LATTICES WITH P-IDEAL TOPOLOGIES

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Introduction

The topology of a Euclidean space is usually defined by the concept of a metric. Its usual metric is defined by various concepts : addition, multiplication, square root and order of real numbers.

On the other hand, the Euclidean space can be considered as a lattice : the cardinal product of real chains (linearly ordered set), which have the order in a natural sense. It is natural to inquire whether a topology, which is homeomorphic to the usual metric topology, is definable in terms of the order relation (or lattice operation) alone. More generally, we can ask whether it is possible or not to introduce a class of topologies on a lattice L which are compatible, in some sense, with its order.

G. Birkhoff has first discussed the topologies on lattices which are defined by an order-convergence. Thereafter, various topologies on lattices were introduced by several authors and their properties were discussed by many writers : O. Frink [8], [9] B.C. Rennie [22] E.S. Northam [19], E.S. Wolk [24] A.J. Ward [23] T. Naito [15], [16], [17], [18] E.E. Floyd [10].

In ([5] p. 242), G. Birkhoff has proved that the operations of groups are continuous in the sense of order-convergence. E.E. Floyd has shown a counter example that the group operations are not continuous with respect to the order-topology. This example shows that there exists no compatible topology, in his sense, such that the group operations are continuous with respect to the topology. (See [10].)

In this paper, I will define several topologies such that the above condition holds : group-operations are continuous. Of course, these topologies are not compatible, in Floyd's sense, with the order of a lattice, but it seems to me that those topologies are rather interesting and useful. It is the purpose of this paper to give topologies satisfying the following conditions and applications of these topologies.

1. *In the Euclidean space, the topology, which is defined by the usual metric, is homeomorphic to our topologies.*

2. *The lattice operations and the group operations are continuous with*

respect to our topologies.

I gave one of topologies satisfying these conditions in a lecture at the 1956 autumn meeting in Kyoto, of the Math. Soc. of Japan; in this paper this topology is called a CP-ideal topology.

In Chapter I, we shall present the definitions of P-, CP-, MP- ideal topology and prove the continuity of lattice operations. This chapter also contains the study of conditions of topologies in lattices. The main results of this chapter are described in Theorems $1 \sim 5$. As an application of these topologies, we shall establish here a representation space, smaller than Birkhoff's, for distributive lattice making essential use of Birkhoff's proof of his representation.

In Chapter II, we concern ourselves with properties of our topologies on conditionally complete lattices. We shall show here that join-, meet-irreducible elements and CP-ideal topologies have very close connections. These results are of importance to our later investigation. In a CP-ideal topology, any bounded closed set is bicompact. As an application of this theorem, I shall prove that the unit sphere of a Banach space is bicompact.

In Chapter III, we shall show that the group operations of any commutative l-group are continuous with respect to the CP-ideal topology.

Moreover, we shall concern ourselves with the structure of l-groups using CP- and MP-ideal topologies.

The main results of this chapter are several representations which are described in Theorems 12, 13: any commutative l-group is isomorphic to a sublattice of the cardinal product of chains. [12], [13], [14].

Moreover, in Chapter IV, we shall show that under some conditions any l-group is isomorphic to a perfect sublattice of the cardinal product of chains.

NOTATIONS. We shall use L to denote a lattice and L', L'',......to denote sublattices of L. I, I_α, I_β...... will be used to denote ideals or dual ideals of a lattice. We shall use lower case Latin letters to denote elements of a lattice and Latin capitals to denote subsets of a lattice. However families of sets will usually be denoted by German capitals. We shall denote the join and the meet of two elements x and y of a lattice by $x \cup y$ and $x \cap y$ respectively, the join and the meet of all elements of a set M by sup M and inf M. The set of all elements such that $a \leqq x \leqq b$, will be denoted by $[a, b]$. The expressions

$$A \vee B \text{ and } \bigvee_{\alpha \in \Delta} X_\alpha$$

will denote the set union of two sets A and B, or of all sets of family $\{X_\alpha | \alpha \in \Delta\}$. Similarly we define

$$A \wedge B \text{ and } \bigwedge_{\alpha \in \Delta} X_\alpha.$$

The complement of a set A will be denoted by A^c and the empty set by ϕ. Any open set will be denoted by U and any closed set by F.

DEFINITIONS. We shall use the terminologies of G. Birkhoff's "Lattice theory" [5]. A chain M of a lattice is a subset such that $x \in M$, $y \in M$ imply $x \gtreqless y$. A lattice L is infinite-distributive if and only if

$$a \cap (\bigcap_\alpha b_\alpha) = \bigcup_\alpha (a \cap b_\alpha) \text{ and } a \cup (\bigcap_\alpha b_\alpha) = \bigcap_\alpha (a \cup b_\alpha).$$

A subset M of a lattice is bounded if and only if there exist elements a and b, such that $a \leqq m \leqq b$ for all $m \in M$. A lattice L is complete if and only if any subset of L has the least upper bound and the greatest lower bound. A lattice L is said to be conditionally complete if any bounded subset of L has the least upper bound and the greatest lower bound. A sublattice L' of a lattice L is called to be a *perfect sublattice* if and only if $x \in L'$, $y \in L'$ imply $[x, y] \subseteqq L'$.

In this paper, topologies on a set E are defined by a subbase of closed sets. Thus with any family $\mathfrak{F}$ of subsets of a set E containing E and the empty set ϕ, a subset F of E is said to be closed if and only if F can be obtained as an intersection of the sets which are unions of a finite number of elements belonging to $\mathfrak{F}$. The topology is a family of all closed sets. $\mathfrak{F}$ is called to be a sub-base of the topology. A closed set F is represented by

$F = \bigwedge_\alpha \bigvee_{\beta=1}^{n_\alpha} I_{\alpha\beta}$ where $I_{\alpha\beta} \in \mathfrak{F}$ and n_α is an integer which corresponds to α.

It is clear that

1) the intersection of any number of closed sets is closed,
2) the union of any finite number of closed sets is closed,
3) E and ϕ are closed.

I have received kind advices from Professor T. Nakayama at Nagoya University and Professor N. Funayama at Yamagata University to whom I wish to express here my hearty thanks.

Chapter I Introduction of Topologies on Lattices

In this chapter the ideas of P-, CP-, MP-ideal topology are introduced on most general lattices. We begin with some notions and definitions concerning ideals.

1. Definitions. Let L be a lattice. A subset I of a lattice L is called to be an *ideal* if and only if the following conditions hold:

i) $x \leqq y$ and $y \in I$ imply $x \in I$,

ii) $x \in I$ and $y \in I$ imply $x \cup y \in I$.

An ideal I is said to be a *prime ideal* if and only if

iii) $x \cap y \in I$ implies $x \in I$ or $y \in I$.

DEFINITION 1. A prime ideal I is called to be a *CP-ideal* if and only if the following condition holds:

iv) if $\{x_\alpha | \alpha \in \Delta\} \subseteqq I$ and there exists $\sup_{\alpha \in \Delta} x_\alpha$, then $\sup_{\alpha \in \Delta} x_\alpha \in I$.

The family of all CP-ideals is said to be a *CP-family*.

For studies of properties of every sub-basis of closed sets, we now introduce the concept of a P-ideal.

DEFINITION 2. A sub-family $\mathfrak{P}'$ of the family of all prime ideals, is called to be a *P-family* if and only if the following condition holds:

iv)$_1$ $L \in \mathfrak{P}'$, $\phi \in \mathfrak{P}'$.

Each element of a P-family is said to be a *P-ideal.*

Dually, we shall define the concepts of a *dual prime ideal*, a *dual P-ideal*, a *dual CP-ideal*, a *dual P-family* and a *dual CP-family*. We shall denote by $\mathfrak{P}$ the union of the P-family and the dual P-family. Analogously, we shall denote by $C\mathfrak{P}$ the union of the CP-family and the dual CP-family.

Let $\mathfrak{F}$ be the family of all prime ideals containing a fixed element a of L. And $I_1 \leqq I_2$, where I_1, I_2 are elements of $\mathfrak{F}$, means that I_2 includes I_1, as a set. Suppose now that a family $\{I_\alpha\}$ is a chain of $\mathfrak{F}$. Then the intersection $\bigwedge_\alpha I_\alpha$ is an element of $\mathfrak{F}$. Hence, by Zorn's Lemma there exists a minimal prime ideal containing a, which is written $I(a)$. Analogously, there exists a minimal CP-ideal containing a. Similarly for $a \in I \in \mathfrak{F}$ there exists a minimal prime ideal which contains a and is contained in the prime ideal I. In a similar way there are a minimal dual prime ideal and a minimal dual CP-ideal having the properties above.

DEFINITION 3. For any element a of L, a minimal prime ideal containing a is called to be an MP-ideal. The family of all MP-ideals and L, ϕ is said to be an MP-family. And dually, we define a dual MP-ideals and a dual MP-family. The union of the MP-family and the dual MP-family is denoted by $M\mathfrak{P}$.

As an immediate consequence of the definitions above, the family $M\mathfrak{P}$ is a family $\mathfrak{P}$.

DEFINITION 4. The *P-ideal topology* of a lattice L is that which results from taking $\mathfrak{P}$ as a subbasis for the closed sets of the space L. Analogously, we shall define *CP-* and *MP-ideal topology.*

NOTE: These topologies are not intrinsic in the sense that are introduced by the authors, G.Birkhoff, O. Frink, B.C. Rennie, E.S. Northam, E.E. Floyd, E.S. Wolk etc.

A so-called intrinsic (or compatible with $\leqq$) topology is that which satisfies some of the following conditions:

1) *whenever* $\{x_i\}$ *is a sequence in* L *with* $x_1 \leqq x_2 \leqq \ldots\ldots$ *and* $\bigcup_i x_i = x$, *or* $x_1 \geqq x_2 \geqq \ldots$ *and* $\bigcap_i x_i = x$, *then the sequence* x_i *converges to* x,

1′) *whenever* $\{x_\alpha\}$ *is an up-directed subset of* L *and* $y = \sup_\alpha x_\alpha$ *or* $\{x_\alpha\}$ *is a down-directed subset of* L *and* $y = \inf_\alpha x_\alpha$, *then* x_α *converges to* y,

2) *any interval* $\{x \mid a \leqq x \leqq b\}$ *is a closed set.*

Our topologies, which are introduced above, do not always satisfy the conditions above. This fact is shown by the following example.

