ON INFINITESIMAL HOLOMORPHICALLY PROJECTIVE TRANSFORMATIONS IN KÄHLERIAN MANIFOLDS

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Recently, T. Ōtsuki and Y. Tashiro [9][1] have studied holomorphically projective correspondences of Kählerian manifolds. Further, Y. Tashiro [11] has introduced the notion of such a correspondence of almost-complex manifolds endowed with a symmetric φ-connection, i. e. a symmetric affine connection with respect to which the almost-complex structure is covariant constant. He has defined the holomorphically projective curvature tensor $P_{kji}{}^h$ which is invariant under holomorphically projective correspondences, and characterized a Kählerian manifold of constant holomorphic sectional curvature by the condition $P_{kji}{}^h = 0$. One of the present authors[2] has introduced the notion of the holomorphically projective changes of a φ-connection of some type, called a half-symmetric φ-connection, and the notion of the infinitesimal holomorphically projective transformation, which will be briefly called an *HP*-transformation.

We shall devote this paper to *HP*-transformations in Kählerian manifolds of some types. In § 1, we shall give some preliminary facts concerning Kählerian manifolds and infinitesimal transformations for the later use. We shall characterize in § 2 the analytic *HP*-transformation as an infinitesimal transformation preserving holomorphically planar curves. In § 3, we shall discuss the properties of analytic *HP*-transformations.

T. Sumitomo [10] and K. Yano and T. Nagano [13] have recently studied infinitesimal projective transformations in a Riemannian manifold and obtained valuable results. We shall consider analogous problems concerning *HP*-transformations. In § 4, we shall deal with a Kählerian manifold admitting an analytic *HP*-transformation which leaves the covariant derivative of the holomorp hically projective curvature tensor. We shall prove in § 5 that a Kählerian manifold which satisfies $\nabla_k R_{ji} = 0$ and admits a non-trivial analytic *HP*-transformation is necessarily an Einstein one.

As will be proved in § 5, the existence of a non-trivial analytic *HP*-transformation in a Kählerian manifold satisfying $\nabla_k R_{ji} = 0$ reduces the manifold to an Einstein one. So, it might become a problem to investigate

1) The number in brackets [ ] refers to the Bibliography at the end of the paper.
2) Ishihara, S. [2].

HP-transformations in Kähler-Einstein manifold. We shall prove in § 6 some theorems on HP-transformations in a Kähler-Einstein manifold, for example, that in a Kähler-Einstein manifold any analytic HP-transformation is uniquely decomposed as a sum of a Killing vector and a gradient analytic HP-transformation.

In the last § 7, we shall discuss HP-tranformations in a compact Kählerian manifold having the constant holomorphic sectional curvature.

1. Preliminaries. We shall first give preliminary formulas on the Kählerian manifold and the infinitesimal transformation, isometric, affine, or holomorphically projective. Let us consider an n $(= 2m > 2)$ real dimensional Kählerian manifold with local coordinates $\{x^i\}$[3]. Then the (positive definite) Riemannian metric g_{ji} and the complex structure $\varphi_i{}^h$ satisfy the following equations.

$$\varphi_j{}^r\varphi_r{}^i = -\delta_j^i, \quad g_{rs}\varphi_j{}^r\varphi_i{}^s = g_{ji},$$
$$\nabla_k\varphi_j{}^h = 0, \qquad \nabla_k g_{ji} = 0,$$

where ∇_k denotes the operator of the covariant differentiation with respect to $\left\{ {h \atop ji} \right\}$.

Let $R_{kji}{}^h$ be the Riemannian curvature tensor and put $R_{ji} = R_{rji}{}^r$, $R_{kjih} = R_{kji}{}^r g_{rh}$, $R = R_{ji}g^{ji}$ and

$$S_{ji} = \varphi_j{}^r R_{ri},$$

then the following identities are valid[4]

$$(1.1) \qquad R_{kji}{}^r\varphi_r{}^h = R_{kjr}{}^h\varphi_i{}^r, \qquad R_{kjir}\varphi_h{}^r = R_{kjhr}\varphi_i{}^r,$$
$$R_{kjih} = R_{kjtr}\varphi_i{}^t\varphi_h{}^r,$$

$$(1.2) \qquad R_{ji} = R_{tr}\varphi_j{}^t\varphi_i{}^r,$$

$$(1.3) \qquad S_{ji} + S_{ij} = 0, \qquad S_{ji} = S_{tr}\varphi_j{}^t\varphi_i{}^r,$$
$$S_{ji} = -\frac{1}{2}\varphi^{tr}R_{trji}.$$

The holomorphically projective curvature tensor[5] $P_{kji}{}^h$ which will be briefly called HP-curvature tensor, P is given by

3). In the present paper we shall restrict our attention to manifolds which are real representations of (complex) Kählerian manifolds, i. e. pseudo-Kählerian one. As to the notations, we follow Yano, K. [12]. We shall represents any quantit es in terms of their components with respect to natural frames $\partial/\partial x^i$. Indices run over $1, 2, \ldots\ldots, n = 2m$.

4) Yano, K. [12].

5) Tashiro, Y. [11].

$$P_{kji}{}^h = R_{kji}{}^h + \frac{1}{n+2}(R_{ki}\delta_j^h - R_{ji}\delta_k^h + S_{ki}\varphi_j{}^h - S_{ji}\varphi_k{}^h + 2\,S_{kj}\varphi_i{}^h).$$

We can obtain easily the following identities.

$$(1.5)\qquad P_{(kj)i}{}^h = 0, \qquad P_{[kji]}{}^h = 0,$$

$$(1.6)\qquad P_{rji}{}^r = 0,$$

$$(1.7)\qquad P_{kji}{}^r\varphi_r{}^h = P_{kjr}{}^h\varphi_i{}^r, \qquad P_{rji}{}^h\varphi_k{}^r = P_{rki}{}^h\varphi_j{}^r,$$

from which we get

$$(1.8)\qquad P_{kjr}{}^r = 0,$$

$$(1.9)\qquad P_{rji}{}^t\varphi_t{}^r = 0, \quad P_{kjr}{}^t\varphi_t{}^r = 0.$$

A necessary and sufficient condition for $P_{kji}{}^h = 0$ is that the manifold is a space of constant holomorphic curvature[6], i.e. a space whose curvature tensor $R_{kji}{}^h$ takes the form

$$(1.10)\qquad R_{kji}{}^h = k(g_{ki}\delta_j^h - g_{ji}\delta_k^h + \varphi_{ki}\varphi_j{}^h - \varphi_{ji}\varphi_k{}^h + 2\,\varphi_{kj}\varphi_i{}^h),$$

where we put

$$k = -\frac{R}{n(n+2)}.$$

For a vector field v^i and a tensor field $a_i{}^h$, the following identities are known[7],

$$(1.11)\qquad \underset{v}{\mathfrak{L}}\nabla_j a_i{}^h - \nabla_j \underset{v}{\mathfrak{L}}\, a_i{}^h = a_i{}^r \underset{v}{\mathfrak{L}}\left\{{h \atop jr}\right\} - a_r{}^h \underset{v}{\mathfrak{L}}\left\{{r \atop ji}\right\},$$

$$(1.12)\qquad \nabla_k \underset{v}{\mathfrak{L}}\left\{{h \atop ji}\right\} - \nabla_j \underset{v}{\mathfrak{L}}\left\{{h \atop ki}\right\} = \underset{v}{\mathfrak{L}} R_{kji}{}^h,$$

where $\underset{v}{\mathfrak{L}}$ denotes the operator of Lie differentiation with respect to v^i.

A Killing vector or an infinitesimal isometry v^i is defined by $\underset{v}{\mathfrak{L}}\, g_{ji} = \nabla_j v_i + \nabla_i v_j = 0$[8]. An infinitesimal affine transformation v^i is defined by

$$\underset{v}{\mathfrak{L}}\left\{{h \atop ji}\right\} \equiv \nabla_j\nabla_i v^h + R_{rji}{}^h v^r = 0.$$

We shall say a vector field v^i an infinitesimal holomorphically projective transformation or, for simplicity, an *HP*-transformation, if it satisfies

6) Tashiro, Y. [11].
7) Yano, K. [12].
8) We shall identify a contravariant vector v^i with a covariant vector $v_i = g_{ir}\, v^r$. Hence we shall say v_i is a Killing vector, or that ρ^i is gradient, for example.

$$\underset{v}{\pounds}\{{}^{h}_{ji}\} = \rho_j\delta_i^h + \rho_i\delta_j^h - \tilde{\rho}_j\varphi_i^h - \tilde{\rho}_i\varphi_j^h,$$ [9]

where ρ_i is a certain vector and $\tilde{\rho}_i = \varphi_i^r\rho_r$. In this case, we shall call ρ_i the associated vector of the transformation. If ρ_i vanishes, then the HP-transformation reduces to an affine one.

