ON DIFFERENTIABLE MANIFOLDS WITH (ϕ, ψ)-STRUCTURES

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1. Introduction. In a previous paper,[1] we have defined for some odd dimensional manifolds two kinds of structures which we have called (ϕ, ξ, η)-structure and (ϕ, ξ, η, g)-structure. The latter is a (ϕ, ξ, η)-structure with a positive definite Riemannian metric g which stands in a notable relation with the (ϕ, ξ, η)-structure. These structures are remarkable in the sense that any differentiable manifold with (ϕ, ξ, η)-structure is an almost contact manifold and any almost contact manifold admits (ϕ, ξ, η, g)-structure.

In this paper, we shall study two kinds of structures for differentiable manifolds of any dimension, the first one ((ϕ, ψ)-structure) may be regarded as generalizations of almost complex structure, almost product structure and (ϕ, ξ, η)-structure, and the second one ((ϕ, ψ, g)-structure) may be regarded as generalizasions of almost Hermitian structure, almost product metric structure and (ϕ, ξ, η, g)-structure. We shall confine ourselves only to algebraic considerations, analytic considerations will be published in later papers.

2. (ϕ, ψ)-structures.

1°. Let M^n be a differentiable manifold of dimension n. Suppose first that there exist over M^n two tensor fields ϕ^i_j and ψ^i_j[2] of type (1, 1) which satisfy the following conditions:

$$\text{rank } |\phi^i_j| = l, \tag{2.1}$$

$$\text{rank } |\psi^i_j| = m, \tag{2.2}$$

$$\phi^i_j \psi^j_k = 0, \tag{2.3}$$

$$\psi^i_j \phi^j_k = 0, \tag{2.4}$$

1) S. Sasaki, On differentiable manifolds with certain structures which are closely related to almost contact structure I, Tôhoku Math. Journ. 12(1960) pp. 459-476.

2) We assume, unless otherwise stated, that the indices run the following range of integers:

$i, j, k, \alpha, \beta, \gamma = 1, 2, \ldots\ldots, n(=l+m)$,
$a, b, c = 1, 2, \ldots\ldots, l$,
$p, q, r = l+1, \ldots\ldots, n$
$A = 1, 2, \ldots\ldots, l'$, $\quad A^* = l'+A$,
$E, F = 1, \ldots\ldots, l_1$, $\quad H, K = l_1+1, \ldots\ldots, l(=l_1+l_2)$
$L = l+1, \ldots\ldots, l+m'$, $\quad L^* = l+m'+L$,
$M, N = l+1, \ldots\ldots, l+m_1$,
$S, T = l+m_1+1, \ldots\ldots, n$.

(2.5) $$\varepsilon\phi^i_j\phi^j_k + \varepsilon'\psi^i_j\psi^j_k = \delta^i_k,$$

where l, m are non negative integers such that

(2.6) $$l + m = n,$$

and $\varepsilon, \varepsilon'$ are $+1$ or -1. In such case we say that the manifold M^n in consideration has a (ϕ, ψ)-*structure of type* (sgn ε, sgn ε') or M^n is a differentiable manifold with a (ϕ, ψ)-structure of type (sgn ε, sgn ε'). From our definition we see that if a differentiable manifold M^n admits a (ϕ, ψ)-structure of type (sgn ε, sgn ε'), then it admits also a $(-\phi, \psi)$-structure, a $(\phi, -\psi)$-structure and a $(-\phi, -\psi)$-structure, all of type (sgn ε, sgn ε'). Hence, we identify all of these structures.

2°. First we shall prove the following

THEOREM 1. *Suppose M^n be a differentiable manifold with a (ϕ, ψ)-structure of type* (sgn ε, sgn ε'). *Then in every sufficiently small coordinate neighborhood U of M^n, we can find frames (ξ^i_a, ξ^i_p) $(a = 1,\ldots\ldots, l; p = l+1,\ldots\ldots, n)$ such that*

(2.7) $$\begin{cases} \phi^i_j = \lambda^a_b\xi^i_a\eta^b_j, \\ \psi^i_j = \mu^p_q\xi^i_p\eta^q_j, \end{cases}$$

where (η^a_j, η^p_j) are the inverse matrix of (ξ^i_a, ξ^i_p) and λ^a_b, μ^p_q are scalars such that

(2.8) $$\begin{cases} \lambda^a_b\lambda^b_c = \varepsilon\,\delta^a_c, \\ \mu^p_q\mu^q_r = \varepsilon'\delta^p_r. \end{cases}$$