EXAMPLE 1. Let R^2 be the Cartesian plane, in which $(x, y) \leqq (x', y')$ if and only if $x \leqq x'$ and $y \leqq y'$. Then it is well known that R^2 is a conditionally complete infinitely distributive lattice. Let L_1 be the sublattice of R^2 such that $\{(x, y) \mid 0 < x < 1,\ 0 < y < 1\}$. Let us denote by L_2 the sublattice consisting of (0, 0), (1, 1) and the points in $\{(x,y) \mid 0 < x < 1,\ 0 < y < 1\}$. We denote by L_3 the sublattice consisting of (1, 1) and the points in $\{(x,y) \mid 0 \leqq x < 1,\ 0 \leqq y < 1\}$. And let L_4 be the sublattice of R^2 which is the union of $\{(x,y) \mid 0 \leqq x < 1,\ 0 < y \leqq 1\}$, (0, 0) and (1, 1). Sublattices L_1, L_2, L_3 and L_4 are expressed as Fig. 1, Fig. 2, Fig. 3 and Fig. 4.

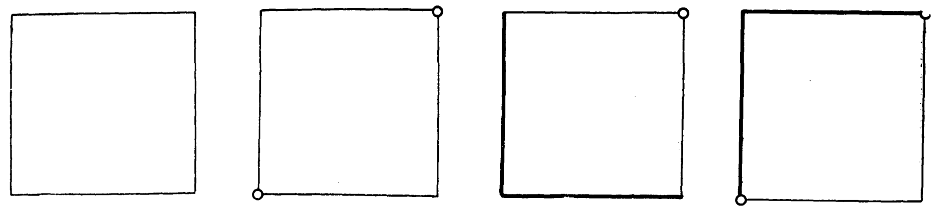

Fig 1 Fig 2 Fig 3. Fig 4.

In Figure 1, an CP-ideal and an MP-ideal are sets of the forms $\{(x,y) \mid x \leqq c\} \wedge L_1$ and $\{(x,y) \mid y \leqq c\} \wedge L_1$.

In Figure 2, every set of the form $\{(x,y) \mid x \leqq c\} \wedge L_2$ or $\{(x,y) \mid y \leqq c\} \wedge L_2$ is an MP-ideal but not a CP-ideal. In this lattice CP-ideals are only two sets L and ϕ. We can easily prove that the sequence $a_n = \left(\frac{1}{n},\ 1 - \frac{1}{n}\right)$ converges to all elements of L_2 with respect to the interval topology and the CP-ideal topology on L_2. In Figure 3, every set of the form $\{(x,y) \mid x \leqq c\} \wedge$

L_3 or $\{(x,y)|y \leqq c\} \wedge L_3$ is an MP-ideal but not a CP-ideal, but each set of the form $\{(x,y)|x \geqq c\} \wedge L_3$ or $\{(x,y)|y \geqq c\} \wedge L_3$ is a dual MP- and a dual CP-ideal.

In Figure 4, every set of the form $\{(x,y)|x \leqq c\} \wedge L_4$ or $\{(x,y)|y \geqq c\} \wedge L_4$ is a CP-, MP-ideal, or a dual CP-, MP-ideal. $\{(x,y)|y \leqq c\} \wedge L_4$ is an MP-ideal but not a CP-ideal. $\{(x,y)|x \geqq c\} \wedge L_4$ is a dual MP-ideal but not a dual CP-ideal.

1) of the conditions above is not satisfied in L_2, L_3, L_4 with its MP-ideal topology. In fact, $\left\{(x,y)|y \leqq \frac{1}{2}\right\} \wedge L_i$ is an MP-ideal and a sequence $\left(1-\frac{1}{n}, \frac{1}{2}\right)$ has (1, 1) as its least upper bound.

2) of the conditions above is not satisfied in L_2 with its CP-ideal topology, because $C\mathfrak{P}$ consists only of L_2 and ϕ. (Also see [$\wedge$, a] of Ex. 2.)

Sublattices $L_1 \sim L_4$ of R^2 are important as examples to illustrate topologies of lattices.

2. Properties of *P*-Ideal Topologies. In this section we attempt to make a contribution to the properties of P-ideal topologies. We begin with the proof of the continuity of lattice operations with respect to the P-ideal topology. Since if a proposition in the P-ideal topology is true, then the proposition in CP- or MP-ideal topology is always true, we shall not especially infer the proposition to CP- or MP-ideal topology.

THEOREM 1. *Let L be a lattice. The lattice operations of L are continuous in its P-ideal topology.*

PROOF. Let $U(a \cup b)$ be any neighborhood of $a \cup b$. Then $U(a \cup b)^c$ can be obtained as an intersection of the sets which are unions of a finite number of elements belonging to $\mathfrak{P}$:

$$U(a \cup b) = \bigvee_{\alpha} \bigwedge_{\beta=1}^{n_\alpha} I^c_{\alpha\beta}, \quad I_{\alpha\beta} \in \mathfrak{P}.$$

Hence, there exists α such that $a \cup b \in \bigwedge_{\beta} I^c_{\alpha\beta}$, which is denoted by α_0.

a) If $I_{\alpha_0\beta}$ is a prime ideal then by the definition of prime ideal either $a \in I^c_{\alpha_0\beta}$ or $b \in I^c_{\alpha_0\beta}$ holds.
We now put $U_\beta(a) = I^c_{\alpha_0\beta}$, $U_\beta(b) = L$ for $a \in I^c_{\alpha_0\beta}$ and

$U_\beta(a) = L$, $U_\beta(b) = I^c_{\alpha_0\beta}$ for $a \notin I^c_{\alpha_0\beta}$. (then $b \in I^c_{\alpha_0\beta}$).

b) If $I_{\alpha_0\beta}$ is a dual prime ideal then we have $a \in I^c_{\alpha_0\beta}$ and $b \in I^c_{\alpha_0\beta}$.
We put $U_\beta(a) = U_\beta(b) = I^c_{\alpha_0\beta}$. In both cases of a) and b), we have $U_\beta(a)$

$U_\beta(b) = I^c_{\alpha_0\beta}$ for all β.

Let $U(a)$ and $U(b)$ be $\bigwedge_{\beta=1}^{n_{\alpha_0}} U_\beta(a)$ and $\bigwedge_{\beta=1}^{n_{\alpha_0}} U_\beta(b)$, then both $U(a)$ and $U(b)$ are neighborhoods of a and b, respectively. We can easily see the fact that $U(a) \cup U(b)$ is contained in $U(a \cup b)$. Thus, the lattice operation $\cup$ is continuous with respect to its P-ideal topology. By duality, the operation $\cap$ is continuous.

NOTE. (1) $x_\alpha \to x$ (P-ideal topology) implies $x_\alpha \cap a \to x \cap a$ (P-ideal topology), but $x_\alpha \uparrow x$ does not imply $x_\alpha \cap a \uparrow x \cap a$.

(2) $x_\alpha \uparrow x$ implies $x_\alpha \to x$ (CP-ideal topology) but not always $x_\alpha \to x$ (MP-ideal topology).

Any lattice is a T_1-space with respect to its interval topology, its order topology and its ideal topology, but not always a T_0-aspce in its P-ideal topology. We shall now give an example which is not a T_0-space with respect to its P-ideal topology.

EXAMPLE 2. Both Fig. $5a$ and $5b$ indicated below are not T_0-spaces with respect to its P-ideal topology. In fact, since prime ideals of Fig.$5a$ are L, ϕ, $\{\wedge, c\}$ and $\{a, b, \wedge\}$ and dual prime ideals L, ϕ, $\{c, \vee\}$ and $\{a, b, \vee\}$ we have $\bar{a} = \bar{b}$. In Fig.$5b$, since prime ideals are L and ϕ, we obtain $\overline{\wedge} = \overline{\vee} = \bar{a} = \bar{b} = \bar{c} = L$.

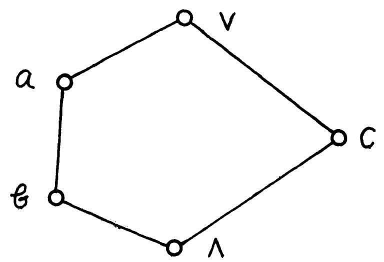

Fig 5a

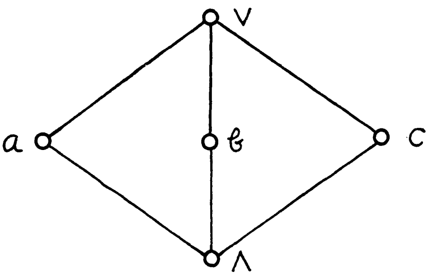

Fig 5b

3. Elementary Properties of Topological Spaces. To simplify the statements of theorems, we shall always assume, otherwise specifically stated, that each topology of lattices is a P-ideal topology.

LEMMA 1. *A lattice L is a T_0-space if and only if for given any two elements $a < b$ there exists either an element of sub-basis which contains a but not b, or an element of sub-basis which contains b but not a.*

PROOF. We shal lnow prove the sufficiency of the condition. For each pair of distinct elements p and q, we have $p \cup q \neq p$ or $p \cup q \neq q$. We may assume that $p \cup q \neq p$, then put $p \cup q = b$ and $p = a$. From the condition there exists an element I of sub-basis which contains a but not b, not or b but not a. If I contains b but not a, then I is a dual prime ideal. By $p \cup q = b$, we have $I \ni q$. If I contains a but not b, then I is a prime ideal. By $p \cup q = b$, we have $I \not\ni q$.

Therefore L is a T_0-space.

LEMMA 2. *A lattice L is a T_1-space if and only if given for any two elements $a < b$ there exist an element of the sub-basis which contains a but not b, and an element of the sub-basis which contains b but not a.*

LEMMA 3. *A lattice L is a T_2-space if and only if for $a > b$ the space L is covered by a finite number of elements of $\mathfrak{P}$ which contain at most one of a and b.*

PROOF. We shall only prove the sufficiency. Let p, q be arbitrary distinct elements of L. We put $p \cup q = a$ and $p \cap q = b$, then there is a finite number of elements of $\mathfrak{P}$ such that $\bigvee_{i=1}^{n} I_i = L$, where I_i contains at most one of a and b. Since I_i is an ideal or a dual ideal, I_i contains at most one element of p and q. The union of all I_k which do not contain p is expressed by F_1. And the union of all I_k which contain p is denoted by F_2. Then we have $F_1^c \ni p$, $F_2^c \ni q$ and $F_1^c \wedge F_2^c = \phi$.