Contracting the last equation with respect to h and i, we get $\nabla_j\nabla_r v^r = (n+2)\rho_j$, which shows that the associated vector is gradient.

A vector field v^i is called contravariant analytic or, for simplicity, analytic, if it satisfies

$$\underset{v}{\pounds}\varphi_i^h \equiv -\varphi_i^r\nabla_r v^h + \varphi_r^h\nabla_i v^r = 0.$$

2. A geometrical interpretation of an analytic HP-transformation. In a differentiable manifold M, we consider a tensor valued function V depending not only on a point P of M but also on k vectors $u_1,\ldots\ldots, u_k$ at the point and denote it by $V(P, u_1,\ldots\ldots, u_k)$. We assume that the value of this function V lies in the tensor space associated to the tangent space of M at P and that it depends differentiably on all its arguments.

Assuming the manifold M to be affinely connected, we take an arbitrary curve $C: x^i = x^i(t)$ and denote its successive derivatives by

$$\frac{dx^i}{dt}, \quad \frac{\delta^2 x^i}{\delta t^2}, \ldots\ldots \tag{2. 1}$$

Then if we substitute (2. 1) into the function V instead of $u_1, u_2,\ldots\ldots, u_k$, we have a family of tensors

$$V(C) = V\left(x, \frac{dx}{dt},\ldots\ldots, \frac{\delta^k x}{\delta t^k}\right)$$

along the curve C.

Let v^i be an infinitesimal transformation, i. e. a vector field, and $'x^i = x^i + \varepsilon v_i$ be the infinitesimal point-transformation determined by v^i, ε being an infinitesimal constant. Given a curve $C: x^i = x^i(t)$, the image $'C$ of C is expressed by

$$x^i = x^i(t) + \varepsilon v^i(x(t)).$$

We shall call the limiting value

$$\underset{v}{\pounds}V(C) \equiv \lim_{\varepsilon\to 0}\frac{V('C) - 'V(C)}{\varepsilon}$$

9) The definition of the HP-transformation is different from that given in Ishihara, S. [2].

the Lie derivative of $V(C)$ with respect to v^i, where we have denoted by $'V(C)$ the family of tensors induced from $V(C)$ by the transformation $'x^k = x^i + \varepsilon v^i$.

In a Kählerian manifold, a curve $x^i = x^i(t)$ defined by

$$(2.2) \qquad \frac{d^2x^h}{dt^2} + \{{}^h_{ji}\}\frac{dx^j}{dt}\frac{dx^i}{dt} = \alpha\frac{dx^h}{dt} + \beta\varphi_j{}^h\frac{dx^j}{dt}$$

is, by definition, a holomorphically planar curve[10] or an H-plane curve, where α and β are certain functions of t.

Let v^i be an infinitesimal transformation and assume that for any ε the infinitesimal point-transformation $'x^i = x^i + \varepsilon v^i$ maps any H-plane curve into an H-plane curve. Then we say that v^i preserves the H-plane curves.

Now we ask for the condition that v^i preserve the H-plane curves. For such a vector v^i, taking account of (2.2), we have

$$(2.3) \qquad \underset{v}{\mathfrak{L}}\left[\frac{d^2x^h}{dt^2} + \{{}^h_{ji}\}\frac{dx^j}{dt}\frac{dx^i}{dt} - \alpha\frac{dx^h}{dt} - \beta\varphi_j{}^h\frac{dx^j}{dt}\right] = \gamma\frac{dx^h}{dt} + \delta\varphi_j{}^h\frac{dx^j}{dt}$$

along any H-plane curve, where γ and δ are certain functions of t.

Denoting the Lie derivative of the Christoffel's symbols and the complex structure $\varphi_i{}^h$, respectively, by

$$t_{ji}{}^h = \underset{v}{\mathfrak{L}}\{{}^h_{ji}\}, \qquad a_i{}^h = \underset{v}{\mathfrak{L}}\varphi_i{}^h,$$

we have from (2.3)

$$(2.4) \qquad t_{ji}{}^h\dot{x}^j\dot{x}^i + a\dot{x}^h + b\varphi_j{}^h\dot{x}^j - \beta\, a_j{}^h\dot{x}^j = 0,$$

where we have put

$$a = -(\gamma + \underset{v}{\mathfrak{L}}\alpha), \qquad b = -(\delta + \underset{v}{\mathfrak{L}}\beta), \qquad \dot{x}^i = dx^i/dt.$$

Since the relation (2.4) holds for any H-plane curve C, it must hold identically for any values of x^i and $\dot{x}^i$.

By means of the definition of the H-plane curves, we see further that the identity (2.4) holds for any value of the coefficient β.

Taking account of these arguments, we can easily see that the relation

$$(2.5) \qquad a_j{}^h\dot{x}^j = f\dot{x}^h + g\varphi_j{}^h\dot{x}^j,$$

$$(2.6) \qquad t_{ji}{}^h\dot{x}^j\dot{x}^i = p\dot{x}^h + q\varphi_j{}^h\dot{x}^j$$

hold for any values x^i and $\dot{x}^i$, where f, g, p and q are certain functions of x^i and $\dot{x}^i$.

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10) Ishihara, S. [2], Otsuki, T. and Tashiro, Y. [9], Tashiro, Y. [11].

If we take account of Lemma 1 given in Appendix I, we obtain by means of (2. 5)

$$a_i^{\ h} \equiv \underset{v}{\pounds}\varphi_i^{\ h} = 0. \tag{2.7}$$

On the other hand, if we substitute (2. 7) and $\nabla_j\varphi_i^{\ h} = 0$ into the identity

$$\nabla_j\underset{v}{\pounds}\varphi_i^{\ h} - \underset{v}{\pounds}\nabla_j\varphi_i^{\ h} = \varphi_r^{\ h}\underset{v}{\pounds}\{{}_{j\,i}^{\,r}\} - \varphi_i^{\ r}\underset{v}{\pounds}\{{}_{j\,r}^{\,h}\},$$

then we get

$$t_{ji}^{r}\varphi_r^{\ h} = t_{jr}^{\ \ h}\varphi_i^{\ r}. \tag{2.8}$$

From (2. 6) and (2. 8), taking account of Lemma 2 given in Appendix I, we have

$$t_{ji}^{\ \ h} = \underset{v}{\pounds}\{{}_{j\,i}^{\,h}\} = \rho_j\delta_i^h + \rho_i\delta_j^h - \widetilde{\rho}_j\varphi_i^{\ h} - \widetilde{\rho}_i\varphi_j^{\ h}, \tag{2.9}$$

where ρ_i is a certain vector field. Therefore the infinitesimal transformation v^i is an analytic *HP*-transformation.

Conversely, it is obvious that an analytic *HP*-transformation preserves the *H*-plane curves. Thus we have the following

THEOREM 2. 1. *In a Kählerian manifold, an infinitesimal transformation preserves the H-plane curves, if and only if it is an analytic HP-transformation.*

3. Some properties of *HP*-transformations. Let v^i be an *HP*-transformation, then it holds

$$\underset{v}{\pounds}\{{}_{j\,i}^{\,h}\} \equiv \nabla_j\nabla_i v^h + R_{rji}^{\ \ \ \ h}v^r = \rho_j\delta_i^h + \rho_i\delta_j^h - \widetilde{\rho}_j\varphi_i^{\ h} - \widetilde{\rho}_i\varphi_j^{\ h}. \tag{3.1}$$

Transvecting (3. 1) with g^{ji}, we have

$$\nabla^r\nabla_r v^h + R_r^{\ h}v^r = 0. \tag{3.2}$$

Hence, by virtue of the well known theorem on an analytic vectors,[11] we have the following

THEOREM 3. 1. *In a compact Kählerian manifold, an HP-transformation is analytic.*

In a compact Kählerian manifold M, it holds that $\int_M (R_{ji}v^jv^i)\,d\sigma \geqq 0$ for an analytic vector v^i, where $d\sigma$ denoted the volume element of M and the equality holds when and only when v^i is parallel. Therefore, if the Ricci's

11) Lichnerowicz, A. [4], Yano, K. [12].

form $R_{ji}\xi^j\xi^i$ is negative definite, then there exists no non-trivial HP-transformation provided that the manifold is compact.

