ON LINEAR LIE ALGEBRAS OF A CERTAIN DIMENSION

TERUKIYO SATÔ

(Received August 21, 1959)

Let C be a complex number field, and K be an arbitrary field of characteristic 0. And we denote by $\mathfrak{Gl}(n, K)$ (or briefly by $\mathfrak{G}$) a Lie algebra of all matrices of degree n with coefficients in K, and by $\mathfrak{Sl}(n, K)$ (or by $\mathfrak{S}$) Lie subalgebra of $\mathfrak{Gl}(n, K)$ composed of all matrices of trace 0.

On Lie subalgebras of $\mathfrak{Gl}(n, C)$, G. B. Gurevič [2] proved the following:[1)]
There exist (n^2-1)-dimensional Lie subalgebras of $\mathfrak{Gl}(n, C)$ different from $\mathfrak{Sl}(n, C)$ if and only if $n=2$, and they are all conjugate to one another with respect to inner automorphism. And there exist (n^2-2)-dimensional Lie subalgebras if and only if $n=2$ or $n=3$, and for $n=3$ they are divided into two conjugate classes with respect to inner automorphism.

In the present note, we shall investigate Lie subalgebras of $\mathfrak{Gl}(n, K)$ in regard to larger n and more general field K. Then our main result will be stated as follows:

Except for $\mathfrak{Sl}(n, K)$, there is no Lie subalgebra of dimension larger than n^2-n+1, and for $n \geqq 3$, (n^2-n+1)-dimensional Lie subalgebras are transformed to one another by automorphism of $\mathfrak{Gl}(n, K)$ and they are divided into two conjugate classes with respect to inner automorphism.

Here (and hereafter) we mean by inner automorphism of $\mathfrak{Gl}(n, K)$ (or $\mathfrak{Sl}(n, K)$) the automorphism $x \to XxX^{-1}$ of $\mathfrak{Gl}(n, K)$ (or $\mathfrak{Sl}(n, K)$) obtained by a matrix X which is regular (or whose determinant is 1) respectively (C. Chevalley [1]).

I wish to express my sincere gratitude to Prof. N.Iwahori for his invaluable suggestions.

LEMMA 1. *Let $\mathfrak{H}$ be a proper Lie subalgebra of $\mathfrak{G}$ different from $\mathfrak{S}$, then we may find a sequence of matrix units $e_{i_1 j_1}, e_{i_2 j_2}, \cdots, e_{i_{n-1} j_{n-1}}$ such that*

$$e_{i_\nu j_\nu} \overline{\in} \{\{\mathfrak{H}, e_{i_1 j_1}, \cdots, e_{i_{\nu-1} j_{\nu-1}}\}\},$$

and either $i_\nu = i_{\nu-1}$ *and* $j_\nu \overline{\in} \{i_1, \cdots, i_{\nu-1}, j_1, \cdots, j_{\nu-1}\}$

1) This is due to Mathematical Reviews Vol. 19, and the author has no opportunity yet to see that paper.

or $\quad j_\nu = j_{\nu-1}$ *and* $i_\nu \overline{\in} \{i_1,\dots, i_{\nu-1}, j_1,\dots, j_{\nu-1}\}$,

where $\{\{*\}\}$ *means a linear subspace generated by* $*$.

PROOF. We shall construct $e_{i_\nu j_\nu}$ by induction on ν. First, if all $e_{ij}(i \neq j)$ belong to $\mathfrak{H}$, then we have $\mathfrak{H} \supseteqq \mathfrak{S}$ because of $e_{ii} - e_{jj} = [e_{ij}, e_{ji}]$. But this contradicts to our assumption, whence we may find $e_{i_1 j_1} \overline{\in} \mathfrak{H}$. Next, we suppose that $e_{i_1 j_1},\dots, e_{i_{\nu-1} j_{\nu-1}}$ were already found ($\nu \leqq n-1$). Since the set $\{i_1,\dots, i_{\nu-1}, j_1,\dots\dots, j_{\nu-1}\}$ consists of ν elements, we may choose k from the set $\{1,\dots, n\}$ which does not belong to $\{i_1,\dots, i_{\nu-1}, j_1,\dots, j_{\nu-1}\}$. With respect to this k, if both $e_{i_{\nu-1}k}$ and $e_{kj_{\nu-1}}$ belong to $\{\{\mathfrak{H}, e_{i_1 j_1},\dots, e_{i_{\nu-1} j_{\nu-1}}\}\}$, then we may express them as follows:

