TENSOR PRODUCTS OF COMMUTATIVE BANACH ALGEBRAS

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In [2] and [5], A. Hausner and G. P. Johnson announced the following theorem: Let G be a locally compact abelian group and X a complex commutative Banach algebra. Then the space of all regular maximal ideals topologized in the Gelfand sense of $L^1(G, X)$, the space of all X-valued Bohner integrable functions on G, is homeomorphic with the Cartesian product of $\mathfrak{M}(X)$ and $\hat{G}$ in the product topology, where $\mathfrak{M}(X)$ means the space of all regular maximal ideals and $\hat{G}$ the dual group of G. Moreover improving the result by B.Yood [10], A.Hausner has proved that the analogous result also holds valid for the space of all vector-valued continuous functions on a compact Hausdorff space [4]: that is for a commutative Banach algebra X and a compact Hausdorff space Ω, the space of all regular maximal ideals in $C(\Omega, X)$, the space of all X-valued continuous functions on Ω, is homeomorphic with the Cartesian product of $\mathfrak{M}(X)$ and Ω in the product topology.

Now it is known by Grothendieck [1] that $L^1(G, X)$ is isometric-isomorphic to $L^1(G) \hat{\otimes}_{\gamma} X$, the tensor product of $L^1(G)$ and X with γ-norm and $C(\Omega, X)$ to $C(\Omega) \hat{\otimes}_{\lambda} X$, the tensor product of $C(\Omega)$ and X with λ-norm. This fact suggests us that the tensor productorial treatment of the problem may be fruitful and make the background of their discussions clear.

Our first result is the following simultaneous generalization of the theorems by Hausner-Johnson and Yood-Hausner: Let A and B be commutative Banach algebras and suppose that $A \hat{\otimes}_{\alpha} B$, the tensor product of A and B for the cross norm α not less than λ-norm, is a Banach algebra, then the space of all regular maximal ideals in $A \hat{\otimes}_{\alpha} B$ topologized with the usual weak topology is homeomorphic with the product space of $\mathfrak{M}(A)$ and $\mathfrak{M}(B)$.

Next are shown some theorems about the regularity and semi-simplicity of the tensor product of A and B, which are also the generalizations of the results announced also in [2] and [5] in the case of $A = L^1(G)$ where G is a locally compact abelian group and the treated cross norm is γ-norm. In our arguments, the following family of linear mappings from $A \hat{\otimes}_{\alpha} B$ to A(or B) plays an essential rôle. In fact there exists a family of linear mappings $\{\theta_\psi | \psi \in B^*\}$ from $A \hat{\otimes}_\alpha B$ to A associated with each linear functional $\psi \in B^*$, having the property ${}^t\theta_\psi(\varphi) = \varphi \otimes_{\alpha'} \psi$ for every $\varphi \in A^*$. The property of this family is closely related to the so-called metrical approximation property of A or B in the sense of Grothendieck [1], which causes some natural restrictions to our Theorem 4.

Throughout this paper, we concern mainly with commutative Banach algebras and for a Banach algebra A, $\mathfrak{M}(A)$ denotes the space of all regular maximal ideals in A with the usual weak topology. For each element $a \in A$, we mean by $\hat{a}$ the abstract Fourier transform of a and its value on an element M in $\mathfrak{M}(A)$ is denoted by $\hat{a}(M) = \phi_M(a)$ where ϕ_M shows the canonical homomorphism associated with M. Sometimes we write $\hat{a}(M) = \langle a, \phi_M \rangle$ regarding ϕ_M as a linear functional on A.