Taking account of the identity (1. 11), we have for a vector field v^i

$$\underset{v}{\mathfrak{L}}\nabla_j\varphi_i{}^h - \nabla_j\underset{v}{\mathfrak{L}}\varphi_i{}^h = \varphi_i{}^r\underset{v}{\mathfrak{L}}\left\{{h \atop jr}\right\} - \varphi_r{}^h\underset{v}{\mathfrak{L}}\left\{{r \atop ji}\right\},$$

which implies

$$\nabla_j\underset{v}{\mathfrak{L}}\varphi_i{}^h = \varphi_r{}^h\underset{v}{\mathfrak{L}}\left\{{r \atop ji}\right\} - \varphi_i{}^r\underset{v}{\mathfrak{L}}\left\{{h \atop jr}\right\},$$

because of $\nabla_j\varphi_i{}^h = 0$. If the vector field v^i is an HP-transformation, it is easily verified that the right-hand side of the last equation vanishes. Thus we have the following theorem by virtue of Obata's theorem[12].

THEOREM 3. 2. *In an irreducible Kählerian manifold admitting no quaternion structure, any HP-transformation is analytic.*

It is known that in a Kählerian manifold admitting a quaternion structure the Ricci tensor vanishes identically[13]. Thus we have

THEOREM 3. 3. *In an irreducible Kählerian manifold having non-vanishing Ricci tensor, any HP-transformation is analytic.*

COROLLARY 3. 4. *In an irreducible Kähler-Einstein manifold if its scalar curvature is non-vanishing, any HP-transformation is analytic.*

In the following part of this section we shall give some formulas on analytic HP-transformations which will be useful in the later sections.

Let v^i be an HP-transformation. Substituting (3. 1) into the identity

$$\nabla_k\underset{v}{\mathfrak{L}}g_{ji} - \underset{v}{\mathfrak{L}}\nabla_k g_{ji} = g_{ri}\underset{v}{\mathfrak{L}}\left\{{r \atop kj}\right\} + g_{jr}\underset{v}{\mathfrak{L}}\left\{{r \atop ki}\right\},$$

we find

$$\nabla_k\underset{v}{\mathfrak{L}}g_{ji} = \rho_j g_{ki} + \rho_i g_{kj} - \tilde{\rho}_j\varphi_{ki} - \tilde{\rho}_i\varphi_{kj} + 2\rho_k g_{ji}, \tag{3. 3}$$

which will be used in § 5.

If we substitute (3. 1) into (1. 12), then we have

$$\underset{v}{\mathfrak{L}}R_{kji}{}^h = \delta_j^h\nabla_k\rho_i - \delta_k^h\nabla_j\rho_i - \varphi_j{}^h\nabla_k\tilde{\rho}_i + \varphi_k{}^h\nabla_j\tilde{\rho}_i - (\nabla_k\tilde{\rho}_j - \nabla_j\tilde{\rho}_k)\varphi_i{}^h. \tag{3. 4}$$

Contracting the last equation with respect to h and k, we find

$$\underset{v}{\mathfrak{L}}R_{ji} = -n\nabla_j\rho_i - 2\varphi_j{}^r\varphi_i{}^t\nabla_r\rho_t. \tag{3. 5}$$

12) Obata, M. [8].
13) Obata, M. [8].

Now we shall assume that v^t is an analytic HP-transformation. Then we have $\underset{v}{£} R_{ji} = \underset{v}{£}(R_{rt}\varphi_j^{\ r}\varphi_i^{\ t})$ by virtue of (1. 2). Hence from (3. 5) it follows

$$\nabla_j \rho_i = \varphi_j^{\ r}\varphi_i^{\ t}\nabla_r\rho_t \tag{3.6}$$

since $n > 2$. The last equation also is written in the form

$$\underset{\rho}{£}\varphi_i^{\ h} \equiv -\varphi_i^{\ r}\nabla_r\rho^h + \varphi_r^{\ h}\nabla_i\rho^r = 0,$$

which shows that ρ^t is analytic. Moreover, according to (3. 6) we have

$$\nabla_j\widetilde{\rho}_i + \nabla_i\widetilde{\rho}_j = \varphi_i^{\ r}(\nabla_j\rho_r - \varphi_j^{\ t}\varphi_r^{\ s}\nabla_t\rho_s) = 0, \tag{3.7}$$

which means that $\widetilde{\rho}^t$ is a Killing vector. Thus we get the following

THEOREM 3. 5. *If a vector ρ_i is the associated vector of an analytic HP-transformation, then ρ^t is analytic and $\widetilde{\rho}^t$ is a Killing vector.*

From (3. 5) and (3. 6) it follows

$$\underset{v}{£} R_{ji} = -(n+2)\nabla_j\rho_i, \tag{3.8}$$

from which we have

$$\underset{v}{£} S_{ji} = (n+2)\nabla_j\widetilde{\rho}_i. \tag{3.9}$$

On the other hand, from (3. 4) and (3. 7) we get

$$\underset{v}{£} R_{kji}^{\ \ \ h} = \delta_j^h\nabla_k\rho_i - \delta_k^h\nabla_j\rho_i - \varphi_j^{\ h}\nabla_k\widetilde{\rho}_i + \varphi_k^{\ h}\nabla_j\rho_i - 2\varphi_i^{\ h}\nabla_k\widetilde{\rho}_j. \tag{3.10}$$

If we substitute (3. 8) and (3. 9) into (3. 10), then we can verify

$$\underset{v}{£} P_{kji}^{\ \ \ h} = 0.\text{[14)]} \tag{3.11}$$

In the next place, substituting (3. 1) and (3. 8) into the identity

$$\underset{v}{£}\nabla_k R_{ji} - \nabla_k \underset{v}{£} R_{ji} = -R_{ri}\underset{v}{£}\left\{{r \atop kj}\right\} - R_{jr}\underset{v}{£}\left\{{r \atop ki}\right\},$$

we have

$$\underset{v}{£}\nabla_k R_{ji} = -(n+2)\nabla_k\nabla_j\rho_i - R_{ki}\rho_j - R_{kj}\rho_i + S_{ki}\widetilde{\rho}_j + S_{kj}\widetilde{\rho}_i - 2R_{ji}\rho_k. \tag{3.12}$$

Hence if we put

$$P_{kji} = \frac{1}{n+2}(\nabla_k R_{ji} - \nabla_j R_{ki}), \tag{3.13}$$

14) Ishihara, S. [2].

it holds

$$\underset{v}{\pounds} P_{kji} = P_{kji}{}^r \rho_r. \tag{3.14}$$

4. An analytic *HP*-transformation which leaves invariant the covariant derivative of the *HP*-curvature tensor. In this section we shall show an analogous theorem to the one obtained by T. Sumitomo for an infinitesimal projective transformation in a Riemannian space[15].

Let v^i be an analytic *HP*-transformation. If we substitute (3. 1) and (3. 11) into the identity

$$\begin{aligned}&\underset{v}{\pounds}\nabla_l P_{kji}{}^h - \nabla_l \underset{v}{\pounds} P_{kji}{}^h \\ &\quad = P_{kji}{}^r \underset{v}{\pounds}\left\{{}^{\,h}_{lr}\right\} - P_{rji}{}^h \underset{v}{\pounds}\left\{{}^{\,r}_{lk}\right\} - P_{kri}{}^h \underset{v}{\pounds}\left\{{}^{\,r}_{lj}\right\} - P_{kjr}{}^h \underset{v}{\pounds}\left\{{}^{\,r}_{li}\right\},\end{aligned}$$

then we obtain

$$\underset{v}{\pounds}\nabla_l P_{kji}{}^h = T_{lkji}{}^h,$$

where we have put

$$\begin{aligned}T_{lkji}{}^h = \delta_l^h P_{kji}{}^r \rho_r - 2\rho_l P_{kji}{}^h - \rho_k P_{lji}{}^h - \rho_j P_{kli}{}^h - \rho_i P_{kjl}{}^h \\ - \varphi_l{}^h P_{kji}{}^r \bar{\rho}_r + \varphi_l{}^r(\bar{\rho}_k P_{rji}{}^h + \bar{\rho}_j P_{kri}{}^h + \bar{\rho}_i P_{kjr}{}^h).\end{aligned}$$

Now we shall assume that $\underset{v}{\pounds}\nabla_l P_{kji}{}^h = 0$. Then we have

$$T_{lkji}{}^h = 0. \tag{4.1}$$

Contracting this equation with respect to h and l, we can verify

$$P_{kji}{}^r \rho_r = 0,$$

by virtue of (1. 5) ~ (1. 9).