$$e_{i_{\nu-1}k} = h_1 + \sum_{\mu=1}^{\nu-1} * e_{i_\mu j_\mu}$$

$$e_{kj_{\nu-1}} = h_2 + \sum_{\mu=1}^{\nu-1} * e_{i_\mu j_\mu}$$

where h_1 and h_2 are elements of $\mathfrak{H}$, and we mean by $*$'s suitable elements of K. Since $\{i_1,\dots, i_{\nu-1}\} \cap \{j_1,\dots, j_{\nu-1}\} = \phi$, we get $[e_{i_\alpha j_\alpha}, e_{i_\beta j_\beta}] = 0$ for $1 \leqq \alpha, \beta \leqq \nu-1$. Then it follows that

$$\mathfrak{H} \ni [h_1, h_2] = [e_{i_{\nu-1}k} - \sum_{\mu=1}^{\nu-1} * e_{i_\mu j_\mu}, e_{kj_{\nu-1}} - \sum_{\mu=*}^{\nu-1} * e_{i_\mu j_\mu}] = e_{i_{\nu-1} j_{\nu-1}}.$$

But this contradicts to our induction hypothesis. Hence either $e_{i_{\nu-1}k}$ or $e_{kj_{\nu-1}}$ does not belong to $\{\{\mathfrak{H}, e_{i_1 j_1},\dots, e_{j_{\nu-1} j_{\nu-1}}\}\}$. Therefore, if we put $i_\nu = i_{\nu-1}, j_\nu = k$ or $j_\nu = j_{\nu-1}$, $i_\nu = k$, then we conclude our induction. q.e.d.

From this lemma, easily we obtain

THEOREM 1. *Except for* $\mathfrak{Sl}(n, K)$, *dimension of any Lie subalgebra of* $\mathfrak{Gl}(n, K)$ *is at most* $n^2 - n + 1$.

LEMMA 2. *If* $\mathfrak{H}$ *is a* $(n^2 - n + 1)$*-dimensional Lie subalgebra of* $\mathfrak{G}$, *then we have the following expression*

$$\mathfrak{G} = \mathfrak{H} \oplus \{\{e_{ij_1},\dots, e_{ij_{n-1}}\}\}$$

$$\text{or} \qquad = \mathfrak{H} \oplus \{\{e_{i_1 j},\dots, e_{i_{n-1} j}\}\} \qquad (\textit{direct sum}),$$

where $i, j_1,\dots, j_{n-1}$, *or* $i_1,\dots, i_{n-1}, j$ *mean the mutually different integers from* 1 *to* n.

PROOF. According to Lemma 1, we have

$$\mathfrak{G} = \mathfrak{H} \oplus \{\{e_{i_1 j_1},\dots, e_{j_{n-1} j_{n-1}}\}\}$$

where either $\quad i_\nu = i_{\nu-1}, j_\nu \overline{\in} \{i_1,\dots, i_{\nu-1}, j_1,\dots, j_{\nu-1}\}$

or $\qquad j_\nu = j_{\nu-1},\ i_\nu \overline{\in} \{i_1,\dots, i_{\nu-1}, j_1,\dots, j_{\nu-1}\}.$