Let A and B be Banach algebras and φ and ψ linear functionals on A and B respectively. We use the product linear functional $\varphi \otimes \psi$ as usual, that is, $\varphi \otimes \psi$ is the linear functional on $A \odot B$, algebraic tensor product of A and B, such that

$$\left\langle \sum_{i=1}^{n} a_i \otimes b_i, \varphi \otimes \psi \right\rangle = \sum_{i=1}^{n} \langle a_i, \varphi \rangle \langle b_i, \psi \rangle.$$

However, for the cross norm α in $A \odot B$ which is not less than λ-norm, $\varphi \otimes \psi$ is continuous, so that we can extend this to the whole space $A \hat{\otimes}_\alpha B$. We denote this extended funcional by $\varphi \otimes_{\alpha'} \psi$.

Other details of tensor products of Banach spaces are referred to Schatten [7] and Grothendieck [1].

1. We set forth with the following theorem which is essentially due to Turumaru [8].

THEOREM 1. *Let A and B be Banach algebras then $A \hat{\otimes}_\gamma B$ is again a Banach algebra.*

We omit this proof.

REMARK. We know some examples of A and B that $A \hat{\otimes}_\alpha B$ becomes a Banach algebra for some cross norm $\alpha\, (\alpha \geqq \lambda)$ such as $C(\Omega) \hat{\otimes}_\lambda B \simeq C(\Omega, B)$. But, generally, we do not know whether $A \hat{\otimes}_\alpha B$ is a Banach algebra or not except for γ-norm.

LEMMA 1. *For any regular maximal ideal* $M \in \mathfrak{M}(A \hat{\otimes}_{\gamma} B)$, *there exists uniquely a pair of regular maximal ideals* $M_1 \in \mathfrak{M}(A)$, $M_2 \in \mathfrak{M}(B)$ *such as* $\phi_M = \phi_{M_1} \otimes_{\gamma'} \phi_{M_2}$.

PROOF. Take an element $x \otimes y$ such as $\phi_M(x \otimes y) \neq 0$ and put $\phi_1(a) = \phi_M(ax \otimes y)/\phi_M(x \otimes y)$ for $a \in A$. ϕ_1 is continuous on A because we have $\|ax \otimes y\|_\gamma \leqq \|a\| \ \|x \otimes y\|_\gamma$ by the property of γ-norm. Moreover, ϕ_1 is defined independently from the choice of $x \otimes y$. In fact, take an another element $x' \otimes y'$ with $\phi_M(x' \otimes y') \neq 0$. We get

$$\phi_M(ax \otimes y)\phi_M(x' \otimes y') = \phi_M(axx' \otimes yy') = \phi_M(x \otimes y)\phi_M(ax' \otimes y')$$

which implies

$$\frac{\phi_M(ax \otimes y)}{\phi_M(x \otimes y)} = \frac{\phi_M(ax' \otimes y')}{\phi_M(x' \otimes y')}.$$

Since, for $a, b \in A$,

$$\phi_1(ab) = \frac{\phi_M(ab\, x^2 \otimes y^2)}{\phi_M(x^2 \otimes y^2)} = \frac{\phi_M(ax \otimes y)\phi_M(bx \otimes y)}{\phi_M(x \otimes y)\phi_M(x \otimes y)} = \phi_1(a)\phi_1(b),$$

ϕ_1 is a multiplicative linear functional on A. Similarly we get a continuous multiplicative linear functional ϕ_2 on B and we have

$$\phi_1 \otimes \phi_2(a \otimes b) = \phi_1(a)\phi_2(b) = \frac{\phi_M(ax \otimes y)}{\phi_M(x \otimes y)} \cdot \frac{\phi_M(x \otimes by)}{\phi_M(x \otimes y)}$$

$$= \frac{\phi_M(ax^2 \otimes by^2)}{\phi_M(x^2 \otimes y^2)} = \phi_M(a \otimes b) \text{ for every } a \in A,\ b \in B,$$

so that we get $\phi_M = \phi_1 \otimes_{\gamma'} \phi_2$.