Substituting the last equation into (4. 1) and taking account of $P_{kji}{}^r \bar{\rho}_r = 0$, we obtain the equation

$$\begin{aligned}2\rho_l P_{kji}{}^h + \rho_k P_{lji}{}^h + \rho_j P_{kli}{}^h + \rho_i P_{kjl}{}^h \\ = \varphi_l{}^r(\bar{\rho}_k P_{rji}{}^h + \bar{\rho}^l P_{kri}{}^h + \bar{\rho}_i P_{kjr}{}^h).\end{aligned}$$

Transvecting this equation with $\rho^l P^{kji}{}_h = \rho^l P_{rsth}\, g^{rk} g^{sj} g^{ti}$ and taking account of (1. 5) ~ (1. 9), we obtain

$$(\rho_l P_{kji}{}^h)(\rho^l P^{kji}{}_h) + 2(\rho^l P_{ljih})(\rho_r P^{rjih}) + (\rho^l P_{kjlr})(\rho_l P^{kjtr}) = 0,$$

after some complicated calculation.

15) Sumitomo, T. [10], Yano, K. and Nagano, T. [13].

Since the each term in the left-hand side of the last equation is non-negative, it must hold $\rho_l P_{kji}{}^h = 0$, from which we get the following

THEOREM 4.1. *If a Kählerian manifold admits an analytic non-affine HP-transformation which leaves invariant the covariant derivative of the HP-curvature tensor, then the manifold is a space of constant holomorpic curvature.*

In a symmetric Kählerian manifold, i.e. in a Kählerian manifold satisfying $\nabla_l R_{kji}{}^h = 0$, the equation $\underset{v}{\pounds}\nabla_l P_{kji}{}^h = 0$ trivially holds, so we have

COROLLARY 4.2. *If a symmetric Kählerian manifold admits an analytic non-affine HP-transformation, then the manifold is a space of constant holomorphic curvature.*

5. An analytic *HP*-transformation in a Kählerian manifold satisfying $\nabla_k R_{ji} = 0$. In this section we shall obtain a theorem on an analytic *HP*-transformation in a Kählerian manifold satisfying $\nabla_k R_{ji} = 0$. The method used here is analogous to the one used by T. Sumitomo [10] for an infinitesimal projective transformation in a Riemannian space.

At the first place, we have a well known[16)]

LEMMA 5.1. *A necessary and sufficient condition for a Riemannian manifold to be an Einstein one is that the following equation holds:*

$$R_{ji}R^{ji} = \frac{R^2}{n}.$$

This follows from the identity $Z_{ji}Z^{ji} = R_{ji}R^{ji} - \frac{R^2}{n}$, where $Z_{ji} = R_{ji} - (R/n)g_{ji}$.

Now consider a Kählerian manifold such that $\nabla_k R_{ji} = 0$ and let v^i be an analytic *HP*-transformation. Then, from (3.12) we have

$$(n+2)\nabla_k\nabla_j\rho_i = -R_{ki}\rho_j - R_{kj}\rho_i + S_{ki}\tilde{\rho}_j + S_{kj}\tilde{\rho}_i - 2R_{ji}\rho_k. \tag{5.1}$$

Transvecting this equation with g^{kj}, we get

$$\nabla^r\nabla_r\rho_i = -\frac{1}{n+2}(2R_i{}^r\rho_r + R\rho_i). \tag{5.2}$$

On the other hand, since ρ^i is analytic, we have

$$\nabla^r\nabla_r\rho_i + R_i{}^r\rho_r = 0.$$

16) For example, Sumitomo, T. [10].

Compairing the last two equations, we find

$$(5.3) \qquad R_i{}^r \rho_r = \frac{R}{n} \rho_i,$$

which shows that ρ^i is a Ricci's direction. Thus it follows

$$(5.4) \qquad R^{tr} \nabla_t \rho_r = \frac{R}{n} \nabla_t \rho^t.$$

LEMMA 5.2. *If a Kählerian manifold satisfying $\nabla_k R_{ji} = 0$ is not an Einstein manifold, then the associated vector ρ^i of an analytic HP-transformation satisfies $\nabla_i \rho^i = 0$.*

PROOF. By applying the Ricci's identity to R_{ji}, we find

$$(5.5) \qquad R_{lkj}{}^r R_{ri} + R_{lki}{}^r R_{jr} = 0.$$

Transvecting this with g^{ki}, we have

$$(5.6) \qquad R_{ltrj} R^{tr} = R_l{}^r R_{jr}.$$

From (5.5) it follows

$$(\underset{v}{\pounds} R_{lkj}{}^r) R_{ri} + R_{lkj}{}^r \underset{v}{\pounds} R_{ri} + (\underset{v}{\pounds} R_{lki}{}^r) R_{jr} + R_{lki}{}^r \underset{v}{\pounds} R_{jr} = 0.$$

If we transvect this equation with $R^{jk} g^{il}$, then we get

$$(R_h{}^k R^{ij} + R_{th} R^{tj} g^{ik}) \underset{v}{\pounds} R_{kji}{}^h = 0$$

by virtue of (5.6).

Now let v^i be an analytic *HP*-transformation. If we substitute (3.10) into the last equation, then we can verify

$$(5.7) \qquad (\nabla_r \rho^r) R_{ji} R^{ji} - R R_{ji} \nabla^j \rho^i = 0,$$

after some calculation.

From (5.4) and (5.7) it follows

$$\left(R_{ji} R^{ji} - \frac{R^2}{n} \right) \nabla_r \rho^r = 0,$$

which implies together with Lemma 5.1 the lemma. q. e. d.

THEOREM 5.3. *If a Kählerian manifold satisfying $\nabla_k R_{ji} = 0$ admits an analytic non-affine HP-transformation, it is a Kähler-Einstein manifold.*

PROOF. Since $\nabla_k R_{ji} = 0$, $R_{ji} R^{ji}$ is constant, and so we have

$$0 = \underset{v}{\pounds}(R_{ji} R^{ji}) = (\underset{v}{\pounds} R_{ji}) R^{ji} + R_{ji} \underset{v}{\pounds}(R_{rt} g^{rj} g^{ti})$$

$$= 2\,[(\underset{v}{\pounds}R_{ji})R^{ji} + R_j{}^r R_{rt}\underset{v}{\pounds}g^{jt}],$$

where v^i is a vector field.

Now let v^i be an analytic non-affine HP-transformation. If we substitute (3. 8) and (5. 4) into the last equation, then we find

$$-\frac{n+2}{n}R\nabla_r\rho^r + R_j{}^r R_{rt}\underset{v}{\pounds}g^{jt} = 0.$$

If we assume that our manifold is not an Einstein one, then we have

$$R_j{}^r R_{rt}\underset{v}{\pounds}g^{jt} = 0, \tag{5.8}$$

by virtue of Lemma 5. 2. By means of $\underset{v}{\pounds}(g^{jt}g_{ti}) = 0$, (5. 8) can be written in the form

$$R^{jr}R_r{}^t\underset{v}{\pounds}g_{jt} = 0.$$

Operating ∇_i to the both sides and then substituting (3. 3), we get

$$\frac{1}{2}(\rho_i\rho^i)(R_{jr}R^{jr}) + (R_{jr}\rho^j)(R_i{}^r\rho^i) = 0.$$

Since the each term of the left-hand side is non-negative, we have $\rho_i R_{jr} = 0$, which contradicts to our assumption. q. e. d.

6. A Kähler-Einstein manifold with non-vanishing scalar curvature. We have given in the last section that the existence of an analytic non-affine HP-transformation in a Kählerian manifold satisfying $\nabla_k R_{ji} = 0$ reduces the manifold to an Einstein one. In this section we shall devote ourselves to discuss such a transformation in a Kähler-Einstein manifold with $R \neq 0$.

We shall first prove the following

LEMMA 6. 1. *In a Kählerian manifold with positive (or negative) definite Ricci's form, any infinitesimal affine transformation is analytic and hence so is any Killing vector field.*

PROOF. When the given manifold V is irreducible, the Ricci tensor being non-zero, the manifold admits no quaternion structure. Thus, taking account of Obata's theorem[17], we see that in V any affine transformation is analytic.

When the given manifold is reducible, for any point there exists a neighbourhood U of the point which is a Pythagorean product of irreducible Kählerian manifolds, say V_1, V_2,……, V_p. In each of these Kählerian manifolds

17) Obata, M. [8].

$V_1, \ldots\ldots, V_p$ its Ricci's form is positive (or negative) definite, because of the assumption on the Ricci's form of V. Then, each of these manifolds $V_1, \cdots, V_p$ admits no quaternion structure.