Since it is clear in the case $n \leqq 3$, we may assume that $n > 3$. Now we suppose that $i_1 = i_2 = \dots = i_{\mu-1} \neq i_\mu (\mu > 2)$, then, in virtue of property of i_ν, j_ν as the above, we have $j_1 \neq j_2$, $j_\mu = j_{\mu-1} \neq j_1$ Now we express $e_{i_\mu j_1}$ and $e_{j_1 j_\mu}$ as follows,

$$e_{i_\mu j_1} = h_1 + \sum *e_{i_\nu j_\nu}$$

$$e_{j_1 j_\mu} = h_2 + \sum *e_{i_\nu j_\nu}$$

where h_1 and h_2 are elements of $\mathfrak{H}$, and $*$'s are suitable elements of K. Then it holds that

$$[h_1, h_2] = [e_{i_\mu j_1} - \sum *e_{i_\nu j_\nu},\ e_{j_1 j_\mu} - \sum *e_{i_\nu j_\nu}]$$

$$= e_{i_\mu j_\mu} - *e_{i_1 j_\mu} = e_{i_\mu j_\mu} - *e_{i_{\mu-1} j_{\mu-1}} \in \mathfrak{H}.$$

However this is impossible. Hence $i_1 = i_2$ implies that $i_1 = i_2 = \dots = i_{n-1}$. In the case $j_1 = j_2$, it may be proved similarly.

LEMMA 3. *For any $(n^2 - n + 1)$-dimensional Lie subalgebra $\mathfrak{H}$, we can express $\mathfrak{G}$ as follows:*

$$\mathfrak{G} = \sigma\tau_0{}^a \mathfrak{H} \oplus \{\{e_{12},\dots, e_{1n}\}\}$$

where σ is an inner automorphism, τ_0 is a certain automorphism of $\mathfrak{G}$ independent of $\mathfrak{H}$, and a is an integer equal to 0 or 1.

PROOF. Put $X_{ij} = e_{ij} - e_{ji} + \sum_{k \neq i, j} e_{kk}\ (i \neq j)$, then we have $X_{ij}\ e_{kl}\ X_{ij}^{-1} = \pm\, e_{\sigma_{ij}(k)\sigma_{ij}(l)}$, where σ_{ij} means transposition (i, j). We also denote by σ_{ij} the automorphism $x \to X_{ij} x X_{ij}^{-1}$, and by τ_0 the automorphism $x \to -{}^t x$. Then by operating an automorphism $\sigma\tau_0{}^a$ which is represented as product of some σ_{ij}'s (and τ_0 if necessary) on the both sides of equality in Lemma 2, we may obtain our lemma.

LEMMA 4. *For $(n^2 - n + 1)$-dimensional Lie subalgebra $\mathfrak{H}$ if it holds that*

$$\mathfrak{G} = \mathfrak{H} \oplus \{\{e_{12},\dots, e_{1n}\}\},$$

then we can find an inner automorphism σ such that σ leaves invariant $\{\{e_{12},\dots, e_{1n}\}\}$ and $\sigma\mathfrak{H}$ contains $\mathfrak{h}$, where we mean by $\mathfrak{h}$ the abelian Lie subalgebra $\{\{e_{11},\dots, e_{nn}\}\}$.

PROOF. Put

$$e_{ii} = e'_{ii} + \sum_{l=2}^{n} \alpha_{il} e_{1l} \qquad (e'_{ii} \in \mathfrak{H},\ \alpha_{il} \in K).$$

If $i \neq 1$, $j \neq 1$, and $i \neq j$, it holds that

$$\mathfrak{H} \ni [e'_{ii}, e'_{jj}] = [e_{ii} - \sum \alpha_{il} e_{1l},\ e_{jj} - \sum \alpha_{jl} e_{1l}] = \alpha_{ji} e_{1i} - \alpha_{ij} e_{1j}.$$

Hence we have $\alpha_{ji} = \alpha_{ij} = 0$, and consequently

$$e'_{ii} = e_{ii} - \alpha_{ii} e_{1i} \qquad (i \neq 1).$$

On the other hand, for $i \neq 1$ similarly we have

$$[e'_{ii}, e'_{11}] = [e_{ii} - \alpha_{ii} e_{1i}, e_{11} - \sum \alpha_{1l} e_{1l}] = \alpha_{1i}\, e_{1i} + \alpha_{ii}\, e_{1i} = 0.$$

Therefore we obtain

$$e'_{11} = e_{11} + \sum_{l=2}^{n} \alpha_{ll}\, e_{1l}.$$

Now if we denote by σ an automorphism $x \to (E + \sum \alpha_{ll}\, e_{1l}) x (E + \sum \alpha_{ll}\, e_{1l})^{-1}$ (where E is a unit matrix), then we have $\sigma e'_{ii} = e_{ii}\ (1 \leqq i \leqq n)$ and $\sigma e_{1j} = e_{jj}$ $(j \neq 1)$. Hence we get our assertion.

