ON A CONJECTURE OF KAPLANSKY

SHÔICHIRO SAKAI

(Received August. 14, 1959)

Prof. Kaplansky stated a conjecture that any derivation of a C^*-algebra would be automatically continuous [1]. In this note, we shall show that this conjecture is in fact true.

THEOREM. *Any derivation of a C^*-algebra is automatically continuous.*

PROOF. Let A be a C^*-algebra, $'$ a derivation of A. It is enough to show that the derivation is continuous on the self-adjoint portion A_s of A. Therefore if it is not continuous, by the closed graph theorem there is a sequence $\{x_n\}$ $(x_n \neq 0)$ in A_s such that $x_n \to 0$ and $x_n' \to a + ib(\neq 0)$, where a and b are self-adjoint. First, suppose that $a \neq 0$ and there exists a positive number $\lambda(>0)$ in the spectrum of a (otherwise consider $\{-x_n\}$). It is enough to assume that $\lambda = 1$.

Then there is a positive element h $(\|h\| = 1)$ of A such that $hah \geqq \frac{1}{2}h^2$. Put $y_n = x_n + 3 \cdot \|x_n\| \cdot I$, then $y_n \to 0$, $y_n' = x_n'$ and $(hy_nh)' = h'y_nh + hy_n'h + hy_nh'$; hence $(hy_nh)' \to h(a+ib)h$.

Therefore

$$\|(hy_{n_0}h)' - h(a+ib)h\| < \frac{1}{8} \qquad \text{for some } n_0 \quad \cdots\cdots\cdots\cdots(1).$$

On the other hand

$$hy_nh \leqq 4\|x_n\|h^2 \text{ and } \frac{1}{2} \cdot \frac{hy_nh}{4\|x_n\|} \leqq hah \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots(2)$$

Since $\|x_n\| \cdot I + x_n \geqq 0$, $\dfrac{hy_nh}{4\|x_n\|} \geqq \dfrac{1}{2}h^2$.

Hence

$$\left\| \frac{hy_nh}{4\|x_n\|} \right\| \geqq \frac{1}{2}\|h\|^2 = \frac{1}{2} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(3)$$

Let C be a C^*-subalgebra of A generated by $hy_{n_0}h$ and I, then by the (3) there is a character φ of C such that $\varphi\left(\dfrac{hy_{n_0}h}{4\|x_{n_0}\|}\right) \geqq \dfrac{1}{2}$.

Let $\bar{\varphi}$ be an extended state of φ on A, and $\mathfrak{m} = \{x \mid \bar{\varphi}(x^*x) = 0,\ x \in A\}$, then $C \cap \mathfrak{m}$ is a maximal ideal of C; it can be written $hy_{n_0}h - \varphi(hy_{n_0}h) \cdot I = u^2 - v^2$ with $u, v \in C \cap \mathfrak{m}\,(u, v \geqq 0)$; hence $(hy_{n_0}h)' = u'u + uu' - v'v - vv'$, so that by the Schwartz's inequality

$$\bar{\varphi}((hy_{n_0}h)') = 0 \qquad\qquad (4)$$

Then by the (1) and (4)

$$|\bar{\varphi}(h(a + ib)h)| < \frac{1}{8} \qquad\qquad (5)$$

On the other hand by the (2)

$$|\bar{\varphi}(h(a + ib)h)| \geqq \bar{\varphi}(hah)$$
$$= \frac{1}{2}\bar{\varphi}\left(\frac{hy_{n_0}h}{4\|x_{n_0}\|}\right) \geqq \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

; hence $|\bar{\varphi}(h(a + ib)h)| \geqq \frac{1}{4}$.

This contradicts the above inequality (5), so that $a = 0$.

Next suppose that $b \neq 0$ and there exists a positive number $\mu(> 0)$ in the spectrum of b (otherwise consider $\{-x_n\}$). It is enough to assume that $\mu = 1$. Then there is a positive element k ($\|k\| = 1$) of A such that $kbk \geqq \frac{1}{2} k^2$; moreover $\|(ky_{n_1}k)' - k(a + ib)k\| < \frac{1}{8}$ for some n_1.

Let C_1 be a C^*-subalgebra of A generated by $ky_{n_1}k$ and I, then there is a character φ_1 of C_1 such that $\varphi_1\left(\frac{ky_{n_1}k}{4\|x_{n_1}\|}\right) \geqq \frac{1}{2}$. Let $\bar{\varphi}_1$ be an extended state of φ_1 on A, then $\bar{\varphi}_1((ky_{n_1}k)') = 0$; hence $|\bar{\varphi}_1(k(a + ib)k)| < \frac{1}{8}$.

On the other hand

$$|\bar{\varphi}_1(k(a + ib)k)| \geqq \bar{\varphi}_1(kbk) \geqq \bar{\varphi}_1\left(\frac{1}{2} k^2\right)$$
$$\geqq \frac{1}{2}\bar{\varphi}_1\left(\frac{ky_{n_1}k}{4\|x_{n_1}\|}\right) \geqq \frac{1}{4}$$

; hence $|\bar{\varphi}_1(k(a + ib)k)| \geqq \frac{1}{4}$.

This contradicts the above inequality; hence $b = 0$, so that $a + ib = 0$.

Now we obtain a contradiction and this completes the proof.

REFERENCES

[1] I. KAPLANSKY, Some aspects of analysis and probability, New York, 1958.
[2] S. SAKAI, On some problems of C^*-algebras, Tôhoku Math. J. 11, (1959) 453–455.

MATHEMATICAL INSTITUTE, TÔHOKU UNIVERSITY.