Let v^i be an infinitesimal affine transformation in the given V. In the neighbourhood U, v^i is decomposed in such a way that $v^i = \sum_{\alpha=1}^{p} v_{(\alpha)}{}^i$, where $v_{(\alpha)}{}^i$ is an infinitesimal affine transformation in the Kählerian manifold V_α $(\alpha = 1, \ldots\ldots, p)$. Since the manifold V_α is irreducible and has no quaternion structure, by means of Obata's theorem, $v_{(\alpha)}{}^i$ is analytic. Accordingly, it is obvious that the given v^i is analytic in U. This proves the lemma. q. e. d.

This lemma implies immediately the following

LEMMA 6. 2. *In a Kähler-Einstein manifold, if its scalar curvature does not vanish, any Killing vector is analytic.*

We notice here that in an Einstein manifold with $R \neq 0$ any infinitesimal affine transformation is a Killing vector. In fact, for an infinitesimal affine transformation v^i we have $\underset{v}{\pounds} R_{ji} = 0$. The manifold being Einsteinian, it follows $\underset{v}{\pounds} g_{ji} = (n/R)\underset{v}{\pounds} R_{ji} = 0$.

By virtue of these lemmas we have the following

THEOREM 6. 3. *In a Kähler-Einstein manifold with non-vanishing scalar curvature, an HP-transformation is analytic if and only if its associated vector is analytic.*

PROOF. The associated vector of an analytic *HP*-transformation is analytic by means of Theorem 3. 5. Conversely, we suppose that the associated vector ρ_i of an *HP*-transformation v^i is analytic. Then, it follows from (3. 5) and (3. 6)

$$\underset{v}{\pounds} R_{ji} = -(n+2)\nabla_j \rho_i.$$

The manifold being an Einstein one, this implies

$$\underset{v}{\pounds} g_{ji} = \frac{1}{k} \nabla_j \rho_i, \qquad k = -\frac{R}{n(n+2)}. \tag{6. 1}$$

Taking account of (6. 1), if we put

$$p_i = v_i - \frac{1}{2k}\rho_i, \tag{6. 2}$$

we have $\nabla_j p_i + \nabla_i p_j = 0$, which means that the vector p^i is a Killing one. According to Lemma 6, 2, the vector p^i is analytic. Therefore the given *HP*-

transformation v^i is analytic. q. e. d.

Let v^i be an analytic *HP*-transformation in a Kähler-Einstein manifold. Then from (3. 8) we have (6. 1). If we define a vector field p^i by (6. 2), we see that p^i is a Killing vector. Next, if we put $q^i = (1/2k)\widetilde{\rho^i} = -(1/2k)\varphi_r{}^i\rho^r$, then we have

$$\widetilde{q}^i = -\frac{1}{2k}\rho^i \tag{6.3}$$

$$v^i = p^i + \varphi_r{}^i q^r. \tag{6.4}$$

Thus we have, taking account of Theorem 3. 5, the following

THEOREM 6. 4[18]. *In a Kähler-Einstein manifold with* $R \neq 0$, *an analytic HP-transformation* v^i *is uniquely decomposed in the form*

$$v^i = p^i + \varphi_r{}^i q^r,$$

where p^i *and* q^i *are Killing vectors.*

On Theorem 6. 4 we remark the following fact. The equation $\underset{u}{\pounds}\varphi_i{}^h = 0$ is equivalent to

$$\nabla_j u_i - \nabla_i u_j = \varphi_j{}^r(\nabla_r \widetilde{u}_i + \nabla_i \widetilde{u}_r).$$

Hence, in a Kählerian manifold, a necessary and sufficient condition in order that an analytic vector u_i is gradient is that $\widetilde{u}^i$ is a Killing vector.

Now, if q^i is a Killing vector, taking account of Lemma 6. 2, it is analytic, and hence so is $\widetilde{q}^i$. If we put $u^i = \widetilde{q}^i$ in the above arguments, we get $\nabla_j \widetilde{q}_i = \nabla_i \widetilde{q}_j$. Thus $\varphi_r{}^i q^r$ in Theorem 6. 4 is gradient analytic.

Thus the uniqueness follows from the fact that an Einstein manifold with $R \neq 0$ can not admit a non-trivial parallel vector field.

Next we have from (6. 4)

$$\underset{v}{\pounds}\left\{ {h \atop j\,i} \right\} = -\underset{\widetilde{q}}{\pounds}\left\{ {h \atop j\,i} \right\}.$$

If we substitute (3. 1) and (6. 3) into the last equation, we find

$$\nabla_j\nabla_i\rho^h + R_{rji}{}^h\rho^r = 2k(\rho_j\delta_i^h + \rho_i\delta_j^h - \widetilde{\rho}_j\varphi_i{}^h - \widetilde{\rho}_i\varphi_j{}^h). \tag{6.5}$$

Thus we have the following

18) In the compact case, this theorem is trivially contained in Matsushima's theorem on analytic vectors. [5].

COROLLARY 6.5. *In a Kähler-Einstein manifold with* $R \neq 0$, *the associated vector of an analytic HP-transformation is a (gradient analytic) HP-transformation.*

Let L_H, L_I and L' be the Lie algebra consisting of all analytic *HP*-transformations, the Lie algebra consisting of all Killing vector fields and the vector space of all analytic gradient *HP*-transformations, respectively. Then Theorem 6.4 asserts that a direct sum $L_H = L_I + L'$ holds good. After some computation, we can verify the following

COROLLARY 6.6. *In a Kähler-Einstein manifold with* $R \neq 0$, *the following relations hold*:

$$L_H = L_I + L' \ (\textit{direct sum}),$$

$$[L_I, L_I] \subset L_I, \qquad [L_I, L'] \subset L', \qquad [L', L'] \subset L_I.$$

From (6.5) we have

$$\nabla_j \nabla_i \rho_h + R_{tjih}\rho^t = 2k(g_{jh}\rho_i + g_{ih}\rho_j - \varphi_{jh}\tilde{\rho}_i - \varphi_{ih}\tilde{\rho}_j). \tag{6.6}$$

Let v^t be non-affine and consider a geodesic $x^t = x^t(s)$ at a point on which $\rho_i(dx^t/ds) \neq 0$, where s is the arc lengh. If we define a function $f(s)$ along the geodesic by $f(s) = \rho_i(dx^t/ds)$, then we have $f''(s) = 4k\, f(s)$ because of (6.6). Now we assume that $R < 0$. Then we find $f(s) = Ae^{2\sqrt{k}s} + Be^{-2\sqrt{k}s}$, where A and B are constant. Thus we obtain

THEOREM 6.7. *In a complete Kähler-Einstein manifold with* $R < 0$, *the length of the associated vector of an analytic non-affine HP-transformation is not bounded.*

Next, if we take the alternating part of (6.6) with respect to i and h, we get

$$R_{tjih}\rho^t = k(g_{ti}g_{jh} - g_{ji}g_{th} + \varphi_{ti}\varphi_{jh} - \varphi_{ji}\varphi_{th} + 2\varphi_{tj}\varphi_{ih})\rho^t, \tag{6.7}$$

from which, taking account of the theorem given in Appendix II, we obtain

THEOREM 6.8. *If a Kähler-Einstein manifold with* $R \neq 0$ *admits an analytic non-affine HP-transformation, then its local homogeneous holonomy group at any point is the full unitary group* $U(n/2)$.

Let ρ^t be an analytic gradient *HP*-transformation, then (6.7) is valid. Hence if L' is transitive[19] at each point of the manifold, then (1.10) holds

19) Let L be a vector space of vector fields in an n dimensional manifold. Denoting by $v(P)$ the value of a vector field v at P, we consider the vector space $L_P = \{v(P) \mid v \in L\}$ of vectors at P. When dim $L_P = n$, we say that the vector space L is transitive at P.

good. Therefore we get the following

THEOREM 6.9. *In a Kähler-Einstein manifold with* $R \neq 0$, *if the vector space consisting of all analytic gradient HP-transformations is transitive at each point, then the manifold is a space of constant holomorphic curvature.*

This theorem can also be proved in the following way. We consider a Kähler-Einstein manifold, then it holds $P_{kji} = 0$, where the tensor P_{kji} is defined by (3.13). Let v^i be a non-affine analytic HP-transformation and ρ_i be its associated vector. Then, from (3.14) it follows $P_{kji}{}^r\rho_r = 0$. If the vector space L' is transitive at each point of the manifold, then we have $P_{kji}{}^h = 0$. This proves that the manifold has constant holomorphic curvature.