PROOF. As the rank of $|\phi^i_j|$ is equal to l, there exist m linearly independent vector fields over U which are solutions of the equation

(2.9) $$\phi^i_j\xi^j = 0.$$

Let us denote any such vector fields by ξ^i_p and take n vector fields ξ^i_a, ξ^i_p over U so that they are linearly independent. If we put the inverse matrix of (ξ^i_a, ξ^i_p) by (η^a_j, η^p_j), then ϕ^i_j can be written as

$$\phi^i_j = \lambda^\alpha_\beta\xi^i_\alpha\eta^\beta_j.$$

By virtue of the construction, we have

(2.10) $$\phi^i_j\xi^j_p = 0.$$

Therefore, we can easily see that $\lambda^\alpha_p = 0$ and hence we get

(2.11) $$\phi^i_j = \lambda^a_b\xi^i_a\eta^b_j + \lambda^p_b\xi^i_p\eta^b_j.$$

Next, ψ^i_j can be written also as

$$\psi^i_j = \mu^\alpha_\beta \xi^i_\alpha \eta^\beta_j.$$

If we put this into (2. 3), we get

$$\phi^i_j \psi^j_k = \mu^\alpha_\beta (\phi^i_j \xi^j_\alpha) \eta^\beta_k = 0.$$

However, $\phi^i_j \xi^j_a$'s do not vanish for any value of a, so we get $\mu^a_\beta \eta^\beta_k = 0$. Hence, we get $\mu^a_\beta = 0$. Therefore, ψ^i_j has the following form:

$$\psi^i_j = \mu^p_q \xi^i_p \eta^q_j + \mu^p_b \xi^i_p \eta^b_j. \tag{2. 12}$$

Thirdly, putting (2. 11) and (2. 12) into (2. 4), we get

$$\psi^i_j \phi^j_k = (\mu^p_a \lambda^a_b + \mu^p_q \lambda^q_b) \xi^i_p \eta^b_k = 0.$$

Hence, we see that the relation

$$\mu^p_a \lambda^a_b + \mu^p_q \lambda^q_b = 0 \tag{2. 13}$$

holds good.

Finally, putting (2. 11) and (2. 12) into the left hand side of (2. 5), we get

$$\varepsilon \phi^i_j \phi^j_k + \varepsilon' \psi^i_j \psi^j_k = \varepsilon \lambda^\alpha_b \lambda^b_c \xi^i_\alpha \eta^c_k + \varepsilon' \mu^p_q \mu^q_r \xi^i_p \eta^r_k.$$

Comparing this with

$$\delta^i_k = \xi^i_\alpha \eta^\alpha_k,$$

we see, by virtue of (2. 5), that the relations

$$\begin{cases} \varepsilon \lambda^a_b \lambda^b_c = \delta^a_c, \\ \varepsilon' \mu^p_q \mu^q_r = \delta^p_r, \\ \varepsilon \lambda^p_b \lambda^b_c + \varepsilon' \mu^p_q \mu^q_c = 0 \end{cases} \tag{2. 14}$$

hold good.

Now, we take another frame $\overline{\xi^i_\alpha}$ which are given by

$$\begin{cases} \overline{\xi^i_b} = \lambda^a_b \xi^i_a + \lambda^p_b \xi^i_p, \\ \overline{\xi^i_q} = \qquad \varepsilon' \mu^p_q \xi^i_p. \end{cases} \tag{2. 15}$$

Then, the inverse matrix $\overline{\eta^\alpha_j}$ of $\overline{\xi^i_\alpha}$ is easily seen to be

$$\begin{cases} \overline{\eta^a_j} = \varepsilon \lambda^a_b \eta^b_j, \\ \overline{\eta^p_j} = \mu^p_b \eta^b_j + \mu^p_q \eta^q_j. \end{cases} \tag{2. 16}$$

The equation (2. 15) can be solved with respect to ξ^i_α giving

$$\begin{cases} \xi^i_b = \varepsilon \lambda^a_b \overline{\xi^i_a} - \varepsilon \lambda^a_b \lambda^q_a \mu^p_q \overline{\xi^i_p}, \\ \xi^i_q = \qquad \mu^p_q \overline{\xi^i_p}, \end{cases} \tag{2. 17}$$

and the equation (2. 16) can be solved with respect to η^α_j giving

(2.18)
$$\begin{cases} \eta_j^a = \lambda_b^a \overline{\eta_j^b}, \\ \eta_j^p = -\varepsilon' \mu_q^p \mu_b^q \lambda_c^b \overline{\eta_j^c} + \varepsilon' \mu_q^p \overline{\eta_j^q}. \end{cases}$$

