SPECIAL VARIATIONS

ALINE H. FRINK

(Received August 16, 1959)

A simple calculus of variations problem involving second derivatives, i. e., the problem of minimizing

$$J=\int_a^b f(x,y,y',y'')\,dx$$

in a class of admissible functions, $y = y(x)$, does not seem to have been treated very fully in the literature [1]. Possibly this is because the problem can be reduced to one involving only first order derivatives by considering the problem of minimizing

$$\int_a^b f(x,y,z,z')\,dx$$

in a class of admissible functions, $y = y(x)$, $z = z(x)$, satisfying the differential equation, $y' - z = 0$. However, the problem treated this way, as a so-called Lagrange problem, is not necessarily equivalent to the original problem [2]. Possibly, too, it is assumed that the usual method of deriving necessary conditions for problems involving only first order derivatives can be used just as easily to obtain similar conditions for higher order problems. The usual method would be to derive the Euler equation

$$f_y - df_{y'}/dx + d^2f_{y''}/dx^2 = 0$$

and then derive the Weierstrass and Legendre conditions by methods depending on the validity of the Euler equation. There is some disadvantage in this as Graves [3] has pointed out He has derived the Weierstrass condition for problems involving first order derivatives without making use of the Euler equation. It would be convenient to have necessary conditions for problems involving second order derivatives derived directly and independently.

By using a special variation we obtain a necessary condition which yields the Weierstrass condition [4]. Another variation gives a generalization of the Legendre condition by assuming only the existence of the generalized partial derivative $f_{y''y''}$. Still another variation gives the usual Euler equation in integrated form [5]. These variations were suggested by polygonal variations used for problems involving first order derivatives [6]. The proofs are quite elementary. No use is made of the notion of fields or Hilbert invariant integral. Only simple limit arguments are used.

It will be assumed that $f(x, y, y', y'')$ is continuous in a region W of sets of values (x, y, y', y''). Sets of values in W are called admissible sets. Arcs, $y = y(x)$, of class $D''(y''$ piece-wise continuous) such that $y(a) = c, y(b) = d$, $y'(a) = p$, $y'(b) = q$ are called admissible arcs. A variation $\eta(x)$ containing a parameter ε is called weak if $\eta(x)$ and its first and second derivatives with respect to x approach zero with ε. It is called strong if η'' does not approach zero with ε.

1. A generalization of the Legendre condition.

THEOREM. *If $f(x, y, y', y'')$ is continuous and if $y = y(x)$ makes J a weak relative minimum in the class of admissible arcs and if*

$$L = \lim_{m \to 0} [f(x, y(x), y'(x), y''(x) + m) - 2f(x, y(x), y'(x), y''(x)) + f(x, y(x), y'(x), y''(x) - m)]/m^2$$

exists for $a \leqq x \leqq b$, then at a point of continuity of y'', $L \geqq 0$.

The proof is given first for $x = a$ which we assume a point of continuity of y''. With m to approach zero, $\eta'(x)$ is defined as follows: $\eta'(x) = m(x - a)$, $a \leqq x \leqq a + \varepsilon$; $\eta'(x) = m(a + 2\varepsilon - x)$, $a + \varepsilon \leqq x \leqq a + 3\varepsilon$; $\eta'(x) = m(x - a - 4\varepsilon)$, $a + 3\varepsilon \leqq x \leqq a + 4\varepsilon$; $\eta'(x) = 0$, $a + 4\varepsilon \leqq x \leqq b$. Then $\eta(x)$ is the unique function which vanishes at $x = a$ and whose derivative is $\eta'(x)$. Note that $\eta(a) = \eta(b) = \eta'(a) = \eta'(b) = 0$. From the continuity of f, y, y', and y'', for every value of m there exists an $\varepsilon(m) > 0$ with $\varepsilon(m) < |m|$ such that if $|x - a| < 4\varepsilon$, $|\delta| < \varepsilon$, and $|\rho| < \varepsilon$ then

$$\begin{aligned} &|f(x, y(x) + \delta, y'(x) + \rho, y''(x) + m) - f(a, y(a), y'(a), y''(a) + m)| < m^4, \\ (*) \quad &|f(x, y(x) + \delta, y'(x) + \rho, y''(x) - m) - f(a, y(a), y'(a), y''(a) - m)| < m^4, \\ &|f(a, y(a), y'(a), y''(a)) - f(x, y(x), y'(x), y''(x))| < m^4. \end{aligned}$$