COROLLARY 6.1. *If a homogeneous Kähler-Einstein manifold with* $R \neq 0$ *admits an analytic non-affine HP-transformation, and if its linear isotropy group is irreducible, then it is a space of constant holomorphic curvature.*

PROOF. Since the linear isotropy group of the manifold is irreducible, taking account of the formula $[L_I, L'] \subset L'$ which is given in Corollary 6.6, we see that the vector space L' is transitive at any point. Accordingly, Theorem 6.9 implies that the given manifold is of constant holomorphic curvature. q. e. d.

7. An HP-transformation in a compact space of constant holomorphic curvature. Let us consider a compact space of constant holomorphic curvature with $R > 0$[20]. Then the manifold being an Einstein one, an analytic vector v is decomposed uniquely in the form

$$v^i = p^i + \varphi_r{}^i q^r \text{ [21]},$$

where p^i and q^i are Killing vectors. Hence we have

$$\underset{v}{\pounds}\{{}^{\,h}_{ji}\} = -\underset{\tilde{q}}{\pounds}\{{}^{\,h}_{ji}\} = -(\nabla_j\nabla_i\tilde{q}^h + R_{rji}{}^h\underset{q}{\tilde{q}^r}). \tag{7.1}$$

Since q^i is a Killing vector, we have

$$\nabla_j\nabla_i q^h + R_{rji}{}^h q^r = 0.$$

Substituting the last equation into (7.1), we find

$$\underset{v}{\pounds}\{{}^{\,h}_{ji}\} = (-\varphi_t{}^h R_{rji}{}^t + \varphi_r{}^t R_{tji}{}^h)q^r.$$

Next if we substitute (1.10) into the last equation, then we get

20) In a compact Kähler-Einstein manifold with $R<0$ there exists no non trivial analytic vector.
21) Matsushima, Y. [5].

$$\underset{v}{\pounds}\{{}^{\,h}_{ji}\} = \rho_j\delta_i^h + \rho_i\delta_j^h - \tilde{\rho}_j\varphi_i^h - \tilde{\rho}_i\varphi_j^h, \qquad \rho_i = -2k\tilde{q}_i,$$

from which we see that v^i is an *HP*-transformation. Thus on taking account of Theorem 3. 1, we have

THEOREM 7. 1. *In a compact space of constant holomorphic curvature with $R > 0$, a necessary and sufficient condition for a vector field to be analytic is that it is an HP-transformation.*

This theorem can be also proved in the following way. Let L_H and A be the Lie algebras of all *HP*-transformations and of all analytic vector fields respectively. Then it follows $L_H \subset A$ from Theorem 3. 1 and

$$A = L_I + \widetilde{L_I}, \text{ (direct sum)},$$

from Matsushima's theorem[22] where L_I denotes the Lie algebra of all Killing vector fields in the manifold and $\widetilde{L_I}$ is defined by $\widetilde{L_I} = \{\widetilde{p^i} \mid p^i \in L_I\}$.

It is known that $\dim L_I = m^2 + 2m$[23], which implies

$$\dim A = 2(m^2 + 2m).$$

On the other hand, it is known[24]

$$\dim L_H = 2(m^2 + 2m).$$

Consequently, we have $\dim L_H = \dim A$, hence we get $L_H = A$.

APPENDIX I

Consider an $n\ (= 2m > 2)$ dimensional real vector space V and let $e_1, \dots, e_n$ be a fixed base. We shall represent any quantities on V in terms of their components with respect to the base e_i. Since n is even, V admits a complex structure, i.e. a tensor φ_i^h such that $\varphi_i^r\varphi_r^h = -\delta_i^h$. In the following we shall consider a fixed complex structure.

LEMMA 1. *Let a_j^i be a tensor on V such that*

$$\varphi_j^r a_r^i + a_j^r \varphi_r^i = 0. \tag{1}$$

Moreover, if it satisfies the equation

$$a_j^i y^j = a y^i + b\varphi_j^i y^j \tag{2}$$

for any vector y^i, then a_j^i must be a zero tensor, where a and b are real-valued functions of y^i.

22) Matsushima, Y. [5].
23) Ishihara, S. [2].
24) Ishihara, S. [3].

PROOF. At the first place, we notice that $a_j^{\ i}$ satisfies the relations

$$a_r^{\ r} = 0, \qquad a_r^{\ i}\varphi_i^{\ r} = 0$$

by virtue of (1). Next, for convenience, we shall introduce an Hermitian metric g_{ji}[25]. Then the equation $g_{rs}\varphi_j^{\ r}\varphi_i^{\ s} = g_{ji}$ holds true and $\varphi_{ji} = \varphi_j^{\ r}g_{ri}$ is skew-symmetric with respect to j and i.

Transvecting (2) with $y_i = g_{ir}y^r$, we have $a_j^{\ t}y^jy_t = ay^ty_t$, and with $\varphi_i^{\ t}y_t$, we get $\varphi_i^{\ t}y_ta_j^{\ i}y^j = -by^ty_t$. Making use of these equations, we can eliminate a and b in (2) and we have

$$t_{jhr}^{\quad t}y^jy^hy^r = 0, \tag{3}$$

where

$$t_{jhr}^{\quad t} = a_j^{\ t}g_{hr} - a_{jh}\delta_r^{\ t} + \varphi_{pj}a_h^{\ p}\varphi_r^{\ t}, \qquad a_{jh} = a_j^{\ r}g_{rh}.$$

Since (3) holds for arbitrary y^t, it follows $t_{(jhr)}^{\quad t} = 0$. If we write down this explicitely, it becomes the following equation:

$$\begin{aligned} (4) \qquad & 2\,(a_j^{\ t}g_{hr} + a_h^{\ t}g_{rj} + a_r^{\ t}g_{jh}) \\ & - (a_{jh}\delta_r^{\ t} + a_{hr}\delta_j^{\ t} + a_{rj}\delta_h^{\ t} + a_{hj}\delta_r^{\ t} + a_{rh}\delta_j^{\ t} + a_{jr}\delta_h^{\ t}) \\ & + \varphi_{pj}a_h^{\ p}\varphi_r^{\ t} + \varphi_{ph}a_r^{\ p}\varphi_j^{\ t} + \varphi_{pr}a_j^{\ p}\varphi_h^{\ t} \\ & + \varphi_{ph}a_j^{\ p}\varphi_r^{\ t} + \varphi_{pr}a_h^{\ p}\varphi_j^{\ t} + \varphi_{pj}\,a_r^{\ p}\varphi_h^{\ t} = 0. \end{aligned}$$

By contraction with respect to i and j we have $(n-2)\,(a_{hr} + a_{rh}) = 0$, which implies

$$a_{jt} + a_{tj} = 0. \tag{5}$$

On the other hand, transvecting (4) with $g^{hr}g_{it}$, we have $na_{jt} - 2\,a_{tj} = 0$, after some calculation. Hence we obtain $a_{jt} = 0$ on taking account of (5). This implies $a_j^{\ t} = 0$. q. e. d.

LEMMA 2. *Let $t_{ji}^{\ \ h}$ be a tensor on V such that*

$$t_{ji}^{\ \ h} = t_{ij}^{\ \ h} \tag{6}$$

$$t_{ji}^{\ \ r}\varphi_r^{\ h} = t_{jr}^{\ \ h}\varphi_i^{\ r}. \tag{7}$$

Moreover, if it satisfies the equation

$$t_{ji}^{\ \ h}y^jy^i = ay^h + b\varphi_j^{\ h}y^j \tag{8}$$

for any vector y^i, a and b being functions of y^i, then $t_{ji}^{\ \ h}$ takes the following form :

25) It is always possible to introduce an Hermitian metric. See, Frölicher, A. [1].

$$t_{ji}{}^h = \alpha_j\delta_i^h + \alpha_i\delta_j^h - \tilde{\alpha}_j\varphi_i{}^h - \tilde{\alpha}_i\varphi_j{}^h, \tag{9}$$

where α_i is a certain vector and $\tilde{\alpha}_i = \varphi_i{}^r\alpha_r$.

PROOF. In the same manner as in the proof of Lemma 1, we shall introduce an Hermitian metric g_{ji}.

From (6) and (7) it follows

$$t_{ji}{}^h + t_{cb}{}^h\varphi_j{}^c\varphi_i{}^b = 0,$$

from which we obtain the equation

$$g^{ji}t_{ji}{}^h = 0, \tag{10}$$

because of $g^{ji}\varphi_j{}^c\varphi_i{}^b = g^{cb}$.