If we put (2.17) and (2.18) into (2.11) and (2.12), we can easily see that the relations

(2.19)
$$\begin{cases} \phi_j^i = \lambda_b^a \overline{\xi_a^i} \overline{\eta_j^b}, \\ \psi_j^i = \mu_q^p \overline{\xi_p^i} \overline{\eta_j^q} \end{cases}$$

hold good, where λ_b^a, μ_q^p are scalars over U such that

$$\begin{cases} \lambda_b^a \lambda_c^b = \varepsilon \delta_c^a, \\ \mu_q^p \mu_r^q = \varepsilon' \delta_r^p. \end{cases}$$

Consequently, if we change our notations and write ξ_α^i, η_j^α instead of $\overline{\xi_\alpha^i}$, $\overline{\eta_j^\alpha}$, we see that our theorem is true. Q. E. D.

We call the frame such that the tensors ϕ_j^i, ψ_j^i take the form (2.7) satisfying (2.8) an *adapted frame of the first order.*

REMARK 1. The above demonstration shows that the conditions (2.2) and (2.6) follow from the conditions (2.1), (2.3), (2.4) and (2.5).

REMARK 2. From (2.8) we see that if $\varepsilon = -1$, then l is even and if $\varepsilon' = -1$, m is even.

3°. Suppose that M^n be a differentiable manifold with a (ϕ, ψ)-structure. Then, at every point P of M^n, the set of vectors such that

(2.20)
$$\phi_j^i \xi^j = 0$$

is an m-dimensional vector subspace V_m spanned by ξ_p^i at P. In the same way we can see from (2.2), that the set of vectors at P such that

(2.21)
$$\psi_j^i \xi_a^j = 0$$

is an l-dimensional vector subspace V_l spanned by ξ_a^i. Hence, V_l and V_m are disjoint and complementary. In other words, if we denote the tangent space at P by T_P, then

(2.22)
$$T_P = V_l \oplus V_m.$$

The correspondence $P \in M^n$ to V_l at P and the correspondence $P \in M^n$ to V_m at P define the so-called l- and m-dimensional distributions over M^n. We call them D_l and D_m. Then we get the following

THEOREM 2. *Suppose M^n be a differentiable manifold with a (ϕ, ψ)-structure. Then, the two distributions D_l and D_m are disjoint and complementary.*

Adapted frames of the first order in a coordinate neighborhood U are nothing but frames whose first l vectors span the vector space V_l of D_l and whose last m vectors span the vector space V_m of D_m at every point of U.

4°. Now, we consider a transformation of adapted frames of the first order

$$(2.23)\qquad \begin{cases} \bar{\xi}^i_b = \alpha^a_b \xi^i_a, \\ \bar{\xi}^i_q = \beta^p_q \xi^i_p, \end{cases}$$

where α and β are non-singular matrices. Then, it induces a transformation of η of the form

$$(2.24)\qquad \begin{cases} \bar{\eta}^a_j = {}'\alpha^a_b \eta^b_j, \\ \bar{\eta}^p_j = {}'\beta^p_q \eta^q_j, \end{cases}$$

where $'\alpha$ and $'\beta$ are inverse matrices of α and β respectively. Putting (2.23) and (2.24) into

$$\begin{aligned} \phi^i_j &= \bar{\lambda}^a_b \bar{\xi}^i_a \bar{\eta}^b_j \\ &= \lambda^a_b \xi^i_a \eta^b_j, \end{aligned}$$

we see that

$$(2.25)\qquad \alpha^c_a \bar{\lambda}^a_b = \lambda^c_a \alpha^a_b,$$

which shows that λ^a_b's transform like components of a mixed tensor under transformations (2.23). Hence, if $\varepsilon = -1$, we can take α so that the matrix λ takes the form

$$(2.26)\qquad \lambda = \begin{pmatrix} 0 & -E_{l'} \\ E_{l'} & 0 \end{pmatrix},$$

where $E_{l'}$ is a unit matrix of dimension $l' = l/2$. In this case ϕ^i_j reduces to the form

$$(2.27)\qquad \phi^i_j = -\bar{\xi}^i_A \bar{\eta}^{A^*}_j + \bar{\xi}^i_{A^*} \bar{\eta}^A_j,$$

where A runs over $1, 2, \ldots\ldots, l'$ and $A^* = A + l'$. If we take β arbitrary, then with respect to the frame $(\bar{\xi}^i_A, \bar{\xi}^i_{A^*}, \bar{\xi}^i_p)$ thus determined, ϕ has the following components:

$$(2.28)\qquad \phi = \left(\begin{array}{cc|c} 0 & -E_{l'} & \\ E_{l'} & 0 & 0 \\ \hline 0 & & 0 \end{array}\right)$$

Especially, we can see that the characteristic roots of the matrix ϕ are equal to 0, i and $-i$ with multiplicities m, l' and l' respectively.