Next, suppose that ϕ_M has another expression $\phi_1' \otimes_{\gamma'} \phi_2'$ for two continuous homomorphisms from A and B to the field of complex numbers. If $\phi_1 \neq \phi_1'$, then there exists an element $a \in A$ such as $\phi_1(a) \neq 0$ and $\phi_1'(a) = 0$. On the other hand we can take an element $b \in B$ such as $\phi_2(b) \neq 0$. Then $\phi_1 \otimes \phi_2 (a \otimes b) = \phi_1(a)\phi_2(b) \neq 0$ and $\phi_1' \otimes \phi_2'(a \otimes b) = \phi_1'(a)\phi_2'(b) = 0$ which is a contradiction.

A symmetric argument for ϕ_2 and ϕ_2' also shows that $\phi_2 \neq \phi_2'$ implies a contradiction. Hence the expression of ϕ_M is unique.

Since ϕ_1 and ϕ_2 are both non-zero continuous multiplicative linear functionals on A and B respectively, there exists two regular maximal ideals $M_1 \in \mathfrak{M}(A)$ and $M_2 \in \mathfrak{M}(B)$ such as $\phi_1 = \phi_{M_1}$ and $\phi_2 = \phi_{M_2}$.

THEOREM 2. *Let A and B be commutative Banach algebras. If* $A \hat{\otimes}_{\alpha} B$ *is a Banach algebra for the cross norm* α *not less than* λ*-norm, then* $\mathfrak{M}(A \hat{\otimes}_{\alpha} B)$ *is homeomorphic with the product space of* $\mathfrak{M}(A)$ *and* $\mathfrak{M}(B)$.

When $\alpha = \gamma$ and $A = L^1(G)$ for a lacally compact abelian group G, we get the result by Hausner-Johnson; the case where $\alpha = \lambda$ and $A = C(\Omega)$ for a compact Hausdorff space Ω is due to Yood-Hausner. Similar statements also hold for forthcoming theorems.

PROOF. Consider the canonical homomorphism Φ from $A \hat{\otimes}_{\gamma} B$ to $A \hat{\otimes}_{\alpha} B$ and take an element M in $\mathfrak{M}(A \hat{\otimes}_{\alpha} B)$. It is easy to see that $\phi_M \Phi$ is a non-zero continuous homomorphism from $A \hat{\otimes}_{\gamma} B$ to the field of complex numbers. Hence, by Lemma 1, there exists uniquely a pair of regular maximal ideals M_1 and M_2 in $\mathfrak{M}(A)$ and $\mathfrak{M}(B)$ such that $\phi_M \Phi = \phi_{M_1} \otimes_{\gamma'} \phi_{M_2}$. Therefore we have $\phi_M = \phi_{M_1} \otimes \phi_{M_2}$ on $A \odot B$, which implies $\phi_M = \phi_{M_1} \otimes_{\alpha'} \phi_{M_2}$.

Now, consider the mapping Ψ from $\mathfrak{M}(A \hat{\otimes}_{\alpha} B)$ to $\mathfrak{M}(A) \times \mathfrak{M}(B)$ defined by $\Psi(M) = (M_1, M_2)$. By Lemma 1 and the assumption $\alpha \geqq \lambda$ we see that Ψ is one-to-one and onto. Since the topology of $\mathfrak{M}(A \hat{\otimes}_{\alpha} B)$ is equivalent to the topology induced by $A \odot B$, it is now easy to see that Ψ is a homeomorphism between $\mathfrak{M}(A \hat{\otimes}_{\alpha} B)$ and $\mathfrak{M}(A) \times \mathfrak{M}(B)$ (cf. Loomis [6: p. 12]).

REMARK. If we denote by $\widehat{A \hat{\otimes}_{\alpha} B}$ the space of all functions of the abstract Fourier transform of the elements of $A \hat{\otimes}_{\alpha} B$, an easy calculation shows that this space is isometric into $\hat{A} \hat{\otimes}_{\lambda} \hat{B}$.

