p$ *is the number of linearly independent harmonic* p*-forms satisfying* $\Lambda_{(1)}^{r_1} \Lambda_{(2)}^{r_2} \Lambda_{(3)}^{r_3} \tau^p = 0 \quad (r_1 + r_2 + r_3 = p/2)$.

REFERENCES

AKIZUKI, Y.

[1] Theory of harmonic integrals, I, II (in Japanese). Tokyo (1955 and 1956)

BERGER, M.:

[1] Sur les groupes d'holonomie des variétés riemanniennes. C. R. Acad. Sci., Paris, 237 (1953), 472-474.

[2] Sur les groupes d'holonomie des variétés riemanniennes non symétriques., C. R. Acad. Sci., Paries, 237 (1953), 1036-1038.

[3] Groupes d'holonomie des variétés riemanniennes. Applications. C. R. Acad. Paris, 238 (1954), 985-986.

BOCHNER, S.:

[1] Vector fields and Ricci curvature. Bull. A. M. S., 52 (1946), 776-797.

[2] Curvature in Hermitian metric. Bull. A. M. S., 52 (1946), 179-195.

[3] Curvature and Betti numbers, I. Ann. of Math., 49 (1948), 379-390.

[4] Curvature and Betti numbers, II. Ann. of Math., 50 (1949), 77-93.

BIDAL, P. et G. DE RHAM:

[1] Les forms différentielles harmoniques. Comm. Math. Helv., 19 (1940), 1-49.

CALABI, E. AND D. C. SPENCER:

[1] Completely integrable almost complex manifolds. Bull. A. M. S., 57 (1951), 254-255.

CHERN, S. S.:

[1] On Riemannian manifold of four dimensions. Bull. A. M. S., 51 (1945), 964-971.

[2] Characteristic classes of Hermitian manifolds, Ann. of Math., 47 (1946), 85-121.

[3] Topics in differential geometry. Princeton (1951),

CARTAN, E.:

[1] Les groupes d'holonomie des espaces généralisées. Acta Math., 48 (1926), 1-42.

DE RHAM, G.:

[1] Variétés différentiables. Act. Sci. et Indust. 1222., Paris (1955).

ECKMANN, B.:

[1] Quelque propriétés globales des variétés kähleriennes. C. R. Acad. Sci., Paris, 229 (1949), 577-579.

ECKMANN, B. AND H. GUGGENHEIMER.:

[1] Formes différentielles et métrique hermitienne sans torsion. I., C. R. Acad. Sci., Paris 229 (1949), 464-496.

[2] Formes différentielles et métrique hermitienne sans torsion. II. C. R. Acad. Sci., Paris, 229 (1949) 489-491.
[3] Sur les variétés closes à métrique hermitienne sans torsion. C. R. Acad. Sci., Paris, 229 (1949), 503-505.

EHRESMANN, C:
[1] Sur la théorie des espaces fibrés. Coll. Top. Alg. (1947).
[2] Sur les variétés presque complexes. Proc. Int. Congr.

EISENHART, L. P.:
[1] Cotinuous groups of tran formations. 38, Princeton (1933).

FRÖLICHER, A.:
[1] Sur l'intégrabilité de structures presque complexes. C. R. Acad. Sci., Paris, 232 (1951), 2284-2286.

GUGGENHEIMER, H.:
[1] Sur les variétés qui possèdent une forme exterieur quadratique fermée. C. R. Acad. Sci., Paris, 232 (1951), 470-472.
[2] A note on curvanre and Betti numbers. Proc. Amer. Math. Soc., 2(1951), 867-870.
[3] Über komplex-analytische Mannigfaltigkeit mit Kählerscher Metrik. Commt. Math. Helv., 25(1951), 257-297.
[4] Über Kählersche und symplektische Differentialalgebren. Tôhoku Math. J., 4, No. 2, (1952), 157-171.
[5] Formes et vecteurs pseudo-analytiques. Ann. Mat. pura appl. (4), 36 (1954), 223-246.
[6] Variétés symplectiques. Coll. Top., Strasbourg, Janvier (1951).