On the other hand, if $\varepsilon = +1$, then the characteristic roots of the matix λ are equal to -1 or $+1$. If multiplicities of the roots -1 and $+1$ are l_1 and $l_2\,(l_1 + l_2 = l)$ respectively, then we can take α so that the matrix λ takes the form

$$\lambda = \begin{pmatrix} -E_{l_1} & 0 \\ 0 & E_{l_2} \end{pmatrix}, \tag{2.29}$$

where E_{l_1} and E_{l_2} are unit matrices of dimensions l_1 and l_2. In this case ϕ takes the form

$$\phi^i_j = -\overline{\xi}^i_E\overline{\eta}^E_j + \overline{\xi}^i_H\overline{\eta}^H_j. \tag{2.30}$$

Even if we take β arbitrary, with respect to the frame $(\overline{\xi}^i_H, \overline{\xi}^i_K, \overline{\xi}^i_p)$ thus determined, ϕ has the following components:

$$\phi = \left(\begin{array}{cc|c} -E_{l_1} & 0 & \\ 0 & E_{l_2} & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right). \tag{2.31}$$

Similar facts hold good for ψ too. Hence, summarizing the above results, we get the following

THEOREM 3. *Suppose M^n be a differentiable manifold with a (ϕ, ψ)-structure of type* (sgn ε, sgn ε'). *Then we can take frames over every coordinate neighborhood U so that ϕ and ψ take the following forms*:

$$\begin{cases} \phi^i_j = -\xi^i_A\eta^{A^*}_j + \xi^i_{A^*}\eta^A_j, & \text{for } \varepsilon = -1 \\ \quad\;\; = -\xi^i_H\eta^H_j + \xi^i_K\eta^K_j, & \text{for } \varepsilon = +1 \\ \psi^i_j = -\xi^i_L\eta^{L^*}_j + \xi^i_{L^*}\eta^L_j, & \text{for } \varepsilon' = -1 \\ \quad\;\; = -\xi^i_M\eta^M_j + \xi^i_S\eta^S_j. & \text{for } \varepsilon' = +1 \end{cases} \tag{2.32}$$

REMARK. If M^n, is a differentiable manifold with a (ϕ, ψ)-structure of type $(-, +)$ such that rank of $|\psi^i_j|$ is equal to 1, then with respect to adapted frames of the first order in a coordinate neighborhood U of M^n, ψ^i_j may take the form $\xi^i\eta_j$ as (ϕ, ψ)-structure is identical with $(\phi, -\psi)$-structure. Hence, the conditions (2.1) to (2.6) reduce in this case to

$$\begin{cases} \text{rank } |\phi^i_j| = 2l', & n = 2l' + 1 \\ \phi^i_j\xi^j = 0, & \phi^i_j\eta_i = 0, \end{cases} \tag{2.33}$$

$$\phi^i_j\phi^j_k = -\delta^i_k + \xi^i\eta_j\xi^j\eta_k.$$

and trivial equations. These combined with $\xi^i\eta_i = 1$ are nothing but the defining equations of the (ϕ, ξ, η)-structure for $M^{2l'+1}$. However, contrary to the case of (ϕ, ξ, η)-structure, our vector fields ξ^i and η_j are defined locally. They do not in general constitute a vector field over $M^{2l'+1}$. Hence, the set of differentiable manifolds with (ϕ, ψ)-structures of type $(-, +)$ such that the rank of $|\psi^i_j|$ is equal to 1 is somewhat wider than the set of differentiable manifolds with (ϕ, ξ, η)-structures.

Formulas in (2. 32) are canonical forms of the tensors ϕ and ψ. We call any frame with respect to which ϕ and ψ take such canonical forms an *adapted frame of the second order* of the given (ϕ, ψ)-structure.