LEMMA 2. *Let E and F be Banach spaces and* α *the cross norm not less than* λ, *then for each* $\psi \in F^*$, *the conjugate space of F, there exists an associated continuous linear mapping* θ_ψ *from* $E \hat{\otimes}_{\alpha} F$ *to E such that* ${}^t\theta_\psi(\varphi) = \varphi \otimes_{\alpha'} \psi$ *for every* $\varphi \in E^*$ *and the same result also holds for each* $\varphi \in E^*$.

PROOF. Define the mapping θ_ψ from $E \odot F$ to E by $\theta_\psi \left(\sum_{i=1}^{n} a_i \otimes b_i\right) = \sum_{i=1}^{n} \langle b_i, \psi \rangle a_i$.

Since θ_ψ is a continuous linear mapping for α-norm, we can extend this to the mapping from $E \hat{\otimes}_\alpha F$ to E, which we also denote by θ_ψ.

Now, for $\varphi \in E^*$ we have

$$\left\langle \sum_{i=1}^n a_i \otimes b_i,\ {}^t\theta_\psi(\varphi) \right\rangle = \left\langle \sum_{i=1}^n <b_i, \psi> a_i,\ \varphi \right\rangle = \sum_{i=1}^n <a_i, \varphi> <b_i, \psi>$$

$$= \left\langle \sum_{i=1}^n a_i \otimes b_i,\ \varphi \otimes \psi \right\rangle.$$

Hence ${}^t\theta_\psi(\varphi) = \varphi \otimes_{\alpha'} \psi$.

A symmetric argument shows that the same result holds for each $\varphi \in E^*$.

THEOREM 3. *Let A and B be commutative Banach algebras and α the cross norm not less than λ, then if $A \hat{\otimes}_\alpha B$ becomes a Banach algebra, $A \hat{\otimes}_\alpha B$ is regular if and only if A and B are regular.*

PROOF. Suppose that $A \hat{\otimes}_\alpha B$ is regular, and consider an open set $G_1 \subset \mathfrak{M}(A)$ and $M_1 \in G_1$. Taking an arbitrary open set $G_2 \subset \mathfrak{M}(B)$ and a point $M_2 \in G_2$ we get, by the hypothesis and Theorem 2, an element $x \in A \hat{\otimes}_\alpha B$ such as

$$\hat{x}(M_1, M_2) = 1 \text{ and } \hat{x}((G_1 \times G_2)^c) = 0$$

where $(G_1 \times G_2)^c$ means the complement of $G_1 \times G_2$. Denoting by ϕ_1 and ϕ_2 the canonical homomorphisms associated with M_1 and M_2 respectively, we have

$$<\theta_{\phi_2}(x), \phi_1> = <x, \phi_1 \otimes_{\alpha'} \phi_2> = \hat{x}(M_1, M_2) = 1$$

and

$<\theta_{\phi_2}(x), \phi_1'> = <x, \phi_1' \otimes_{\alpha'} \phi_2> = 0$ for every ϕ_1' associated with a regular maximal ideal $M_1' \notin G_1$. Thus A is regular and similarly B is regular, too.

Next assume that A and B are regular and consider an open set $G \subset \mathfrak{M}(A \hat{\otimes}_\alpha B)$ and a point $M \in G$. We must show that there exists an element x in $A \hat{\otimes}_\alpha B$ such that $\hat{x}(M) = 1$ and $\hat{x}(M') = 0$ for every $M' \notin G$. By Theorem 2, there exist regular maximal ideals $M_1 \in \mathfrak{M}(A)$ and $M_2 \in \mathfrak{M}(B)$ whose couple (M_1, M_2) is associated with M, besides we can find two open sets $G_1 \subset \mathfrak{M}(A)$ and $G_2 \subset \mathfrak{M}(B)$ such that $(M_1, M_2) \in G_1 \times G_2 \subset G$. Since A and B are regular Banach algebras there exist two elements $a \in A$ and $b \in B$ such that $\hat{a}(M_1) = \hat{b}(M_2) = 1$ and $\hat{a}(M'_1) = \hat{b}(M'_2) = 0$ for every $M'_1 \notin G_1$ and $M'_2 \notin G_2$. Then $x = a \otimes b$ satisfies our requirements. This completes the proof.