This defines ε in terms of m. If $y = y(x)$ makes J a weak relative minimum, and the following limit exists, it must be non negative, i.e.,

$$I = \lim_{m \to 0} \left\{ \int_a^b [f(x, y(x) + \eta(x), y'(x) + \eta'(x), y''(x) + \eta''(x)) - f(x, y(x), y'(x), y''(x))]\, dx \right\} / \varepsilon m^2 \geqq 0.$$

From the mean value theorem for integrals

$$I = \lim_{m\to 0}\frac{1}{m^2}\{f(\alpha, y(\alpha)+\eta(\alpha), \eta'(\alpha)+y'(\alpha), y''(\alpha)+m)-f(\alpha, y(\alpha), y'(\alpha), y''(\alpha))$$
$$+f(\beta, y(\beta)+\eta(\beta), y'(\beta)+\eta'(\beta), y''(\beta)-m)-f(\beta\ y(\beta), y'(\beta), y''(\beta))$$
$$+f(\gamma, y(\gamma)+\eta(\gamma), y'(\gamma)+\eta'(\gamma), y''(\gamma)-m)-f(\gamma, y(\gamma), y'(\gamma), y''(\gamma))$$
$$+f(\theta, y(\theta)+\eta(\theta), y'(\theta)+\eta'(\theta), y''(\theta)+m)-f(\theta, y(\theta), y'(\theta), y''(\theta))\}$$

where $a \leqq \alpha \leqq a + \varepsilon$, $a + \varepsilon \leqq \beta \leqq a + 2\varepsilon$, $a + 2\varepsilon \leqq \gamma \leqq a + 3\varepsilon$, $a + 3\varepsilon \leqq \theta \leqq a + 4\varepsilon$. Now add and subtract

$$2f(a, y(a), y'(a), y''(a) + m) - 4f(a, y(a), y'(a), y''(a)) + 2f(a, y(a), y'(a), y''(a) - m)$$

to the quantity inside the braces above, and regroup to obtain

$$I = \lim_{m\to 0}\{2f(a, y(a), y'(a), y''(a) + m) - 4f(a, y(a), y'(a), y''(a)) + 2f(a, y(a), y'(a), y''(a) - m) + B\}/m^2$$

where B is the sum of eight terms each of which satisfies one of the conditions (*) and hence the limit as m approaches zero of B/m^2 equals zero. We have then that $I = 2L$. Since $I \geqq 0$, $L \geqq 0$. The conclusion of the theorem has been shown to hold at the left hand end point $x = a$. Since $y = y(x)$ minimizes J on any subinterval the same conclusion holds for a replaced by any value of x between a and b, at which y'' is continuous and the theorem has been proved. If $f_{y''y''}$ exists, it is equal to L.

2. A generalization of the Weierstrass condition.

THEOREM. *If $f(x, y, y', y'')$ is continuous and if $y = y(x)$ makes J a minimum in the class of admissible arcs, then at a point of continuity of $y''(x)$, for real numbers $m, k, mk > 0$,*

$$H \equiv f(x, y, y', y'' + m) + f(x, y, y', y'' - k)m/k - (1 + m/k)f(x, y, y', y'') \geqq 0.$$

This is proved first for the left hand end point $x = a$ as in § 1. We define $\eta'(x)$ as follows: $\eta'(x) = m(x - a)$, $a \leqq x \leqq a + \varepsilon$; $\eta'(x) = m\varepsilon - k(x - a - \varepsilon)$, $a + \varepsilon \leqq x \leqq a + \varepsilon + 2m\varepsilon/k$; $\eta'(x) = m(x - a - 2\varepsilon - 2m\varepsilon/k)$, $a + \varepsilon + 2m\varepsilon/k \leqq x \leqq a + 2\varepsilon + 2m\varepsilon/k$; $\eta'(x) = 0$, $a + 2\varepsilon + 2m\varepsilon/k \leqq x \leqq b$. Then $\eta(x)$ is the unique function which vanishes at $x = a$, and whose derivative is $\eta'(x)$. Consider the curves $C_\epsilon : y = y(x) + \eta(x; m, k, \varepsilon)$, $C_0 : y = y(x)$. For ε sufficiently small, C_ϵ is admissible whenever C_0 is admissible. It is easily seen that

$$\lim_{\epsilon\to 0}[J(C_\epsilon) - J(C_0)]/2\varepsilon = H$$

for $x = a$. Since $y = y(x)$ minimizes J on any subinterval the result is valid with a replaced by any value of x between a and b. The relation $H \geqq 0$ gives us the Weierstrass condition. For if $y = y(x)$ makes J a strong relative minimum, then if $f_{y''}$ exists, at a point of continuity of y'',

$$\lim_{k\to 0} H = f(x, y, y', y'' + m) - f(x, y, y', y'') - mf_{y''}(x, y, y', y'') \geqq 0.$$

If we put $m = k$, $H \geqq 0$ also gives us the preceding generalized Legendre condition. However, this establishes it as a necessary condition for a strong relative minimum, whereas the first discussion shows that it is necessary for a weak relative minimum as well.