5°. Now, the tensor fields ϕ and ψ define linear maps of tangent vectors at every point of M^n by $v \to \phi v$ and $v \to \psi v$.

THEOREM 4. *Suppose M^n be a differentiable manifold with a (ϕ, ψ)-structure and V_l, V_m are associated vector spaces at any point P of M^n. Then*

(i) $\phi\phi\, v = \varepsilon\, v$, *for* $v \in V_l$,

(ii) $\psi\psi\, v = \varepsilon'\, v$, *for* $v \in V_m$,

(iii) $\phi\psi\, v = 0,\ \psi\phi\, v = 0$, *for* $v \in T_P$.

PROOF. If $v \in V_l$, then $\psi v = 0$ and the converse is also true. In this case we see easily that

$$\phi\phi\, v = \varepsilon(\delta - \varepsilon'\psi\psi)v = \varepsilon\, v.$$

Hence (i) is proved. In the same way we can prove (ii). (iii) follows immediately from (2. 3) and (2. 4).

THEOREM 5. *The linear maps $v \to (\phi + \psi)v$ and $v \to (\phi - \psi)v$ of tangent spaces are non-singular.*

PROOF. By virtue of (2. 3), (2. 4) and (2. 5), we can verify that

$$(\phi + \psi)(\varepsilon\phi + \varepsilon'\psi) = \varepsilon\phi\phi + \varepsilon'\psi\psi = \delta.$$

Hence, $\phi + \psi$ and $\varepsilon\phi + \varepsilon'\psi$ are non-singular and inverse to each other. Similarly, $\phi - \psi$ is non-singular.

3. Associated Riemannian metric g.

6°. In this section we study if we can associate a positive definite Riemannian metric g to any differentiable manifold M^n with a (ϕ, ψ)-structure or not. We begin with a lemma.

LEMMA 1. *Suppose M^n be a differentiable manifold such that there exist two distributions D_l and D_m of dimensions l and m which are disjoint and complementary. Then there exists a positive definite Riemannian metric h with respect to which the vector spaces V_l and V_m of the distributions at every point of M^n are orthogonal to each other.*

PROOF. First we introduce an arbitrary positive definite Riemannian metric f over M^n. Suppose $\{U_\alpha\}$ be a sufficiently fine open covering of M^n by coordinate neighborgoods.

Now we take l (resp. m) orthonormal vector fields ξ^i_a (resp. ξ^i_p) over U_α with respect to f so that they span the vector space V_l (resp. V_m) of the distribution D_l (resp. D_m) at every point of U_α. Of course, ξ^i_a and ξ^i_p are not orthogonal to each other in general. We define

$$h^{ij}(U_\alpha) = \sum_a \xi^i_a \xi^j_a + \sum_p \xi^i_p \xi^j_p. \tag{3.1}$$

On the other hand, let U_β be another coordinate neighborhood which belongs to $\{U_\alpha\}$ such that $U_\alpha \cap U_\beta$ is not empty and $\overline{\xi}^i_a$, $\overline{\xi}^i_p$ are vector fields over U_β defined in the same way as above. Then, it is evident that

$$\overline{\xi}^i_a = \sum_b u_{ab} \xi^i_b,$$

$$\overline{\xi}^i_p = \sum_q u_{pq} \xi^i_q$$

hold good over $U_\alpha \cap U_\beta$, where (u_{ab}) and (u_{pq}) are orthogonal matrices. We can easily verify that

$$h^{ij}(U_\alpha) = h^{ij}(U_\beta)$$

holds good over $U_\alpha \cap U_\beta$. This shows that the set of tensor fields $h^{ij}(U_\alpha)$, $U_\alpha \in \{U_\alpha\}$, constitutes a global tensor field over M^n. The inverse h_{ij} of the tensor field h^{ij}, then determine s a positive definite Riemannian metric over M^n. We can easily verify that, with respect to $h = (h_{ij})$, the two vector spaces V_l and V_m are orthogonal at every point of M^n. Q. E. D.

LEMMA 2. *With respect to the metric h over M^n defined in the proof of Lemma 1, the relation*

$$h_{ij} \phi^i_h \psi^j_k = 0 \tag{3.2}$$

holds good.

PROOF. Let U be an arbitrary coordinate neighborhood of M^n. We take frames over U so that orthonormal vectors ξ^i_a (resp. ξ^i_p) with respect to the metric h span the vector space V_l (resp. V_m).