LEMMA 4. *Let S be a T_1-space. The space S is a T_3-space if and only if each pair of an element $I \in \mathfrak{P}$ and $p \notin I$, has neighborhoods $U(I)$ and $U(p)$ such that $U(I) \wedge U(p) = \phi$.*

4. Representations and P-Ideal Topologies.

THEOREM 2. *If a lattice L is a T_0-space with respect to its P-ideal topology, then L is isomorphic to a sublattice of a Boolean lattice 2^E where E is the family of dual prime ideals of L. Therefore L is a distributive lattice.*

PROOF. Let $E(x)$ be the set of all dual prime ideals which contain x. By Lemma 1 we have $x \neq y$ if and only if $E(x) \neq E(y)$. By the definition of a dual prime ideal we get $E(x \cup y) = E(x) \vee E(y)$. $I \in E(x \cap y) \leftrightarrow I \ni x \cap y \leftrightarrow I \ni x, I \ni y \leftrightarrow I \in E(x) \wedge E(y)$.

Then we have $E(x \cap y) = E(x) \wedge E(y)$.

Thus the theorem follows.

The converse of the theorem with respect to its CP-ideal topology is not true in general. To illustrate this fact, we shall give an example:

EXAMPLE 3. L_2 of Ex. 1 is a distributive lattice but not a T_0-space with respect to its CP-ideal topology. But this lattice is a T_2-space with respect to its MP-ideal topology.

NOTE. Let $\bar{a}$ be the closure of $a \in L$ with respect to a P-ideal topology. Put $\bar{L} = \{\bar{a} \mid a \in L\}$. We shall now define the lattice operations on $\bar{L}$ such that $\bar{a} \cup \bar{b} = \overline{a \cup b}$ and $\bar{a} \cap \bar{b} = \overline{a \cap b}$. Then a correspondence $a \to \bar{a}$ of L to $\bar{L}$ is a lattice homomorphism. Since $\bar{L}$ is a T_0-space ([21] Th. 6), it is distributive. This fact makes it possible for many purposes to limit the consideration to T_0-spaces and distributive lattices.

THEOREM 2$'$. *In a lattice L with its MP-ideal topology, the following three conditions are equivalent;*

1) *L is a distributive lattice,*
2) *L is a T_0-space,*
3) *L is a T_1-space.*

PROOF. By Theorem 2 we need only to prove that if L is distributive then L is a T_1-space in its MP-ideal topology. Suppose that L is a distributive lattice and $a < b$. Then there exists a maximal prime ideal, which does not contain b, of $[a, b]$, which is denoted by N (Zorn's Lemma).

We shall now put $I = \{x \mid x \cap b \leqq n$ for some n of $N\}$. Then I is a prime ideal such that $I \not\ni b$ and $I \ni a$. In fact, if $(x \cap y) \cap b \leqq n$ for $n \in N$, then $n = \{(x \cap y) \cap b\} \cup n = \{(x \cap b) \cup n\} \cap \{(y \cap b) \cup n\}$ Since N is a prime ideal of $[a, b]$, we have either $x \cap b \leqq (x \cap b) \cup n \in N$ or $y \cap b \leqq (y \cap b) \cup n \in N$: $x \in$ or $y \in I$. The other condition is clear. This proves the fact that L is a T_1-space with respect to its MP-ideal topology.

If in the well known Birkhoff's representation, we take the dual MP-family instead of all dual prime ideals, then we obtain a representation smaller than G. Birkhoff's.

COROLLARY. *Any distributive lattice L is isomorphic to a sublattice of the lattice of all sub-families of a dual MP-family.*

PROOF. When, in the proof of Theorem 2, we take the dual MP-family instead of all dual prime ideals, we have by the proof of Theorem 2$'$ that

$$x \neq y \text{ implies } E(x) \neq E(y).$$

And hence the corollary is clear.

THEOREM 3. *If a lattice L is a T_1-space with respect to its CP-ideal topology, then L is an infinitely distributive lattice.*

PROOF. We shall now prove that if $\sup_\alpha x_\alpha$ exists, then there is $\sup_\alpha (x_\alpha$

$\cap a)$, and $a \cap \sup_{\alpha} x_\alpha = \sup_{\alpha} (a \cap x_\alpha)$. To prove this, let $a \cap \sup_{\alpha} x_\alpha = b$. Then b is an upper bound of $a \cap x_\alpha$ for all α. According to Lemma 2, for $b > c$ there exists a CP-ideal I such that $c \in I$ and $b \notin I$. Since I is a CP-ideal there is α_0 such that $x_{\alpha_0} \notin I$. In fact, if $x_\alpha \in I$ for all α, then $\sup x_\alpha \geqq b \in I$ which is a contradiction. We have $x_{\alpha_0} \cap a \notin I$, then $x_{\alpha_0} \cap a \nleqq c$. Therefore b is the least upper bound of $a \cap x_\alpha$ for all α. Thus we have $a \cap \sup_{\alpha} x_\alpha = \sup_{\alpha} (a \cap x_\alpha)$.

Dually, we can prove that $a \cup \inf_{\alpha} x_\alpha = \inf_{\alpha} (a \cup x_\alpha)$.

In the theorem above, we can not take T_0 instead of T_1. This can be illustrated by the following example.

EXAMPLE 4. L_3 of Ex. 1 is a T_0-space with respect to its CP-ideal topology and a distributive but not an infinitely distributive lattice. In fact, in Fig. 3 we shall put $x_n = \left(\frac{1}{2}, 1 - \frac{1}{n}\right)$ and $a = \left(\frac{2}{3}, \frac{2}{3}\right)$. Then we have $a \cap \sup_n x_n = a$ and $\sup_n (x_n \cap a) = \left(\frac{1}{2}, \frac{2}{3}\right)$.

THEOREM 4. *If a lattice L is a T_2-space with respect to its CP-ideal topology, then L is a T_3-space.*

PROOF. Suppose that L is a T_2-space. By Lemma 4, for any pair $I \in C\mathfrak{P}$ and $a \notin I$, we need only to prove that there exist neighborhoods $U(a)$ and $U(I)$ such that $U(a) \wedge U(I) = \phi$. We assume that I is a CP-ideal. Since a is an upper bound of $\{x \mid x \leqq a\} \wedge I$, there exists an upper bound b of $\{x \mid x \leqq a\} \wedge I$ which is smaller than a. In fact, otherwise $a = \sup \{x \mid x \leqq a\} \wedge I$, which contradicts the hypothesis of a. By Lemma 3, for $a > b$ there exists a finite system $\{I_i\}$, each element of which contains at most one element of a, b, and $L = \bigvee_{i=1}^{n} I_i$. The union of all I_i, each of which does not contain a, is denoted by F_1. The union of all I_k, each of which contains a, is written F_2. Then F_1^c and F_2^c are neighborhoods of a and I, respectively. And $F_1^c \wedge F_2^c = \phi$. To prove this we need only to show that $F_2^c \supseteqq I$. Suppose that $x \in F_2 \wedge I$. Then $x \cap a \in I$, hence $x \cap a \in \{x \mid x \leqq a\} \wedge I$: since b is an upper bound of $\{x \mid x \leqq a\} \wedge I$, $x \cap a \leqq b$. From $x \in F_2$ there exists k such that $x \in I_k$. By $I_k \ni a$ we have $x \cap a \in I_k$. Since I_k is a dual ideal, we have $b \in I_k$ which is a contradiction. Therefore we have $F_2^c \supseteqq I$.

In a similar way we can prove the case that I is a dual CP-ideal.

5. Topological Products and Cardinal Products. We shall first define the concepts of topological products and cardinal products. Suppose that for each member α of an index set Δ there is given a topological space L_α. The topological product of L_α, written $X\{L_\alpha \mid \alpha \in \Delta\}$, is defined as the set of all functions a on Δ such that $a(\alpha) \in L_\alpha$ for all $\alpha \in \Delta$, and having for each closed set C_α of a sub-basis of each L_α, the family of all sets of all functions a with $a(\alpha) \in C_\alpha$, as a sub-basis of closed sets.

The cardinal product of lattices L_α, written $L = \prod_{\alpha \in \Delta} L_\alpha$, is defined as a lattice of all functions a such that $a(\alpha) \in L_\alpha$ for all $\alpha \in \Delta$, where $a \leqq b$ means that $a(\alpha) \leqq b(\alpha)$, in L_α, for all $\alpha \in \Delta$. Then, we can prove the following theorem.

THEOREM 5. *A subset I of $\prod_{\alpha \in \Delta} L_\alpha = L$ is a CP-ideal if and only if I is represented by $\{a \mid a(\alpha_0) \in I_{\alpha_0}\}$ for some α_0 of Δ, where I_{α_0} is a CP-ideal of L_{α_0}. And the dual statement is true.*

PROOF. Let I be a CP-ideal of $\prod_{\alpha \in \Delta} L_\alpha$ and I_{α_0} the projection of I into L_{α_0}. We first show that I_{α_0} is a CP-ideal of L_{α_0}. If $a_{\alpha_0} \in L_{\alpha_0}$ and $b_{\alpha_0} \leqq a_{\alpha_0}$ in L_{α_0} then by the definition of I_{α_0} there exists an element a of L such that $a(\alpha_0) = a_{\alpha_0}$. Let b be the element of L such that $b(\alpha_0) = b_{\alpha_0}$ and $b(\alpha) = a(\alpha)$ for $\alpha \neq \alpha_0$. Since $a \geqq b$, we have $b \in I$ therefore $b_{\alpha_0} = b(\alpha_0) \in I_{\alpha_0}$. Next, let us suppose that $M_{\alpha_0} \subseteqq I_{\alpha_0}$ and sup M_{α_0} exists. Take one element a belonging to I. For $a_{\alpha_0} \in M_0$ there exists an element $b \in I$ such that $b(\alpha_0) = a_{\alpha_0}$. Then we have $a \cup b \in I$. Let c_{α_0} be an element of L such that $c_{\alpha_0}(\alpha_0) = a_{\alpha_0}$ and $c_{\alpha_0}(\alpha) = a(\alpha)$ for $\alpha \neq \alpha_0$, then we have that

$$c_{\alpha_0} \leqq a \cup b \in I \text{ implies } c_{\alpha_0} \in I.$$

Therefore if $a_{\alpha_0} \in M_{\alpha_0}$ then $c_{\alpha_0} \in I$. We shall put $\sup_{a_{\alpha_0} \in M_{\alpha_0}} c_{\alpha_0} = d$, then $d(\alpha_0) = \sup M_{\alpha_0}$ and $d(\alpha) = a(\alpha)$ for all $\alpha \neq \alpha_0$. By $d \in I$, we have sup $M_{\alpha_0} \in I_{\alpha_0}$.