From (8) we can easily obtain the relations

$$t_{ji}{}^h y_h y^j y^i = ay^r y_r, \qquad t_{ji}{}^r\varphi_r{}^t y_t y^j y^i = -by^r y_r.$$

Making use of these relations, we can eliminate a and b in (8) and get

$$t_{jirp}{}^h y^j y^i y^r y^p = 0, \tag{11}$$

where

$$t_{jirp}{}^h = t_{ji}{}^h g_{rp} - t_{jir}\delta_p^h + t_{ji}{}^s\varphi_{sr}\varphi_p{}^h, \qquad t_{jir} = t_{ji}{}^s g_{sr}.$$

Since (11) holds for arbitrary y^i, it follows $t_{(jirp)}{}^h = 0$. Transvecting this with g^{rp} and taking account of (6), (7) and (10), we can obtain

$$t_{ji}{}^h = \alpha_j\delta_i^h + \alpha_i\delta_j^h + \beta_j\varphi_i{}^h + \beta_i\varphi_j{}^h, \tag{12}$$

where

$$\alpha_j = \frac{1}{n+2}t_{jr}{}^r, \qquad \beta_j = \frac{-1}{n+2}t_{jr}{}^s\varphi_s{}^r.$$

In the last place, substituting (12) into (7), we get $\beta_j = -\tilde{\alpha}_j$. q. e. d.

We can generalize Lemma 2 in the following form.

LEMMA 3. *Let $t_{ji}{}^h$ be a tensor on V such that $t_{ji}{}^h = t_{ij}{}^h$. If it satisfies*

$$t_{ji}{}^h y^j y^i = ay^h + b\varphi_j{}^h y^j \tag{13}$$

for any vector y^i, then $t_{ji}{}^h$ takes the following form:

$$t_{ji}{}^h = \alpha_j\delta_i^h + \alpha_i\delta_j^h + \beta_j\varphi_i{}^h + \beta_i\varphi_j{}^h,$$

where a and b are real-valued functions of y^i and α_i, β_i are certain vectors.

Lemma 2 is a direct consequence of Lemma 3, but the proof of the latter is very complicated, so we shall only give the outline of it.

Multiplying the both sides of (13) with $y^k\varphi_c{}^l y^c$, we have

$$t^{hkl}_{jib}y^j y^t y^b y^c = (a\varphi_c{}^l\delta_b{}^k\delta_r{}^h + b\varphi_r{}^h\varphi_c{}^l\delta_b{}^k)y^c y^b y^r,$$

where

$$t^{hkl}_{jibc} = t_{ji}{}^h\delta_b{}^k\varphi_c{}^l.$$

If we take the alternating part with respect to h, k and l of the above equation, we have

$$t^{[hkl]}_{jibc}y^j y^t y^b y^c = 0,$$

which implies

$$t^{[hkl]}_{(jibc)} = 0.$$

By contraction, we get

$$t^{[hkr]}_{(jibr)} = 0, \tag{14}$$

where the left-hand side is the sum of 72 terms because of the symmetry with respect to i and j.

Transvecting (14) with $\varphi_k{}^b$ and putting $t_j = t_{jr}{}^r$, $\rho_j = t_{ji}{}^k\varphi_k{}^t$, we obtain after a complicated calculation

$$(n^2 + 2n)t_{ji}{}^h - 8\,\Phi_3\cdot\Phi_2 t_{ji}{}^h = A_j\delta_i^h + A_i\delta_j^h + B_j\varphi_i{}^h + B_i\varphi_j{}^h, \tag{15}$$

where

$$A_j = \tilde\rho_j + (n+1)t_j, \qquad B_j = \tilde t_j - (n+1)\rho_j,$$
$$\tilde\rho_j = \varphi_j{}^r\rho_r, \qquad \tilde t_j = \varphi_j{}^r t_r.$$

and Φ_2, Φ_3 are the operators defined by

$$\begin{aligned}\Phi_2 A_{ji}{}^h &= \frac{1}{2}(A_{ji}{}^h + A_{jr}{}^t\varphi_t{}^r\varphi_t{}^h)^{26)},\\ \Phi_3 A_{ji}{}^h &= \frac{1}{2}(A_{ji}{}^h - A_{rt}{}^h\varphi_j{}^r\varphi_i{}^t).\end{aligned} \tag{16}$$

These operators Φ_2 and Φ_3 satisfy the following identities[27].

26) If we put $\overset{*}{o}{}^{rh}_{it} = \frac{1}{2}(\delta_i{}^r\delta_t{}^h + \varphi_i{}^r\varphi_t{}^h)$, $o^{rt}_{ji} = \frac{1}{2}(\delta_j{}^r\delta_i{}^t - \varphi_j{}^r\varphi_i{}^t)$, (16) is written in the following form :

$$\Phi_2 A_{ji}{}^h = \overset{*}{o}{}^{rh}_{it}A_{jr}{}^t, \quad \Phi_3 A_{ji}{}^h = o^{rt}_{ji}A_{rt}{}^h.$$

27) Obata, M. [7].

$$\Phi_2\cdot\Phi_2 = \Phi_2, \qquad \Phi_3\cdot\Phi_3 = \Phi_3, \qquad \Phi_2\cdot\Phi_3 = \Phi_3\cdot\Phi_2. \tag{17}$$

After some calculation we get easily

$$\Phi_3\cdot\Phi_2(A_j\delta_i^h + A_i\delta_j^h + B_j\varphi_i^h + B_i\varphi_j^h) = 0.$$

Therefore, if we operate $\Phi_3\cdot\Phi_2$ to the both sides of (15), we get

$$(n^2 + 2n)\Phi_3\cdot\Phi_2\ t_{ji}^h - 8\Phi_3\cdot\Phi_2\cdot\Phi_3\cdot\Phi_2 t_{ji}^h = 0,$$

from which we obtain

$$(n+4)\ (n-2)\Phi_3\cdot\Phi_2 t_{ji}^h = 0$$

by virtue of (17). Consequently, $\Phi_3\cdot\Phi_2 t_{ji}^h = 0$ holds. Thus if we substitute the last equation into (15), we find

$$t_{ji}^h = \alpha_j\delta_i^h + \alpha_i\delta_j^h + \beta_j\varphi_i^h + \beta_i\varphi_j^h,$$

where

$$\alpha_j = \frac{1}{n(n+2)}A_j = \frac{1}{n(n+2)}\{\tilde{\rho}_j + (n+1)t_j\},$$

$$\beta_j = \frac{1}{n(n+2)}B_j = \frac{1}{n(n+2)}\{\tilde{t}_j - (n+1)\rho_j\}. \qquad \text{q. e. d.}$$

We can also prove the following lemma according to Lemma 2. 1 given in Ishihara, S. [2] and Lemma 3.

LEMMA 4. *Let t_{ji}^h be a tensor on V such that*

$$\Phi_2 t_{ji}^h = 0^{28)}, \qquad \Phi_3\cdot\Phi_1(t_{ji}^h - t_{ij}^h) = 0.$$

If t_{ji}^h satisfies

$$t_{ji}^h y^j y^i = ay^h + b\varphi_j^h y^j$$

for any vector y^i, then it takes the following form:

$$t_{ji}^h = \rho_j\delta_i^h + \rho_i\delta_j^h - \tilde{\rho}_j\varphi_i^h - \tilde{\rho}_i\varphi_j^h + \sigma_j\delta_i^h + \tilde{\sigma}_j\varphi_i^h,$$

where ρ_j and σ_j are certain vectors and operator Φ_1 defined by

$$\Phi_1 t_{ji}^h = \frac{1}{2}(t_{ji}^h - t_{jr}^t\varphi_i^r\varphi_t^h).$$

In an almost-complex manifold, an affine connection Γ_{ji}^h is called a φ-connection, if the almost-complex structure φ_i^h is covariant constant with respect to Γ_{ji}^h, i. e. $\nabla_j\varphi_i^h = 0$, where ∇_j is defined by $\nabla_j v^h = \partial_j v^h + \Gamma_{jr}^h v^r$ for

28) This equation is equivalent to (7).

a vector field v^t, for example. A φ-connection is said to be half-symmetric[29], if its torsion tensor $S_{ji}{}^h$ satisfies $\Phi_3 \cdot \Phi_1 S_{ji}{}^h = 0$.