LEMMA 5. *If the Lie subalgebra* $\mathfrak{h}$ *is contained in the Lie subalgebra* $\mathfrak{H}$ *such that*

$$\mathfrak{G} = \mathfrak{H} \oplus \{\{e_{12}, \dots, e_{1n}\}\},$$

then $\mathfrak{H}$ *must be* $\{\{\mathfrak{h}, e_{ij}\ (i \neq 1,\ i \neq j)\}\}$.

(Hereafter this Lie subalgebra $\{\{\mathfrak{h}, e_{ij} (i \neq 1,\ i \neq j)\}\}$ will be denoted by $\mathfrak{H}_{n-1}$).

PROOF. We take an element $x = \sum \xi_k e_{kk}$ of $\mathfrak{h}$. If we put $e_{ij} = \sum \alpha_l e_{1l}$ $+ h\,(i \neq 1,\ i \neq j,\ \alpha_l \in K$, and $h \in \mathfrak{H})$, then we have

$$\begin{aligned}[x, h] &= \left[\sum \xi_k e_{kk},\ e_{ij} - \sum \alpha_l e_{1l}\right] = (\xi_i - \xi_j) e_{ij} - \sum \alpha_l (\xi_1 - \xi_l) e_{1l} \\ &\equiv \sum_l \alpha_l (\xi_i - \xi_j) e_{1l} - \sum_l \alpha_l (\xi_1 - \xi_l) e_{1l} \\ &\equiv \sum_l \alpha_l (\xi_i - \xi_j - \xi_1 + \xi_l) e_{1l} \qquad \mod \mathfrak{H}.\end{aligned}$$

Here, since $[x, h]$ belongs to $\mathfrak{H}$, $\alpha_l(\xi_i - \xi_j - \xi_1 + \xi_l)$ must be equal to 0. Hence, if we had selected $\xi_1, \dots, \xi_n$ previously such as $\xi_i - \xi_j - \xi_1 + \xi_l \neq 0$, then we could get $\alpha_l = 0$, which implies $e_{ij} \in \mathfrak{H}$. q. e. d.

THEOREM 2. *Any (n^2-n+1)-dimensional Lie subalgebras of $\mathfrak{Gl}(n, K)$ (except for $\mathfrak{Sl}(2, K)$ in the case $n=2$) are transformed to each other by automorphisms of $\mathfrak{Gl}(n, K)$. And the number of conjugate classes of these subalgebras with respect to inner automorphism of $\mathfrak{Gl}(n, K)$ is 1 in the case $n=2$, and is 2 in the case $n \geqq 3$.*

PROOF. We denote by $\mathfrak{H}$ the Lie subalgebra as in the theorem. Then from Lemma 3 we have the following expression

$$\mathfrak{G} = \sigma\tau_0{}^a\mathfrak{H} \oplus \{\{e_{12},\dots, e_{1n}\}\},$$

and apparently we may suppose that always $a=0$ in the case $n=2$. Moreover by Lemma 4, there exists an inner automorphism σ' such that

$$\mathfrak{G} = \sigma'\sigma\tau_0{}^a\mathfrak{H} \oplus \{\{e_{12},\dots, e_{1n}\}\} \qquad \text{and } \sigma'\sigma\tau_0{}^a\,\mathfrak{H} \supset \mathfrak{h}.$$