Now we shall prove that if $a_{\alpha_0} \cap b_{\alpha_0} \in I_{\alpha_0}$ then either $a_{\alpha_0} \in I_{\alpha_0}$ or $b_{\alpha_0} \in I_{\alpha_0}$. Then there exists an element c of I such that $c(\alpha_0) = a_{\alpha_0} \cap b_{\alpha_0}$. Let a be an element of L such that $a(\alpha_0) = a_{\alpha_0}$ and $a(\alpha) = c(\alpha)$ for $\alpha \neq \alpha_0$, and b an element of L such that $b(\alpha_0) = b_{\alpha_0}$ and $b(\alpha) = c(\alpha)$, for $\alpha \neq \alpha_0$. Then we have $a \cap b = c \in I$, and then either $a \in I$ or $b \in I$. Thus we have $a_{\alpha_0} \in I_{\alpha_0}$ or $b_{\alpha_0} \in I_{\alpha_0}$. These show that I_{α_0} is a CP-ideal of L_{α_0}.

Now we shall see that I is $\prod_{\alpha \in \Delta} I_\alpha$. To prove this we need only to show $I \supseteqq \prod_{\alpha \in \Delta} I_\alpha$. Let us suppose that $a \in \prod_{\alpha \in \Delta} I_\alpha$, and let b_{α_0} be an element of I such that $b_{\alpha_0}(\alpha_0) = a(\alpha_0)$. Since $a \cap b_{\alpha_0} \in I$ and $a = \bigcup_{\alpha_0 \in \Delta} (a \cap b_{\alpha_0})$ we obtain $a \in I$.

We shall prove that the set $\{\alpha \mid L_\alpha \neq I_\alpha\}$ has at most one element. Let us suppose that $L_\alpha \neq I_\alpha$ and $L_\beta \neq I_\beta$. Then there are two elements $a_\alpha \in L_\alpha - I_\alpha$ and $b_\beta \in L_\beta - I_\beta$. Let d be any element of I. Let a be the element such that $a(\alpha) = a_\alpha$ and $a(\delta) = d(\delta)$ for $\alpha \neq \delta$, and b the element such that $b(\beta) = b_\beta$ and $b(\delta) = d(\delta)$ for $\beta \neq \delta$. By $a \cap b \leqq d$, we have $a \cap b \in I$, hence $a \in I$ or $b \in I$, which contradicts that $a_\alpha \notin I_\alpha$ and $b_\beta \notin I_\beta$. Then we have

$$I = I_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} L_\alpha \text{ for some } \alpha_0 \in \Delta,$$

where I_{α_0} is a CP-ideal of L_{α_0}.

From the theorem above, we can easily prove the following corollaries.

COROLLARY 1. *The topological product $X\{L_\alpha | \alpha \in \Delta\}$ of any collection $\{L_\alpha | a \in \Delta\}$ of lattices, each with a CP-ideal topology, is homeomorphic to the cardinal product $\prod_{\alpha \in \Delta} L_\alpha$, also topologized by its CP-ideal topology, of these lattices.*

PROOF. By Theorem 5 the basis of closed sets of $X\{L_\alpha | \alpha \in \Delta\}$ and that of $\prod_{\alpha \in \Delta} L_\alpha$ are the same. Hence these spaces are homeomorphic.

COROLLARY 2. *A Euclidean space R^n, is homeomorphic to the lattice R^n which is topologized by its CP-ideal topology.*

PROOF. The set R, topologized by metric, is homeomorphic to the lattice with its CP-ideal topology. Hence by Corollary 1 we obtain the proof of Corollary 2.

COROLLARY 3. *The weak topology of the set of all functions of any abstract set X to a lattice, is homeomorphic to the CP-ideal topology of the set considered as a lattice.*

PROOF. In a similar way to Corollary 2, we can prove Corollary 3.

COROLLARY 4. *If lattices L_α, each with its CP-ideal topology, satisfies a condition ϕ then the cardinal product $\prod_{\alpha \in \Delta} L_\alpha$, with its CP-ideal topology, satisfies the condition ϕ, where ϕ is one of T_2-space, bicompact, T_3-space.*

PROOF. The proof of Corollary 4 follows from the well known theorems of topological product space and the theorems above.

Chapter II Conditionally Complete Infinitely Distributive Lattices

In this chapter we shall only consider conditionally complete infinitely distributive lattices.

The purpose of this chapter is to examine the close relationship between irreducible elements and T-spaces, and to prove the compactness (bicompactness) of the lattice above. Throughout this chapter, unless otherwise stated, the word topology is used to refer to a CP-ideal topology.

6. Irreducible Elements and Topologies. Let $[a, b]$ be an interval of a lattice L. An element c of $[a, b]$ is said to be a join-irreducible element in $[a, b]$ if and only if $c = x \cup y$, $x \in [a, b]$ and $y \in [a, b]$ imply $x = c$ or $y = c$. In a similar way we define a meet-irreducible element.

LEMMA 1. *If I is a CP-ideal, $I \ni a$, $I \not\ni b$, $a < b$, then there exists an element c of $[a, b]$ such that $I \wedge [a, b] = [a, c]$, and c is a meet-irreducible element in $[a, b]$.*

PROOF. Let I be a CP-ideal. Since L is conditionally complete, there is $\sup I \wedge [a, b]$, written c. Then we have $[a, c] = I \wedge [a, b]$. In fact, by the definition of I we have $c \in I$ and then $[a, c] \subseteqq I \wedge [a, b]$. By the definition of c we have $[a, c] \supseteqq I \wedge [a, b]$. Thus $[a, c] = I \wedge [a, b]$.

Next, we shall show that c is a meet-irreducible element. Suppose that $x \cap y = c$, $x \in [a, b]$ and $y \in [a, b]$. Then, since I is a prime ideal we have either $x \in I$ or $y \in I$: $x \in I \wedge [a, b]$ or $y \in I \wedge [a, b]$. Hence we have $x \leqq c$ or $y \leqq c$. Thus we have $x = c$ or $y = c$.

THEOREM 6. *L is a T_0-space if and only if for every $a < b$ there exists a join-irreducible element in $[a, b]$, different from a, or a meet-irreducible element in $[a, b]$, different from b.*

PROOF. Suppose that L is a T_0-space. Then for any $a < b$ there exists an element I of the closed sub-basis which contains either a but not b, or b but not a. Suppose that $a \in I$ and $b \notin I$, then I is clearly a CP-ideal. By Lemma 1 we have $[a, c] = I \wedge [a, b]$, where c is a meet-irreducible element in $[a, b]$. In exactly the same way we can show that in the case of $a \notin I$ and $b \in I$ there is a join-irreducible element in $[a, b]$.

Conversely, suppose that for any $a < b$ there exists a meet-irreducible element c, different from b, in $[a, b]$. Then set $\{x \mid x \cap b \leqq c\}$, written I, is a CP-ideal. In fact, if $x_\alpha \in I$ and $\sup_\alpha x_\alpha$ exists, then

$$(\sup_\alpha x_\alpha) \cap b = \sup_\alpha (x_\alpha \cap b) \leqq c.$$

Hence we have $\sup_\alpha x_\alpha \in I$. If $x \cap y \in I$ then we have

$$c = \{(x \cap y) \cap b\} \cup c = \{(x \cap b) \cup c\} \cap \{(y \cap b) \cup c\}.$$

$a \leqq c \leqq (x \cap b) \cup c \leqq b$ and $a \leqq (y \cap b) \cup c \leqq b$ are clear. Since c is a meet-irreducible element in $[a, b]$ then we have

$$c = (x \cap b) \cup c \text{ or } c = (y \cap b) \cup c.$$

And hence we obtain $x \in I$ or $y \in I$. Thus I is a CP-ideal such that $a \in I$ and $b \notin I$.

Analogously we can prove the fact that if a join-irreducible element exists, then there exists a dual CP-ideal I such that $a \notin I$ and $b \in I$. Therefore, by Lemma 1 of Chapter I, L is a T_0-space.

From the proof of the theorem above, we can conclude the theorem below.

THEOREM 7. *A lattice L is a T_1-space with respect to its CP-ideal topology if and only if for every $a < b$, $[a, b]$ contains both a join-irreducible element, different from a, in $[a, b]$ and a meet-irreducible element different from b, in $[a, b]$.*

THEOREM 8. *A lattice L is a T_2-space with respect to its CP-ideal topology if and only if for $a < b$ there exists a finite number of join-irreducible elements $\overline{c_i}$ different from a and meet-irreducible elements c_k different from b such that*

$$[a, b] = (\bigvee_j [\overline{c_i}, b]) \vee (\bigvee_k [a, c_k]).$$

PROOF. We first show that the condition above is necessary. To prove this, suppose that L is a T_2-space. By § 2 Lemma 3, L is covered by a finite number of elements of $C\mathfrak{P}$ which contain at most one of a and b. Then by Lemma 1 we have the necessity of the condition above.

We shall next prove the sufficiency of the condition above. Let us suppose that for $a < b$,

$$[a, b] = (\bigvee_i [\overline{c_i}, b]) \vee (\bigvee_k [a, c_k]).$$

Now we put $I_k = \{x \mid x \cap b \leqq c_k\}$ and $\overline{I_i} = \{x \mid x \cup a \geqq \overline{c_i}\}$. Then $L = (\bigvee_k I_k) \vee (\bigvee_i \overline{I_i})$. In fact, if $x \in L$ then $a \leqq (x \cup a) \cap b \leqq b$. Hence there exists a number k such that $(x \cup a) \cap b \in [a, c_k]$ or $(x \cup a) \cap b \in [\overline{c_k}, b]$. If $(x \cup a) \cap b \in [a, c_k]$ then $(x \cap b) \cup a \cup c_k = c_k$: $x \cap b \leqq c_k$ hence $x \in I_k$. If $(x \cup a) \cap b \in [\overline{c_k}, b]$ then $x \in \overline{I_k}$. Both I_k and $\overline{I_i}$ contain at most one of a and b. On the other hand it is easily shown that I_k is a CP-ideal and $\overline{I_i}$ is a dual CP-ideal. Hence by § 2 Lemma 3 L is a T_2-space.