By the same arguments as in §2, we can prove the following

THEOREM. *In an almost-complex manifold endowed with a half-symmetric φ-connection Γ_{ji}^h, in order that an infinitesimal transformation v^t preserves the H-plane curves*[30], *it is necessary and sufficient that it satisfies the following equations*:

$$\underset{v}{\pounds}\varphi_i{}^h = 0,$$

$$\underset{v}{\pounds}\Gamma_{ji}^h = \rho_j\delta_i{}^h + \rho_i\delta_j{}^h - \tilde{\rho}_j\varphi_i{}^h - \tilde{\rho}_i\varphi_j{}^h + \sigma_j\delta_i{}^h + \tilde{\sigma}_j\varphi_i{}^h,$$

where ρ_j and σ_j are certain vector fields.

APPENDIX II

Let V be an n $(= 2m)$ dimensional real vector space and $(g_{ji}, \varphi_i{}^h)$ an Hermitian structure on it. Then $g_{ji} = g_{rs}\varphi_j{}^r\varphi_i{}^s$ holds, and so $\varphi_{ji} = \varphi_j{}^r g_{ri}$ is skew-symmetric.

Consider a tensor $R_{kji}{}^r$ of type (1, 3) on V and suppose that there exists a non-zero vector ρ^t satisfying

$$\rho^t R_{tji}{}^h = (g_{ti}\delta_j{}^h - g_{ji}\delta_t{}^h + \varphi_{ti}\varphi_j{}^h - \varphi_{ji}\varphi_t{}^h + 2\,\varphi_{tj}\varphi_i{}^h)\rho^t. \tag{1}$$

For a vector σ^t, we denote by $R(\sigma)$ the matrix whose (i, h)-element is $R(\sigma)_i{}^h = \rho^t\sigma^j R_{tji}{}^h$. Then we have

$$R(\sigma)_i{}^h = \rho_i\sigma^h - \sigma_i\rho^h + \tilde{\rho}_i\tilde{\sigma}^h - \tilde{\sigma}_i\tilde{\rho}^h + 2\,\varphi_i{}^h\rho^t\tilde{\sigma}_t, \tag{2}$$

where $\tilde{\rho}_i = \varphi_i{}^r\rho_r$ and $\tilde{\sigma}^t = -\varphi_r{}^t\sigma^r$.

We shall prove the following

LEMMA. *If we choose a suitable base in V, the Lie algebra generated by the matrices $R(\sigma)$, $\sigma \in V$, contains all matrices of the following form*:

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

where A is a skew-symmetric (m, m)-matrix and B is a symmetric (m, m)-matrix.

PROOF. Since we can change the length of ρ^t without loss of generality,

29) Ishihara, S. [2].

30) The definition of H-plane curves in an almost-complex manifold with a φ-connection have given in [2] and [11].

it is possible to choose a base such that

$$\rho^{t} = (\delta_{\alpha 1},\ \delta_{\bar{\alpha}\bar{1}}) \qquad g_{jt} = \delta_{jt},$$

$$\varphi_{\alpha\beta} = \varphi_{\bar{\alpha}\bar{\beta}} = 0, \qquad \varphi_{\alpha\bar{\beta}} = -\delta_{\alpha\beta}, \qquad \varphi_{\bar{\alpha}\beta} = \delta_{\alpha\beta}.$$

where indices α and β run over 1, 2,......, m and $\bar{\alpha} = m + \alpha$.

In the following, we shall only use the lower indices, because the base in consideration is orthogonal. If we take $m-1$ vectors

$$\underset{\gamma}{\sigma_t} = (\delta_{\alpha\gamma},\ \delta_{\bar{\alpha}\bar{\gamma}}), \qquad \gamma = 2,\ldots\ldots, m,$$

then we have

$$\underset{\gamma}{\tilde{\sigma}_t} = (-\delta_{\alpha\gamma},\ \delta_{\bar{\alpha}\bar{\gamma}}), \qquad \rho = (-\delta_{\alpha 1},\ \delta_{\bar{\alpha}\bar{1}}).$$

It is easily seen that $2m$ vectors ρ_t, $\tilde{\rho}_t$, $\underset{\gamma}{\sigma_t}$ and $\underset{\gamma}{\tilde{\sigma}_t}$ are linearly independent.

We shall now put

$$R_1^* = R(\tilde{\rho}), \qquad R_\gamma = R(\underset{\gamma}{\sigma}), \qquad R_\gamma^* = R(\underset{\gamma}{\tilde{\sigma}}),$$

and use the following notations on (m, m)-matrixes:

$$\begin{aligned}
A_{\beta\alpha} &= (a_{\mu\lambda}):\ a_{\beta\alpha} = -a_{\alpha\beta} = 1,\ a_{\mu\lambda} = 0 \text{ for } (\mu, \lambda) \neq (\alpha, \beta), (\beta, \alpha);\\
B_{\beta\alpha} &= (b_{\mu\lambda}):\ b_{\beta\alpha} = b_{\alpha\beta} = 1,\ b_{\mu\lambda} = 0 \text{ for } (\mu, \lambda) \neq (\alpha, \beta), (\beta, \alpha);\\
C &= (c_{\mu\lambda}):\ c_{11} = 2,\ c_{\varepsilon\varepsilon} = 1 \text{ for } \varepsilon = 2,\ldots\ldots, m,\ c_{\mu\lambda} = 0 \text{ for } \mu \neq \lambda;\\
D_\gamma &= (d_{\mu\lambda}):\ -d_{11} = d_{\gamma\gamma} = 1,\ d_{\mu\lambda} = 0 \text{ for } (\mu, \lambda) \neq (1, 1), (\gamma, \gamma),
\end{aligned}$$

where α, β, λ and μ run over 1, 2,......, m and $\gamma = 2,\ldots\ldots, m$.

Using this notation, we can obtain

$$R_\gamma = 2\begin{pmatrix} A_{1\gamma} & 0 \\ 0 & A_{1\gamma} \end{pmatrix},\ R_\gamma^* = 2\begin{pmatrix} 0 & B_{1\gamma} \\ -B_{1\gamma} & 0 \end{pmatrix},\ R_1^* = 4\begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}. \tag{3}$$

By some calculation we get for $\beta, \gamma = 2,\ldots\ldots, m$

$$[R_\beta, R_\gamma] = 4\begin{pmatrix} A_{\beta\gamma} & 0 \\ 0 & A_{\beta\gamma} \end{pmatrix}, \tag{4}$$

$$[R_\beta^*, R_\gamma] = 4\begin{pmatrix} 0 & B_{\beta\gamma} \\ -B_{\beta\gamma} & 0 \end{pmatrix} \qquad \text{for } \beta \neq \gamma, \tag{5}$$

$$[R_\gamma^*, R_\gamma] = 8\begin{pmatrix} 0 & D_\gamma \\ -D_\gamma & 0 \end{pmatrix}.$$

Hence we find

$$P = R_1^* - \frac{1}{2}\sum_{\gamma=2}^{m}[R_\gamma^*, R_\gamma] = 4(m+1)\begin{pmatrix} 0 & B_{11} \\ -B_{11} & 0 \end{pmatrix}, \tag{6}$$

from which it follows

$$P + \frac{m+1}{2}[R_\gamma^*, R_\gamma] = 4(m+1)\begin{pmatrix} 0 & B_{\gamma\gamma} \\ -B_{\gamma\gamma} & 0 \end{pmatrix}. \tag{7}$$

According to (3) $\sim$ (7), we see that the lemma is true. q. e. d.

By means of the above lemma, we have the following

THEOREM. *In a Kählerian manifold of n dimensions, if its curvature tensor* $R_{kji}{}^h$ *satisfies at a point the relation*

$$\rho^t R_{tji}{}^h = (g_{ti}\delta_j^h - g_{ji}\delta_t^h + \varphi_{ti}\varphi_j{}^h - \varphi_{ji}\varphi_t{}^h + 2\,\varphi_{tj}\varphi_i{}^h)\rho^t \tag{1}$$

for a non-vanishing vector ρ^t, *then the local homogeneous holonomy group of the manifold at any point coincides with the full unitary group* $U(n/2)$.

PROOF. We suppose that the relation (1) holds at a point P. We consider for fixed indices j and k a matrix $R(kj\,;\,P)$ whose (i, h)-element is given by the value of $R_{kji}{}^h$ at the point P. Then, it follows from the above lemma that the Lie algebra generated by all the matrices $R(kj\,;\,P)$ is equivalent to the Lie algebra of the full unitary group $U(n/2)$. Therefore, taking account of Nijenhuis' theorem[31], we see that the local homogeneous holonomy group of the manifold at the point P contains the unitary group $U(n/2)$. The manifold being Kählerian, it follows thus that the local homogeneous holonomy group of the manifold at P coincides with $U(n/2)$. q. e. d.

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31) Nijenhuis, A. [6].

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