2. In the following we restrict ourself to the case of $\alpha = \gamma$ and $\alpha = \lambda$. A Banach space E is called to satisfy *the condition of approximation* if for every compact set $K \subset E$ and $\varepsilon > 0$ there exists a continuous linear mapping u of finite rank from E into itself such as $\|u(x) - x\| < \varepsilon$ for all $x \in K$. (cf. Grothendieck [1])

Most of our familiar Banach algebras such that $C(\Omega)$ (Ω is a compact Hausdorff space), $L^\infty(X, \mu)$ (X is a locally compact space and μ a positive measure) and $L^1(G)$ (G is a locally compact abelian group) etc. satisfy this condition.

THEOREM 4. *Let A and B be commutative Banach algebras.*

$1°$ *If either A or B satisfies the condition of approximation then $A \hat{\otimes}_\gamma B$ is semi-simple if and only if A and B are semi-simple.*

$2°$ *Suppose that $A \hat{\otimes}_\lambda B$ becomes a Banach algebra, then $A \hat{\otimes}_\lambda B$ is semi-simple if and only if A and B are semi-simple.*

PROOF. Suppose that $A \hat{\otimes}_\gamma B$ is semi-simple and an element $a \in A$ satisfies $\hat{a} = 0$. We have

$\widehat{a \otimes b}(M_1, M_2) = \hat{a}(M_1)\hat{b}(M_2) = 0$ for all $(M_1, M_2) \in \mathfrak{M}(A) \times \mathfrak{M}(B)$ and $b \in B$. Hence, by Theorem 2, $a \otimes b = 0$ for all $b \in B$, which implies $a = 0$ i. e. A is semi-simple. Similarly, we see that B is semi-simple. This proves the necessity of $1°$. (Therefore the necessity of $1°$ holds without the assumption of approximation condition).

The same argument for $A \hat{\otimes}_\lambda B$ shows that A and B are semi-simple if $A \hat{\otimes}_\lambda B$ is semi-simple.

We shall prove the sufficiency of $2°$. Take a non-zero element $x \in A \hat{\otimes}_\lambda B$. By the property of λ-norm, there exist linear functionals $\varphi \in A^*$ and $\psi \in B^*$ such as $<x, \varphi \otimes_{\lambda'} \psi> = <\theta_\psi(x), \varphi> \neq 0$ which implies $\theta_\psi(x) \neq 0$. As A is semi-simple we can find a regular maximal ideal $M_1 \in \mathfrak{M}(A)$ such that $\widehat{\theta_\psi(x)}(M_1) \neq 0$. Hence we have $<\theta_\psi(x), \phi_1> = <x, {}^t\theta_\psi(\phi_1)> = <x, \phi_1 \otimes_{\lambda'} \psi> = <\theta_{\phi_1}(x), \psi> \neq 0$ where ϕ_1 is a continuous homomorphism associated with M_1. Therefore, by the semi-simplicity of B, there exists a regular maximal ideal $M_2 \in \mathfrak{M}(B)$ such that $\widehat{\theta_{\phi_1}(x)}(M_2) \neq 0$. We have, by Theorem 2, $\hat{x}(M_1, M_2) = \langle x, \phi_{M_1} \underset{\lambda'}{\otimes} \phi_{M_2} \rangle = \langle \theta_{\phi_1}(x), \phi_{M_2} \rangle = \widehat{\theta_{\phi_1}(x)}(M_2) \neq 0$. That is $\hat{x} \neq 0$; this completes the proof.