EXAMPLE 5. In L_4 of § 1 Ex. 1 we put $a = (0, 0)$ and $b = (1, 1)$. The set of all meet-irreducible elements is $\{(x, y) \mid y = 1\}$ and the set of all join-irreducible elements is $\{(x, y) \mid x = 0\}$. Therefore the lattice L_4 is not a T_2-space. In fact, no vicinity of $(1, 0)$ is covered by a finite number of $C\mathfrak{P}$.

COROLLARY. *If a lattice L of finite length is distributive then it is a T_3-space with respect to its CP-ideal topology.*

PROOF. Suppose that L is distributive, then L is an infinitely distributive lattice. It is known that every distributive lattice of finite length has at most a finite number of irreducible elements. (See G. Birkhoff [1] p. 139 Lemma 2) For $a < b$, each element, which is covered by b, is a meet-irreducible element, and each element, which covers a, is a join-irreducible element in $[a, b]$. Hence by Theorem 8 L is a T_2-space. By Theorem 4 L is a T_3-space.

COROLLARY. *If a lattice L of finite length is distributive, then L is a T_3-space with respect to its MP-ideal topology.*

PROOF. In a lattice L of finite length, CP-ideal and MP-ideal are equivalent. In fact, every ideal I is expressed by $I = [0, a]$ for some a of L, and prime ideal $[0, a]$ is a CP-and MP-ideal.

7. Bicompactness. O. Frink proved in his paper [3] that complete lattices are bicompact with respect to its "Interval topology". I defined "*Ideal topologies of lattices*" in a previous paper and proved that each bounded closed set of a conditionally complete lattice is bicompact with respect to its "Ideal topology". But this proposition is not true for the other topologies which are defined by many writers. This proposition is important on its application.

In this section we shall prove that the proposition above is true for CP-ideal topologies. The proof of this proposition follows by essentially the same argument that was given for [8], [18] if we notice that $I \wedge [a, b]$ is represented as $[d, c]$.

THEOREM 9. *Let L be a conditionally complete lattice. If a closed subset M of L with a CP-ideal topology is bounded, then M is bicompact. In particular, each complete lattice is bicompact in its CP-ideal topology.*

PROOF. To prove this theorem it is sufficient that if $\mathfrak{F}'$ is any collection of closed sets having the finite intersection property and $\mathfrak{F}'$ contains M, then there exists a common point to all members of $\mathfrak{F}'$.

Since M is a bounded set, there exist two elements a and b such that $a \leq x \leq b$ for all $x \in M$. If $F_\gamma \in \mathfrak{F}'$ then it can be expressed by $F_\gamma = \bigwedge_\alpha \bigvee_{\beta=1}^{n_\alpha} I^\gamma_{\alpha\beta}$, where $I^\gamma_{\alpha\beta} \in C\mathfrak{P}$. We can extend $\mathfrak{F}'$ to be maximal by Zorn's lemma and call the extended callection $\mathfrak{F}$. From the property of $\mathfrak{F}$ we have $\bigvee_{\beta=1}^{n_\alpha} I^\gamma_{\alpha\beta} \in \mathfrak{F}$ for all α. If $A \vee B \in \mathfrak{F}$, then by the property of $\mathfrak{F}$ we have $A \in \mathfrak{F}$ or B

$\in \mathfrak{F}$. Therefore, for each α there exists β such that $I^{\gamma}_{\alpha\beta} \in \mathfrak{F}$, which is denoted by I^{γ}_{α}. Then we have

$$F_{\gamma} \supseteqq \bigwedge_{\alpha} I^{\gamma}_{\alpha},\ [a, b] \supseteqq M \text{ and } \bigwedge_{\gamma} F_{\gamma} \supseteqq \bigwedge_{\gamma} \bigwedge_{\alpha} I^{\gamma}_{\alpha}.$$

By $M \in \mathfrak{F}$, then $[a, b] \in \mathfrak{F}$, and then $I^{\gamma}_{\alpha} \wedge [a, b] \in \mathfrak{F}$. By § 5 Lemma 1 and its dual

$$I^{\gamma}_{\alpha} \wedge [a, b] = [c_{\alpha\gamma},\ d_{\alpha\gamma}].$$

By the finite intersection properties of $\mathfrak{F}$, $[c_{\alpha\gamma},\ d_{\alpha\gamma}] \wedge [c_{\beta\delta},\ d_{\beta\delta}] \neq \phi$. Then we have

$$c_{\alpha\gamma} \leqq d_{\beta\delta} \text{ for all } \alpha, \beta, \gamma \text{ and } \delta.$$

Since L is conditionally complete there exists $\bigcup_{\alpha,\gamma} c_{\alpha\gamma}$. Therefore we have

$$\bigwedge_{\gamma} F_{\gamma} \ni \bigcup_{\alpha,\gamma} c_{\alpha\gamma}.$$

This proves the theorem.

Let $\mathfrak{C}$ be the set of all real valued continuous functions defined on $[0, 1]$ of R, in which $x \leqq y$ if and only if $x(t) \leqq y(t)$ for all t of $[0, 1]$. It is well known that $\mathfrak{C}$ is a vector lattice. Let $\mathfrak{F}$ be the set of linear continuous functionals on $\mathfrak{C}$, in which $f \leqq g$ if and only if $f(x) \leqq g(x)$ for all x of $\mathfrak{C}$. Then it is well known that $\mathfrak{F}$ is a vector lattice and a closed subset of $R^{\mathfrak{C}}$ with its weak topology: the topology taking sets of type $\{f \mid |f(x_0) - g(x_0)| < \varepsilon\}$ as a sub-basis for open set (See [15]). Then the following theorem is established.

COROLLARY. *A subset* $\{f \mid |f| \leqq 1, f \in \mathfrak{F}\}$ *of* $\mathfrak{F}$ *is bicompact. More generally, for any* $f_1 \in R^{\mathfrak{C}}$ *and* $f_2 \in R^{\mathfrak{C}}$ *a subset* $\{f \mid f_1 \leqq f \leqq f_2, f \in \mathfrak{F}\}$ *of* $\mathfrak{F}$ *is a bicompact set.*

PROOF. By Chapter 1 § 5 Corollary 3 of Theorem 5, the weak topology of $R^{\mathfrak{C}}$ and the CP-ideal topology of $R^{\mathfrak{C}}$ are homeomorphic. Now we have

$$\{f \mid f_1 \leqq f \leqq f_2\} = \bigwedge_{x} \{f \mid f_1(x) \leqq f(x) \leqq f_2(x)\}.$$

Hence $\{f \mid f_1 \leqq f \leqq f_2\}$ is a closed set. Therefore $\{f \mid f_1 \leqq f \leqq f_2\} \wedge \mathfrak{F}$ is a bounded closed subset of $R^{\mathfrak{C}}$. Thus $\{f \mid f_1 \leqq f \leqq f_2\} \wedge \mathfrak{F}$ is bicompact. In particular $\{f \mid |f| \leqq 1\} \wedge \mathfrak{F} = \{f \mid -1 \leqq f \leqq 1\} \wedge \mathfrak{F}$ is bicompact (See [1] pp. 61-63).

Chapter III Commutative L-Groups

We shall be concerned below with lattice ordered group (l-group), in the following sense.

DEFINITION. An *l-group* G is (i) a lattice (ii) a group, in which (iii) the inclusion relation is invariant under all group-translations $x \to a + x + b$.

A vector lattice is a vector space V with real scalars which an l-group under addition, and which for any positive scalar λ, $x \to \lambda x$ is an automorphism.

To simplify the statement of theorems, we shall always assume, unless specifically stated, operation is commutative. We shall use the additive notation for group operation, and the notations and terminologies of G. Birkhoff's "Lattice theory" [pp. 214-258].

8. **Formulae.** We shall extract following formulae from G. Birkhoff's "Lattice theory" and omit the proofs (See [5] pp. 219 and 231).

In any l-group (not necessary commutative) we have the following basic algebraic rules.

(0) $a - (x \cap y) + b = (a - x + b) \cup (a - y + b)$;

$$a - (a \cap b) + b = b \cup a.$$

(0′) $a - (x \cup y) + b = (a - x + b) \cap (a - y + b)$;

$$a - (a \cup b) + b = b \cap a.$$

(1) Any l-group is distributive (See [5] p. 219 Th. 5).

(2) If one of $\sup\limits_{\alpha} x_\alpha$ and $\inf\limits_{\alpha} (- x_\alpha)$ exists, then the other one exists and

$$- \sup_{\alpha} x_\alpha = \inf_{\alpha} (- x_\alpha).$$

(3) If one of $\bigcup\limits_{\alpha} x_\alpha$ and $\bigcup\limits_{\alpha} (a + x_\alpha)$ exists then the other one exists and

$a + \sup\limits_{\alpha} x_\alpha = \sup\limits_{\alpha} (a + x_\alpha)$. Analogously, $a + \inf\limits_{\alpha} x_\alpha = \inf\limits_{\alpha} (a + x_\alpha)$.

(4) If we put $a^+ = a \cup 0$, $a^- = a \cap 0$ then $a = a^+ + a^-$.

In any commutative l-group we have the following formulae.

(5) If $\sup\limits_{\alpha} x_\alpha$ exists then $\sup\limits_{\alpha} (x_\alpha \cap a)$ exists and

$(\sup\limits_{\alpha} x_\alpha) \cap a = \sup\limits_{\alpha} (x_\alpha \cap a)$. (See [22] Th.7 or [5] p. 231).

(6) $x + y = x \cup y + x \cap y$ (See [5] p. 219).

LEMMA 1. *In a commutive l-group G, we have the following formulae,*

$$x \cup y + x \cup y = 2x \cup 2y, \quad 2(x \cap y) = 2x \cap 2y.$$

More generally, we have

$$\sup_{\alpha} x_\alpha = x \textit{ implies } \sup_{\alpha} 2x_\alpha = 2x,$$

$$\inf_\alpha x_\alpha = y \textit{ implies } \inf_\alpha 2x_\alpha = 2y.$$

PROOF. We need only prove that $\sup_\alpha x_\alpha = x$ implies $\sup_\alpha 2x_\alpha = 2x$. $2x_\alpha \leqq 2x$ is clear. For any c such that $2x_\alpha \leqq c$ for all α, we shall show $2x \leqq c$. By (3) we have

$$2x = \sup_\alpha x_\alpha + \sup_\beta x_\beta = \sup_\alpha (x_\alpha + \sup_\beta x_\beta) = \sup_\alpha \sup_\alpha (x_\beta + x_\beta).$$

By (6), (3) we have

$$x + y = (x \cup y) + (x \cap y) = \{x + (x \cap y)\} \cup \{y + (x \cap y)\} \leqq 2x \cup 2y.$$

Hence we get $x_\alpha + x_\beta \leqq 2x_\alpha \cup 2x_\beta \leqq c$. Therefore we obtain $2x \leqq c$. Thus, $\sup_\alpha 2x_\alpha = 2x$.