HODGE, W. V. D.:
[1] Theory and Applications of harmonic integrals. Cambr. (1941).

IWAMOTO, H.:
[1] On the structure of Riemannian spaces whose holonomy grou s fix a null system, Tôhoku Math. J., 1, No. 2 (1950), 109-135.

LIEBERMANN, P.:
[1] Sur les variéfés presque complexes V_{2n} munies d'un champ de n-elemen s reels. C. R. Acad. Sci., Paris, 233 (1951), 1971-1973.
[2] Formes différentielles sur une variété symlectique. C. R. Acad. Sci., Paris, 234 (1952), 395-397.
[3] Sur les structures presque quaternioniennes de deuxieme espèce. C. R. Acad. Sci., Paris, 234 (1952), 1030:1032.
[4] Sur le problème d'équivalence de certaines structures infinitésimales. Thèse (1953).

LICHNEROWICZ, A.:
[1] Un théorème sur les espaces homogènes complexes. Archiv. der Math., 5 (1954), 207-215.
[2] Théorème de réductibilité des variétés kähleriennes et applications. C. R. Acad. Sci., Paris, 231 (1950) 1280-1282.
[3] Généralisations de la géométrie kählerienne globale. Coll. geo. diff., Louvain (1951).
[4] Formes a dérivée covariante nulle sur une variété riemannienne. C. R. Acad. Sci., Paris, 232 (1951), 146-147.
[5] Sur les variétés riemanniennes admettant une forme à dérivée covariante nulle. C. R. Acad. Sci., Paris, 232 (1951), 677-679.
[6] Sur les variétés symplectiques. C. R. Acad. Sci., Paris, 233(1951), 723-725.
[7] Sur les formes harmoniques des variétés riemanniennes localement réductibles. C. R. Acad. Sci., Paris, 232 (1951), 1634-1636.
[8] Espaces homogenes kähleriens. Coll. Inter. C. N. R. S., LII, Géométrie differentielle, Strasbourg, 26 (1953).
[9] Un théorème sur les espaees homogènes complexes. Archiv. der Math., 5(1954), 207-215.
[10] Sur les groupes d'holonomie des variétés riemanniens et kähleriens. Edizioni

Cremonese della S. A. Perrela, Roma (1954).
ŌTSUKI, T.
[1] On some 4-dimensional Riemannian Spaces. I. II. Mem. Fac. Sci. Kyûshu Univ., Ser. A (Math), 4 (1949).
SASAKI, S.:
[1] On the real representations of spaces with Hermitian connection. Sci. Rep. Tôhoku Univ., ser. I, 33 (1949), 52–61.
SCHOUTEN, J. A.
[1] Ricci Calculus. Springer (1954).
SCHOUTEN, J. A. AND K. YANO:
[1] On pseudo-Kählerian spaces admitting a continuous group of motions. Proc. Kon. Ned. Acad. Wet., Amsterdam, A. 54 (1951), 200–212.
THOMAS, T. Y.:
[1] The decomposition of Riemann Spaces in the large. Monatsh. f. Math. Phys., 47 (1939).
WEIL, A.:
[1] Sur la théorie des formes différentielles attachées à une variété analytique complexe. Commt. Math. Helv., 20 (1947), 110–116.
YANO, K.:
[1] On harmonic and Killing vector fields. Ann. of Math., 55 (1952), 38–45.
YANO, K. AND S. BOCHNER:
[1] Curvature and Betti numbers. Ann. of Math. Studies, Princeton (1953).
YANO, K. AND I. MOGI:
[1] Sur les variétés pseudo-käehleriennes à courbure holomorphique constante. C R. Acad. Sci., Paris, 237 (1953), 962–964.
[2] On real representations of Kaehlerian manifolds. Ann. ot Math., 61 (1955), 170–189.

FUKUSHIMA UNIVERSITY.