If either A or B satisfies the condition of approximation, the canonical homomorphism Φ from $A \hat{\underset{\gamma}{\otimes}} B$ to $A \hat{\underset{\lambda}{\otimes}} B$ is one-to-one (cf. [1]). Take a non-zero element $x \in A \hat{\underset{\gamma}{\otimes}} B$. Owing to the sufficiency of 1° we can find a couple of continuous homomorphisms ϕ_1 and ϕ_2 from A and B to the field of complex numbers such as $\langle \Phi(x), \phi_1 \underset{\lambda'}{\otimes} \phi_2 \rangle \neq 0$. Since $\langle \Phi(x), \phi_1 \underset{\lambda'}{\otimes} \phi_2 \rangle = \langle x, \phi_1 \underset{\gamma'}{\otimes} \phi_1 \rangle$, we get $\hat{x}(M_1, M_2) = \langle x, \phi_1 \underset{\gamma'}{\otimes} \phi_2 \rangle \neq 0$ where M_1 and M_2 are regular maximal ideals in $\mathfrak{M}(A)$ and $\mathfrak{M}(B)$ associated with ϕ_1 and ϕ_2 respectively. Hence $A \hat{\underset{\gamma}{\otimes}} B$ is semi-simple.

At the last, we shall discuss the general Wiener's tauberian theorem in terms of the tensor product of commutative Banach algebras.

THEOREM 5. *Let A and B be Banach algebras, both semi-simple and reqular and the elements $a \in A$ and $b \in B$ with $\hat{a}(M_1)$ and $\hat{b}(M_2)$ having compact support in $\mathfrak{M}(A)$ and $\mathfrak{M}(B)$ respectively are dense in each Banach algebra. Then*

1° *if either A or B satisfies the condition of approximation every proper closed ideal in $A \hat{\underset{\gamma}{\otimes}} B$ is contained in a regular maximal ideal ;*

2° *if $A \hat{\underset{\lambda}{\otimes}} B$ is a Banach algebra, the same conclusion as that of case 1° holds for $A \hat{\underset{\lambda}{\otimes}} B$, too.*

In other words if both A and satisfy the usual sufficient condition for the general Wieners tauberian theorem, then the same condition holds for $A \hat{\underset{\gamma}{\otimes}} B$ and $A \hat{\underset{\lambda}{\otimes}} B$ with some restrictions.

PROOF. By the hypothesis and Theorem 3 and 4 it follows that $A \hat{\underset{\gamma}{\otimes}} B$ and $A \hat{\underset{\lambda}{\otimes}} B$ are semi-simple and regular. Therefore, using the general tauberian theorem, we obtain the conclusion of our theorem by showing that the elements x in $A \hat{\underset{\gamma}{\otimes}} B$ (resp. $A \hat{\underset{\lambda}{\otimes}} B$) with $\hat{x}(M)$ having compact support in $\mathfrak{M}(A \hat{\underset{\gamma}{\otimes}} B)$ (resp. $\mathfrak{M}(A \hat{\underset{\lambda}{\otimes}} B)$) are dense in $A \hat{\underset{\gamma}{\otimes}} B$ (resp. $A \hat{\underset{\lambda}{\otimes}} B$).

Now, by the hypothesis and Theorem 2, it is clear that the elements $a \otimes b$ with $\widehat{a \otimes b}$ having compact support in $\mathfrak{M}(A) \times \mathfrak{M}(B)$ are dense in the set $\{a \otimes b \mid a \in A,\ b \in B\}$, which implies that the family $\left\{\sum_{i=1}^{n} a_i \otimes b_i \,\middle|\, \widehat{\sum_{i=1}^{n} a_i \otimes b_i}\right.$ has compact support in $\left.\mathfrak{M}(A) \times \mathfrak{M}(B)\right\}$ is dense in $A \odot B$. Besides, since $A \odot B$ is dense in $A \underset{\gamma}{\hat{\otimes}} B$ (resp. $A \underset{\lambda}{\hat{\otimes}} B$) in γ-norm (resp. λ-norm), the above family is dense in $A \underset{\gamma}{\hat{\otimes}} B$ (resp. $A \underset{\lambda}{\hat{\otimes}} B$) in γ-norm (resp. λ-norm). This completes the proof.

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