9. Continuity of Group Operations. In this section we shall prove that operations are continuous in the CP-ideal topology. To prove this, in an l-group it is sufficient to show that for any neighborhood $U(a \pm b)$ of $a \pm b$ there exist a neighborhood $U(a)$ of a and a neighborhood $U(\mathrm{b})$ of b such that $U(a \pm b) \supseteqq U(a) \pm U(b)$.

In a vector lattice, we shall show that for $U(\lambda a)$ there are $U(\lambda)$, $U(a)$ such that $U(\lambda a) \supseteqq U(\lambda)\cdot U(a)$. Any neighborhood U is represented by

$$U = (\bigwedge_\alpha \bigvee_{\beta=1}^{n\alpha} I_{\alpha\beta})^c = \bigvee_\alpha \bigwedge_{\beta=1}^{n\alpha} I^c_{\alpha\beta}, \text{ where } I_{\alpha\beta} \in \mathfrak{P}.$$

By 0) and 3) we can easily show the following lemma.

LEMMA 2. *a) In any l-group, if I is an element of $C\mathfrak{P}$ or $M\mathfrak{P}$ then each of $I + a$ and $-I$ is an element of $C\mathfrak{P}$ or $M\mathfrak{P}$, respectively.*

b) In a vector lattice, if I is an element of $C\mathfrak{P}$, $M\mathfrak{P}$ or $\mathfrak{P}$ then for a real number λ, λI is an element of $C\mathfrak{P}$, $M\mathfrak{P}$ or $\mathfrak{P}$, respectively.

DEFINITION. A P-ideal topology is said to be a *P_G-ideal topology* if and only if

(G) i) $I \in \mathfrak{P}$ implies $I + a \in \mathfrak{P}$ and $-I \in \mathfrak{P}$.

ii) If $\{I_\alpha\}$ is a chain in $\mathfrak{P}$ and each of $\{I_\alpha\}$ is an ideal, then

$$\bigwedge_\alpha I_\alpha \in \mathfrak{P}.$$

Similarly, we define the concepts of P_G-ideal, P_G-family.

By this definition and Lemma 2, each of CP- and MP- ideal topology is a P_G-ideal topology.

LEMMA 3. *In an l-group G with its P_G-ideal topology,*

a) any neighborhood of an element a of G can be written in the form $U + a$ where U is a neighborhood of zero element 0 of G, *b*) $-U$ *is a neighborhood of* 0.

THEOREM 10. *In any commutative l-group, the group operations are continuous with respect to its CP-ideal topology.*

PROOF. By Lemma 3, it is sufficient to show that for $U+x-y$ there exist U_1+x, U_2+y such that

$$U+(x-y)\supseteqq(U_1+x)-(U_2+y):U\supseteqq U_1-U_2,$$

where U, U_1 and U_2 are neighborhoods of 0. Since $-U_2$ is a neighborhood of 0, we shall show that for any neighborhood U of 0 there exists a neighborhood U_1 of 0 such that $U\supseteqq 2U_1$.

Case 1) U^c is a dual CP-ideal I. Let I_1 be the set of all x such that $2x\in U^c$: $I_1=\{x\,|\,2x\in U^c\}$. If $x\in I_1$ and $x\leqq y$, then $2x\in U^c$ and $2x\leqq 2y$. Hence we have $2y\in U^c$: $y\in I_1$. If $x_\alpha\in I_1$ and $\inf_\alpha x_\alpha=x$, then $2x_\alpha\in U^c$. By Lemma 1 $\inf_\alpha 2x_\alpha=2x$. Since U^c is a dual CP-ideal, we have $2x\in U^c$: $x\in I_1$. If $x\cup y\in I_1$, then $2(x\cup y)\in U^c$. By Lemma 1, $2x\cup 2y\in U^c$. Hence, we have $2x\in U^c$ or $2y\in U^c$: $x\in I_1$ or $y\in I_1$. Thus we conclude that I_1 is a dual CP-ideal.
Now put $I_1^c=U_1$, then U_1 is a neighborhood of 0. If $x\in U_1$, $y\in U_1$, then, from the fact that $2x\cap 2y\leqq x+y\leqq 2x\cup 2y$ (See Lemma 1) $2x\in U$ and $2y\in U$, we have $x+y\in U$. Therefore we obtain $2U_1\subseteqq U$.

Case 2) U^c is a CP-ideal. This case is dual of case 1).

Case 3) U is any neighborhood of 0. U^c can be written in the form $U^c=\bigwedge_\alpha\bigvee_{\beta=1}^{n_\alpha}I_{\alpha\beta}$; $U=\bigvee_\alpha\bigwedge_{\beta=1}^{n_\alpha}I^c_{\alpha\beta}$.

Since U is a neighborhood of 0, then there exists α_0 such that $\bigwedge_{\beta=1}^{n_\alpha}I^c_{\alpha_0\beta}\ni 0$. By case 1) and 2) for each $I^c_{\alpha_0\beta}$ there exists a neighborhood $U_{1\beta}$ of 0 such that $2U_{1\beta}\subseteqq I^c_{\alpha_0\beta}$. We put $U_1=\bigwedge_{\beta=1}^{n_\alpha}U_{1\beta}$ then U_1 is a neighborhood of 0 and $2U_1\subseteqq U$.

This proves the theorem.

THEOREM 11. *In any vector lattice with its CP-ideal topology,*

a) *for fixed* λ_0, $\lambda_0 x$ *is continuous,*

b) *if* $a>0$, $\varepsilon_n\downarrow 0$, *imply* $a\varepsilon_n\downarrow 0$, *then* λx *is continuous, where* ε_n, λ *are real numbers.*

PROOF. Suppose that U is a neighborhood of 0 and λ_0 a real number. We can easily prove that $\lambda_0 U$ is a neighborhood of 0. Let us denote $U_1=$

$\frac{1}{\lambda_0}U$, then $\lambda_0(U_1 + x) \subseteqq U + \lambda_0 x$, completing the proof of a).

Next we shall prove b). Let $U + \lambda_0 a$ be a neighborhood of $\lambda_0 a$. Suppose that U^c is a dual CP-ideal I. By case 1) of Theorem 10 there exists a dual CP-ideal I_1 such that $3 \cdot I_1^c \subseteqq I^c$. From the hypothesis of b), there exists a positive real number ε such that

$$\varepsilon a^+ \in I_1^c \text{ and } \varepsilon(-a)^+ \in I_1^c.$$

Now put $U_1 = \frac{1}{\varepsilon} I_1^c \wedge \frac{-1}{\varepsilon} I_1^c \wedge \frac{1}{\lambda_0} I_1^c \wedge \frac{-1}{\lambda_0} I_1^c$ and $V = (-\varepsilon, \varepsilon)$.

Then U_1 is a neighborhood of 0, and if $x \in U_1$ then $-x \in U_1$. Moreover, we have $x^+ \in U_1$ and $x^- \in U_1$.

If $0 \leqq \lambda < \varepsilon$, then we have

$$\lambda x \leqq \lambda x^+ < \varepsilon x^+ \in \varepsilon U_1 : \varepsilon x^+ \in I_1^c : \lambda x \in I_1^c.$$

If $-\varepsilon < \lambda < 0$, then we have

$$\lambda x = (-\lambda)(-x) \leqq \varepsilon(-x)^+ \in I_1^c : \lambda x \in I_1^c.$$

Thus we have that if $x \in U_1$ and $\lambda \in V$ then $\lambda x \in I_1^c$. Similarly, by $(a^+)\varepsilon \in I_1^c$ and $(-a)^+\varepsilon \in I_1^c$ we have $a\lambda \in I_1^c$ for all $\lambda \in V$. By $x \in U_1$, we have $\lambda_0 x \in I_1^c$. Therefore if $x \in U_1$ and $\lambda \in V$, then we have $(x + a)(\lambda + \lambda_0) = x\lambda + x\lambda_0 + a\lambda + a\lambda_0 \in I_1^c + I_1^c + I_1^c + a\lambda_0 \leqq I^c + a\lambda_0$.

Thus we have $(U_1 + a)(V + \lambda_0) \subseteqq U + a\lambda_0$.

Dually, the case such that U^c is a CP-ideal, can be proved. In the same way as the proof of Theorem 10, we can prove the case such that U is any neighborhood of 0.

10. Structure of *L*-Groups; Representation. In this section we shall be concerned with the representation of a commutative l-group and the study of its properties.

LEMMA 1. *The family of all minimal P_G-ideals containing a fixed element a is represented by $\{I_\alpha + a \mid \alpha \in D\}$, where $\{I_\alpha \mid \alpha \in D\}$ is the family of all minimal P_G-ideals containing* 0.

PROOF. $I_\alpha + a$ is a P_G-ideal containing a. Let $I(a)$ be a minimal P_G-ideal containing a and contained in $I_\alpha + a$. Then we have $0 \in I(a) - a \leqq I_\alpha$. Since I_α is a minimal P_G-ideal containing 0, then we have $I(a) - a = I_\alpha : I(a) = I_\alpha + a$. Conversely, let I be any minimal P_G-ideal containing a. There exists α such that $I_\alpha \subseteqq I - a$, then we have $I_\alpha + a \subseteqq I$. By the hypothesis of I_α we have $I_\alpha + a = I$.

LEMMA 2. *In any l-group with its P_G-ideal topology we have*

a) *$a \leqq b$ implies $I_\alpha + a \subseteqq I_\alpha + b$, where I_α is a P_G-ideal,*

b) *a family $\{I_\alpha + a | a \in G\}$ is a chain in set inclusion.*

PROOF. We shall only prove b). Let a and b be two elements of G. We put $c = a \cap b$. By a) we get $I_\alpha + c \subseteqq I_\alpha + a$ and $I_\alpha + c \subseteqq I_\alpha + b$. Since $I_\alpha + c$ is a prime-ideal and includes c, hence we have $a \in I_\alpha + c$ or $b \in I_\alpha + c$. If $a \in I_\alpha + c$ then we have $I_\alpha + c = I_\alpha + a$, because $I_\alpha + a$ is a minimal P_G-ideal containing a. In a similar way, if $b \in I_\alpha + c$ then we have $I_\alpha + c = I_\alpha + b$. Thus we obtain $I_\alpha + a \subseteqq I_\alpha + b$ or $I_\alpha + b \subseteqq I_\alpha + a$. We can easily show the following lemma.