Then from Lemma 5 we obtain $\sigma'\sigma\tau_0{}^a\,\mathfrak{H} = \mathfrak{H}_{n-1}$, which implies that the number of conjugate classes with respect to inner automorphism is at most 2, and is 1 in the case $n=2$. To prove that surely this is 2 for $n \geqq 3$, we consider two Lie subalgebras $\mathfrak{H}_1 = \{\{e_{11}, e_{ij}(j \neq 1)\}\}$ and $\mathfrak{H}_{n-1}$. Then $\mathfrak{H}_i(i=1$ or $n-1)$ is a whole of all matrices leaving invariant a certain i-dimensional subspace V_i of the space on which $\mathfrak{Gl}(n, K)$ operates, and such V_i is uniquely determined by $\mathfrak{H}_i$. If there exists a regular matrix X such that $X\,\mathfrak{H}_1X^{-1} = \mathfrak{H}_{n-1}$, then $\mathfrak{H}_1$ is a totality of all matrices which leave invariant $(n-1)$-dimensional subspace $X^{-1}V_{n-1}$, which is impossible. Hence there is no X as above, and we complete the proof.

COROLLARY. *Dimension of any Lie subalgebra of $\mathfrak{Sl}(n, K)$ is at most n^2-n. And any (n^2-n)-dimensional Lie subalgebras of $\mathfrak{Sl}(n, K)$ are transformed to each other by automorphism of $\mathfrak{Sl}(n, K)$, and the number of conjugate classes of these subalgebras with respect to inner automorphism of $\mathfrak{Sl}(n, K)$ is 1 in the case $n=2$ and 2 in the case $n \geqq 3$.*

PROOF. We denote by $\mathfrak{K}$ an arbitrary Lie subalgebra of $\mathfrak{S}$, and let m be its dimension. Then $\mathfrak{H} = \{\{E\}\} \oplus \mathfrak{K}$ is a Lie subalgebra of $\mathfrak{G}$ of dimension $m+1$ and we have $\mathfrak{K} = \mathfrak{H} \cap \mathfrak{S}$. Then Theorem 1 implies that $m \leqq n^2-n$. Let m be equal to n^2-n. Then from the proof of Theorem 2, we can see that there exists an automorphism τ_0 and inner automorphisms σ, σ' such that $\sigma'\sigma\tau_0{}^a\,\mathfrak{H} = \mathfrak{H}_{n-1}$. Here, we can easily see that σ', σ and τ_0 leave $\mathfrak{S}$ invariant and σ', σ may be regarded as inner automorphisms of $\mathfrak{S}$. Thus we obtain $\sigma'\sigma\tau_0{}^a\,\mathfrak{K} = \mathfrak{H}_{n-1} \cap \mathfrak{S}$. The rest is obvious. q.e.d.

REMARKS. 1. Any (n^2-n+1)-dimensional Lie subalgebra of $\mathfrak{Gl}(n, K)$ is decomposed into the following form: for example

$$\mathfrak{H}_1 = \{\{E, e_{11}, e_{12}, \dots, e_{1n}\}\} \oplus \Big\{\Big\{ \sum_{i=2}^{n} \xi_i \, e_{ii} \big(\textstyle\sum \xi_i = 0\big), \\ e_{jk}(j \neq 1,\ k \neq 1,\ j \neq k)\Big\}\Big\},$$

where the first term of the right hand side is the radical and the second is a simple subalgebra.

2. Lie subalgebras $\mathfrak{H}_1$ and $\mathfrak{H}_{n-1}$ may be considered as associative algebras, and hence so may be any $(n^2 - n + 1)$-dimensional Lie subalgebra of $\mathfrak{Gl}(n, K)$ (except for $\mathfrak{Sl}(2, K)$). And of course, Theorem 1 also holds replacing "Lie subalgebra" by "associative subalgebra". But Theorem 2 does not hold, i.e. $\mathfrak{H}_1$ is not isomorphic to $\mathfrak{H}_{n-1}$(in the associative sence). In fact, $\{\{e_{11}\}\}$ is a left ideal of $\mathfrak{H}_1$, but there is no 1-dimensional left ideal in $\mathfrak{H}_{n-1}$.

REFERENCES

[1] C. CHEVALLEY, Théorie des groupes de Lie, II(1951).
[2] G. B. GUREVIČ, Linear Lie algebras of dimension n^2-1 or n^2-2. Tul'sk. Meh. Inst. Trudy. (1953). (Russian).

AIZU JUNIOR COLLEGE, AIZU-WAKAMATSU.