3. The Euler equation. By means of another special variation the usual integrated form of the Euler equation can be obtained. For a fixed $\delta > 0$ and r, where $a + \delta < r - \delta < r < r + \delta < b - \delta < b$, $\eta'(x)$ is defined as follows:

$$\eta'(x) = (x - a)/\delta \text{ for } a \leqq x \leqq a + \delta$$
$$\eta'(x) = 1 \text{ for } a + \delta \leqq x \leqq r - \delta$$
$$\eta'(x) = (r - x)/\delta \text{ for } r - \delta \leqq x \leqq r$$
$$\eta'(x) = (r - a - \delta)(r - x)/(b - r - \delta)\delta \text{ for } r \leqq x \leqq r + \delta$$
$$\eta'(x) = (\delta + a - r)/(b - r - \delta) \text{ for } r + \delta \leqq x \leqq b - \delta$$
$$\eta'(x) = (r - a - \delta)(x - b)/(b - r - \delta)\delta \text{ for } b - \delta \leqq x \leqq b.$$

The variation $\eta(x)$ is the unique function which vanishes at $x = a$ and whose derivative is $\eta'(x)$. Note that $\eta(x)$ also vanishes at $x = b$, and that $\eta'(a) = \eta'(b) = 0$. Replace y, y', y'' in J by $y(x) + \varepsilon\eta(x)$, $y'(x) + \varepsilon\eta'(x)$, and $y''(x) + \varepsilon\eta''(x)$ and call the result $I(\varepsilon)$. It follows in the usual way that $I'(0) = 0$ for all $\delta > 0$.

$$I'(0) = \int_a^b (\eta f_y + \eta' f_{y'} + \eta'' f_{y''})\, dx = \int_a^b (G\eta' + f_{y''}\eta'')\, dx = 0$$

where we have set

$$G = f_{y'} - \int_a^x f_y \, dx$$

to simplify the notation. Since this holds for all $\delta > 0$

$$\lim_{\delta\to 0} I'(0) = 0.$$

If f_y, $f_{y'}$, and $f_{y''}$ are continuous the limits of the integrals of $G\eta'$ taken over the intervals $[a, a + \delta]$, $[r - \delta, r]$, $[r, r + \delta]$, $[b - \delta, b]$ are all zero. The limits of the integrals of $f_{y''}\eta''$ taken over the intervals $[a + \delta, r - \delta]$, $[r + \delta, b - \delta]$ are also zero. The limits of the remaining integrals in $I'(0)$ then give us the following:

$$\lim_{\delta \to 0} I'(0) = \int_a^r G\,dx + \left(\int_r^b G\,dx\right)(a-r)/(b-r) + f_{y''}(a) - f_{y''}(r)$$

$$+ f_{y''}(r)(a-r)/(b-r) + f_{y''}(b)(r-a)/(b-r) = 0.$$

Now r was any point such that $a < r < b$. Replace r by x in the preceding equation and multiply by $b - x$. By making use of the fact that

$$\int_b^x G\,dx = \int_b^a G\,dx + \int_a^x G\,dx$$

the Euler equation is obtained:

$$\int_a^x \int_a^x f_y\,dx\,dx - \int_a^x f_{y'}\,dx + f_{y''} = c_1 x + c_2$$

where

$$(b-a)c_1 = \int_a^b \int_a^x f_y\,dx - \int_a^b f_{y'}\,dx + f_{y''}(b) - f_{y''}(a)$$

$$(b-a)c_2 = -a\left[\int_a^b \int_a^x f_y\,dx - \int_a^b f_{y'}\,dx\right] - af_{y''}(b) + bf_{y''}(a).$$

References

[1] LECAT, Calcul des Variations, Encyclopédie des Sciences Math., tome II, vol. 6(1913), p. 43.
[2] BOLZA, Vorlesungen über Variationsrechnung (1909), p. 543.
[3] GRAVES, A proof of Weierstrass condition in the calculus of variations, American Math. Monthly (1934), vol. XLI, no. 8, p. 502.
[4] HADAMARD, Leçons sur le Calcul des Variations (1910), p. 461.
[5] AHIEZER, Lekcii po variacionnomu iscisleniyu (1955), p. 116.
[6] A. H. FRINK AND ORRIN FRINK, Polygonal variations, Bull. American Math. Soc., (1938), p. 539.