LEMMA 3. *If we define $\oplus$ and $\geqq$, in the following sense, $(I_\alpha + a) \oplus (I_\alpha + b) = I_\alpha (a + b)$, $I_\alpha + a \geqq I_\alpha + b$ if and only if $I_\alpha + a \supseteqq I_\alpha + b$, then a family $\{I_\alpha + a | a \in G\}$ is a chain l-group with respect to $\oplus$ and $\geqq$.*

In this chain l-group, it is clear that

$$(I_\alpha + a) \cup (I_\alpha + b) = I_\alpha + a \cup b,\ (I_\alpha + a) \cap (I_\alpha + b) = I_\alpha + a \cap b.$$

By Lemma 3 the cardinal product (direct product) $\prod_{\alpha \in D} \{I_\alpha + a | a \in G\}$ is a commutative l-group. Now if we put

$$f(a) = (I_\alpha + a | \alpha \in D) \in \prod_{\alpha \in D} \{I_\alpha + a | a \in G\},$$

then we have

$$f(a \cup b) = f(a) \cup f(b),\ f(a \cap b) = f(a) \cap f(b)$$

and $$f(a + b) = f(a) \oplus f(b).$$

From this fact we can conclude the following theorem.

THEOREM 12. *Every commutative l-group G is homomorphic with a sub-group of cardinal product (direct product) $\prod_{\alpha \in D} \{I_\alpha + a | a \in G\}$ of all chain l-groups $\{I_\alpha + a | a \in G\}$, where I_α is a minimal P_G-ideal containing* 0.

We can easily prove the following lemma with respect to the kernel of f.

LEMMA 4. a) *$\{x | I_\alpha + x = I_\alpha\}$ is a sub-group of G, which is denoted by G^*_α.*

b) *$x \in G^*_\alpha$ and $y \in G^*_\alpha$ imply $[x \cap y, x \cup y] \subseteqq G^*_\alpha$.*

c) *$G^*_\alpha = I_\alpha \wedge - I_\alpha$.*

d) *$\bigwedge_\alpha G^*_\alpha$ is a sub-group of G, which is written G^*, and $G^* = f^{-1}(0)$.*

THEOREM 12′. *Every commutative l-group is isomorphic with a sub-group of the cardinal product (direct products) $\prod_{\alpha \in D} \{I_\alpha + a | a \in G\}$ of chain* *l-groups*. (See [12], [13], [14]).

PROOF. Since an l-group G is a distributive lattice, by Theorem 2′ G is a T_1-space with respect to its MP-ideal topology. Therefore by Lemma 4c) kernel G^* has only one element 0.

THEOREM 13. *a) The kernel G^* is a minimal closed set in the P_G-ideal topology.*

b) M is a minimal closed set if and only if there exists x such that $M = G^* + x$.

c) $\{G^* + a \mid a \in G\}$ is a partition of G.

PROOF. Since a) and c) is clear, we shall only show b). It is clear that the minimal closed set containing a is set $\bigwedge_{\alpha \in D} (\pm I_\alpha + a)$. Hence we have

$$\bigwedge_{\alpha \in D} (\pm I_\alpha + a) = (\bigwedge_{\alpha \in D} \pm I_\alpha) + a = G^* + x.$$

Chapter IV Conditionally Complete L-Groups

In this chapter we shall confine ourselves to the case such that l-groups are conditionally complete and topologies are CP-ideal topologies. A conditionally complete l-group is infinitely distributive and commutative (See [11], [14], [22]). To study this l-groups we shall introduce the concept of coordinate axes. In Chapter III we have defined the notion of $\{I_\alpha \mid \alpha \in D\}$ of CP-(MP-) ideal topologies. In this chapter we shall be concerned with the connection between the coordinate system and the family $\{I_\alpha \mid \alpha \in D\}$. Using those properties we shall give a representation of a T_1-space with respect to its CP-ideal topology.

11. Introduction of Coordinate Systems.

DEFINITION. An interval $[a, b]$ of a lattice is called a *chain-interval* if and only if $[a, b]$ is a chain. More generally, a chain M of a lattice is called a chain-interval if and only if for any pair of $x \in M, y \in M$ and $x \leqq y$, $[x.y]$ is a chain and $[x, y] \leqq M$.

In any l-group G, we shall denote by $\mathfrak{D}$ the family of all chain-intervals C_α containing 0 which are contained in $G^+ = \{x \mid x \geqq 0\}$, and $C_1 \leqq C_2$ means that C_2 includes C_1, as a set. Then $\mathfrak{D}$ is a non-empty family (For $\{0\} \in \mathfrak{D}$). Suppose now that a sub-family $\{C_\alpha\}$ of $\mathfrak{D}$ is a chain. Then, the set union $\bigvee_\alpha C_\alpha$ is an element of $\mathfrak{D}$. Hence, by Zorn's lemma there exists a maximal chain-interval in G^+. Similarly, there exists a maximal chain-interval containing given chain-interval.

DEFINITION. We shall denote by A^+ any maximal chain-interval containing 0 which contains at least two elements. Let us denote by A^- the set of a such that $-a \in A^+$. The set union of A^+ and A^- is called a coordinate axis, and denoted by A.

The family of all coordinate axes A_α is called a coordinate system, and written $\mathfrak{A} = \{A_\alpha | \alpha \in D'\}$ ($\mathfrak{A}$ may be ϕ).

It is clear that A_α^- is a chain-interval. Now we shall prove the following lemma.

LEMMA 1. *Every coordinate axis A is a chain-interval.*

PROOF. To prove this, it is sufficient to show that for any $a > 0$, $[-a, a] \subseteqq A$. Let x be an element of $[-a, a]$. Then we have $x = x^+ + x^-$, $a \geqq x^+ \geqq 0$ and $a \geqq -(x^-) \geqq 0$.

Since $[0, a]$ is a chain-interval, we have $x^+ \leqq -x^-$ or $x^+ \geqq -x^-$. Hence we get $-a \leqq x \leqq 0$ or $a \geqq x \geqq 0$. Thus we have $x \in A$, which proves a).

LEMMA 2. *If both A_1 and A_2 are coordinate axes such that $A_1 \neq A_2$, then the set intersection $A_1 \wedge A_2$ contains only* 0.

PROOF. If $A_1 \neq A_2$ then since both A_1^+ and A_2^+ are maximal elements of $\mathfrak{D}$, there are two elements a_1 and a_2 such that $a_1 \in A_1^+$, $a_1 \notin A_2^+$, $a_2 \notin A_1^+$ and $a_2 \in A_2^+$. We put $b = a_1 \cap a_2$, then $b \in A_1^+ \wedge A_2^+$, and $0 \leqq a_1 - b \leqq a_1$, $0 \leqq b \leqq a_1$. Since A_1 is a chain-interval we have $a_1 - b \leqq b$ or $a_1 - b \geqq b$: $a_1 \geqq 2b$ or $a_1 \geqq 2b$. Similarly, we have $a_2 \leqq 2b$ or $a_2 \geqq 2b$. From the four possible cases we have $b = 0$.

Thus we have $A_1 \wedge A_2 = 0$.

LEMMA 3. *A coordinate axis A is a subgroup of G.*

PROOF. By Lemma 2 we can prove that if a is an element of A then $2a$ is contained in A. Now, since $a \in A$ implies $-a \in A$, we shall show that the sum $b + c$ of elements b and c, both of which are contained in A, is also an element of A. To show this we put $\max[b, c, -b, -c] = a$. Then $a \in A$, hence $b + c \in [-2a, 2a] \subseteqq A$. Thus A is a subgroup of G.

12. Properties of $\{I_\alpha | \alpha \in D\}$. In Lemmas 2 and 3 Chapter III we have discussed the properties of $\{I_\alpha | \alpha \in D\}$. We can easily prove the following lemmas.

LEMMA 4. *If $[b, c]$ is a chain and $b < c$, then $I = \{x | x \cap c \leqq b\}$ is a minimal CP-ideal containing b and a minimal MP-ideal containing b.*

LEMMA 5. *a) Let I be a CP-ideal. If $a \in I$, $b \notin I$, $a < b$ and* sup $\{[a, b] \wedge I\} = c$, *then $[c, b]$ is a chain-interval.*

b) If c is the cross element of a CP-ideal I and a chain-interval $[c,b]$, then we have $I = \{x \mid x \cap b \leqq c\}$.

LEMMA 6. *If $[a, c]$ is a chain-interval and $a < b < c$, then there exists y such that $I_2 = I_1 + y$, where $I_1 = \{x \mid x \cap c \leqq a\}$ and $I_2 = \{x \mid x \cap c \leqq b\}$.*

LEMMA 7. *If $I_\alpha + a \supseteqq I_\beta$ then $\alpha = \beta$, where both I_α and I_β are elements of $\{I_\alpha \mid \alpha \in D\}$* (See § 10 Lemma 2).

13. Axes and $\{I_\alpha \mid \alpha \in D\}$. In this section we shall be concerned with connection between the coordinate system $\mathfrak{A}$ and the family $\{I_\alpha \mid \alpha \in D\}$ of all minimal CP-ideals containing 0.

THEOREM 14. *Let G be a conditionally complete l-group. Then we have*

a) if A is a coordinate axis, then for any a there exists only one α such that A and $I_\alpha + a$ intersect,

b) if $I_\alpha + a \neq G$ then there exists only one β such that $I_\alpha + a$ and A_β intersect,where $A_\beta \in \mathfrak{A}$.

PROOF. We shall first prove a). Let b be an element of A such that $b > 0$.

We put $I_\alpha = \{x \mid x \cap b \leqq 0\}$. Then

$I_\alpha + a = \{x + a \mid (b + a) \cap (x + a) \leqq a\} = \{x \mid (b + a) \cap x \leqq a\}$.

Case 1) $a \geqq 0$. Let c be the least upper bound of $[0, a] \wedge A$, which exists. By Lemma 3, we have $b + c \in A$. If $b + c \in I_\alpha + a$ then $(b + c) \cup a \in I_\alpha + a$. By $a \leqq (b + c) \cup a \leqq b + a$, we have $(b + c) \cup a = a$: $b + c \leqq a$ which contradicts the definition of c. Hence $b + c \notin I_\alpha + a$.

Case 2) $a \leqq 0$. Let d be the greatest lower bound of $[a, 0] \wedge A$. By Lemma 3, we have $-b + d \in A$. By $-b + d \leqq a \cup (-b + d) \leqq d$, we get $a \cup (-b + d) = d$, hence $(a - d) \cup (-b) = 0 : (d - a) \cap b = 0 : d \cap (b + a) = a$. Thus $d \in I_\alpha + a$.

Case 3) a is any element of G. By § 10 Lemma 2, we have

$$I_\alpha + a^- \subseteqq I_\alpha + a \subseteqq I_\alpha + a^+.$$

By Case 1) and 2) $I_\alpha + a$ and A intersect.

We shall next prove b). Since $I_\alpha + a \neq G$, $I_\alpha + a^+ \neq G$. Hence there exists b such that $b > 0$ and $I_\alpha + a^+ \not\ni b$. We put $\sup \{[a^-, b] \wedge (I_\alpha + a^+)\} = c$ and $\sup \{[a^-,b] \wedge (I_\alpha + a^-)\} = d$. By Lemma 5, $[d, b]$ is a chain-interval and

$$I_\alpha + a^+ = \{x \mid x \cap b \leqq c\},\ I_\alpha + a^- = \{x \mid x \cap b \leqq d\}.$$

Let B be a maximal chain-interval containing $[d, b]$. We put $e = \sup\{(B - c)$

$\wedge [0, c]\}$. Also it is clear that $B - c$ is an axis. Let m be any element of $B - c$ such that $m > 0$. Then $m + e \notin I_\alpha + a$ and $m + e \in B - c$.

By $I_\alpha + a^- \subsetneqq I_\alpha + a \subsetneqq I_\alpha + a^+$, we have $(I_\alpha + a) \wedge (B - c) \ni (d - c)$ and $(I_\alpha + a)^c \wedge (B - c) \ni (m + e)$. Therefore $I_\alpha + a$ and $B - c$ intersect.

Suppose that both coordinate axes A_1 and A_2 intersect with $I_\alpha + a$ whose cross elements are denoted by m_1 and m_2, respectively. Let $(I_\alpha + a)^c \wedge A_1 \ni m_1'$, $(I_\alpha + a)^c \wedge A_2 \ni m_2'$. If $0 \in I_\alpha + a$ then by Lemma 2 $m_1' \cap m_2' = 0$ which is a contradiction to $I_\alpha + a \not\ni m_1'$, $m_2' \notin I_\alpha + a$. If $0 \notin I_\alpha + a$ then $m_1 \cup m_2 = 0$ which is a contradiction to $I_\alpha + a \ni m_1$ and $I_\alpha + a \ni m_2$. This proves b).

COROLLARY. *a) If A_α and $I_\alpha + a$ intersect, then for any b, A_α and $I_\alpha + b$ intersect.*

b) If A_α and $I_\alpha + a$ intersect, then for any b, $A_\alpha + b$ and $I_\alpha + a$ intersect.

PROOF. Suppose that A_α and $I_\alpha + a$ intersect and the cross element is c. By Theorem 14 there exists β such that A_α and $I_\beta + b$ intersect. Let c' be the cross element of A_α and $I_\beta + b$. By Lemma 6 if $c \gtreqqless c'$ then $I_\alpha + a \gtreqqless I_\beta + b$. By Lemma 7 we have $\alpha = \beta$, hence A_α and $I_\alpha + b$ intersect.

Next we shall show that b) is true. By a) A_ι and $I_\alpha + a - b$ intersect, therefore it is clear that $A_\alpha + b$ and $I_\alpha + a$ intersect.

NOTE. From theorems above, a CP-ideal I_α corresponds to only one coordinate axis A, which is written A_α. Then the family $\{I_\alpha | I_\alpha \neq L\}$ of all CP-ideals containing 0 and the family $\{A_\alpha\}$ of all coordinate axes have a one-to-one correspondence. Therefore we may denote $\{I_\alpha | \alpha \in D\}$ and $\{A_\alpha | \alpha \in D\}$. From this convension, I_α and A_α having the same index necessarily intersect.

14. Conditions of Topologies.

THEOREM 15. *In a conditionally complete l-group G with its CP-ideal topology, T_0-, T_1-, T_2- and T_3-space are equivalent.*

PROOF. By Theorem 4 it is sufficient to show that any T_0-space is a T_2-space. Let $a < b$. We may suppose that there exists a join-irreducible element c in $[a, b]$ (See Th. 6). We can easily show that $[a, c]$ is a chain.

Case 1) there exists e such that $a < e < c$. Let us denote sets $\{x | x \cap c \leqq e\}$ and $\{x | x \cup a \geqq e\}$ by I_1 and I_2, respectively.

Case 2) there exists no e such that $a < e < c$. Let us denote sets $\{x | x \cap c \leqq a\}$ and $\{x | x \cup a \geqq c\}$ by I_1 and I_2, respectively. Then, I_1 is a CP-ideal containing no b and I_2 a dual CP-ideal containing no a. And we have $I_1 \vee I_2 = L$.

Dually, we can prove the case such that there exists a meet-irreducible element c in $[a, b]$. By §2 Lemma 3, G is a T_2-space, completing the proof.

From the proof of the theorem above we obtain the following corollary.

COROLLARY. *In any conditionally complete l-group G with its CP-ideal topology, G is a* T_0-$(T_3$-$)$ *space if and only if for* $a < b$ *there exist elements c and d such that* $[c, d]$ *is a chain-interval and* $a \leqq c < d \leqq b$.

15. Representation. Let $\prod_{\alpha \in D} A_\alpha$ be the cardinal (direct) product of all coordinate axes. By Theorem 14, a coordinate axis A_α and the CP-ideal $I_\alpha + a$ intersect. Let a_α be the cross element of A_α and $I_\alpha + a$. Define a mapping f as follows; for any element of G such that $a \geqq 0$, $f(a) = (a_\alpha | \alpha \in D)$.

THEOREM 16. *If a conditionally complete l-group G is a* T_1*-space with respect to its CP-ideal topology then G is isomorphic with a perfect sublattice of the cardinal (direct) product* $\prod_{\in D} A_\alpha$ *of conditionally complete chain l-groups.*

PROOF. By § 10 Lemma 2 we have either $I_\alpha + a \supseteqq I_\alpha + b$ or $I_\alpha + a \subseteqq I_\alpha + b$. We may suppose that $I_\alpha + a \supseteqq I_\alpha + b$. Then we have $I_\alpha + a \ni a \cup b$. Hence by § 10 Lemma 2, $I_\alpha + a = I_\alpha + (a \cup b)$. Thus we obtain $a_\alpha = (a \cup b)_\alpha$; $(a \cup b)_\alpha = a_\alpha \cup b_\alpha$. Then we have

$$f(a \cup b) = ((a \cup b)_\alpha | \alpha \in D) = (a_\alpha \cup b_\alpha | \alpha \in D)$$
$$= (a_\alpha | \alpha \in D) \cap (b_\alpha | \alpha \in D) = f(a) \cup f(b).$$

In a similar way we have $f(a \cap b) = f(a) \cap f(b)$ and $f(a + b) = f(a) + f(b)$.

We shall now show that the mapping f is one-to-one. From the hypothesis that G is a T_1-space, for any pair of elements a and b of G there is a minimal CP-ideal I such that $I \ni a$ and $I \not\ni b$. There exists sup $\{[a \cap b, b] \wedge I\}$, which is denoted by c. Then $[c,b]$ is a chain-interval. By Lemma 6 and § 10 Lemma 1 there is α such that $I = I_\alpha + c$, where I_α is a minimal CP-ideal containing 0. Therefore $a_\alpha \wedge b_\alpha$:

$$a \neq b \leftrightarrow f(a) \neq f(b); \ \sup_\alpha a_\alpha = a.$$

We shall show that the set $\{(a_\alpha | \alpha \in D) | a \in G\}$ is a perfect sublattice of $\prod_{\alpha \in D} A_\alpha$. To prove this it is sufficient to show that if $a_\alpha \leqq g(\alpha) \leqq b_\alpha$ for all $\alpha \in D$ then there exists an element c of G such that $c_\alpha = g(\alpha)$ for all $\alpha \in D$. By the hypothesis there exists $\sup_\alpha g(\alpha)$, which is written c. Then by Lemma 8 we have $a \leqq c \leqq b$. Also $g(\alpha) \leqq c_\alpha$ for all $\alpha \in D$. By Lemma 2 $a_\alpha \cap a_\beta =$

0 for all $\alpha \neq \beta$, hence we have

$$g(\alpha_0) \cap (\bigcup_{\alpha_0 \neq \alpha} g(\alpha)) = \bigcup_{\alpha_0 \neq \alpha} \{g(\alpha_0) \cap g(\alpha)\} = 0 = c_{\alpha_0} \cap (\bigcup_{\alpha_0 \neq \alpha} \cap g(\alpha)),$$

$$g(\alpha_0) \cup (\bigcup_{\alpha_0 \neq \alpha} g(\mathrm{B})) = c = c_\alpha \cup (\bigcup_{\alpha_0 = \alpha} g(\alpha)) = \sup_\alpha c_\alpha.$$

Since G is distributive, we have $g(\alpha_0) = c_{\alpha_0}$. Thus G^+ is lattice-isomorphic with a perfect sublattice of $\prod_{\alpha \in D} A_\alpha^+$. By [5] p. 214 Th. 1, our theorem is true.

THEOREM 17. *A conditionally complete l-group is a T_1-space if and only if G is isomorphic with a perfect sublattice of* $\prod_{\alpha \in D} A_\alpha$.

PROOF. By the above note, the condition is necessary. Conversely, suppose that the condition is fulfilled. By Theorem 5 Cor. 4, $\prod_{\alpha \in D} A_\alpha$ is a T_2-space. By Theorem 15 Cor., for any $(a_\alpha | \alpha \in D) < (b_\alpha | \alpha \in D)$ there is a chain-interval. By the Theorem 15 Cor., $f(G)$ is a T_1-space with respect to its CP-ideal topology.

NOTE. This theorem can be proved from Th. 12′.

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