On the other hand, as (ξ^i_a, ξ^i_p) are adapted frames of the first order, we have

$$\begin{cases} \phi^i_j = \lambda^a_b \xi^i_a \eta^b_j, \\ \psi^i_j = \mu^p_q \xi^i_p \eta^q_j. \end{cases}$$

Hence, making use of the fact

$$h_{ij}\xi^i_a \xi^j_p = 0,$$

we can easily verify that our Lemma is true.

7°. Now let us prove one of our main theorems.

THEOREM 6. *Suppose M^n be a differentiable manifold with a (ϕ, ψ)-structure of type* (sgn ε, sgn ε'). *Then, there exists a positive definite Riemannian metric g over M^n such that the relations*

$$(3.3) \qquad \begin{cases} g_{ij}\phi^i_h \psi^j_k = 0, \\ g_{ih}\phi^h_j = \varepsilon g_{jh}\phi^h_i, \\ g_{ih}\psi^h_j = \varepsilon' g_{jh}\psi^h_i, \\ g_{ij}\phi^i_h \phi^j_k + g_{ij}\psi^i_h \psi^j_k = g_{hk} \end{cases}$$

hold good.

PROOF. We put

$$(3.4) \qquad g_{ij} = \frac{1}{2}(h_{ij} + h_{\alpha\beta}\phi^\alpha_i \phi^\beta_j + h_{\alpha\beta}\psi^\alpha_i \psi^\beta_j).$$

Then, first by virtue of (2.3), (2.4) and (3.2), $(3.3)_1$ is easily seen to be true.

Secondly, we see by virtue of (2.4) and (2.5) that

$$\begin{aligned} g_{ih}\phi^h_j &= \frac{1}{2}h_{ih}\phi^h_j + \frac{1}{2}h_{\alpha\beta}\phi^\alpha_i(\varepsilon\delta^\beta_j - \varepsilon\varepsilon'\psi^\beta_h\psi^h_j) \\ &= \frac{1}{2}h_{ih}\phi^h_j + \frac{\varepsilon}{2}h_{jh}\phi^h_i \\ &= \varepsilon g_{jh}\phi^h_i. \end{aligned}$$

In the same way, $(3.3)_3$ can be proved.

Thirdly, we see by virtue of $(3.3)_{2,3}$ and (2.5) that

$$\begin{aligned} & g_{ij}\phi^i_h\phi^j_k + g_{ij}\psi^i_h\psi^j_k \\ &\qquad = \varepsilon g_{ih}\phi^i_j\phi^j_k + \varepsilon' g_{ih}\psi^i_j\psi^j_k \\ &\qquad = g_{hk}. \end{aligned}$$

Hence, $(3.3)_4$ is proved. Q. E. D.

We shall call the Riemannian metric g whose existence is insured by Theorem 6 the *associated Riemannian metric* of the (ϕ, ψ)-structure in consideration. And the differentiable manifold with the (ϕ, ψ)-structure and its associated metric g is called a *manifold with a (ϕ, ψ, g)-structure.* It is an analogue of the almost Hermitian manifold for almost complex structure.

THEOREM 7. *Suppose M^n be a differentiable manifold with a (ϕ, ψ, g)-structure. Then, tensor equations*

$$(3.5)\qquad \begin{cases} g_{ij}(\phi^i_h + \psi^i_h)(\phi^j_k + \psi^j_k) = g_{hk}, \\ g_{ij}(\phi^i_h - \psi^i_h)(\phi^j_k - \psi^j_k) = g_{hk} \end{cases}$$

hold good.

PROOF. By virtue of (2.3), (2.4) and $(3.3)_4$, we can easily see that

$$\begin{aligned} &g_{ij}(\phi^i_h + \psi^i_h)(\phi^j_k + \psi^j_k) \\ &\quad = g_{ij}\phi^i_h\phi^j_k + g_{ij}\psi^i_h\psi^j_k = g_{hk}. \end{aligned}$$

In the same way $(3.5)_2$ can be proved.

THEOREM 8. *Suppose M^n be a differentiable manifold with a (ϕ, ψ, g)-structure, then the two distributions D_l and D_m are orthogonal with respect to the metric g at every point of M^n.*

PROOF. Take a point P of M^n and ξ^i_a are vectors which span the vector space V_l of D_l at P and ξ^i_p are vectors which span the vector space V_m of D_m at P. Then, by virtue of $(3.3)_4$,

$$\begin{aligned} g_{ij}\xi^i_a\xi^j_p &= (g_{\alpha\beta}\phi^\alpha_i\psi^\beta_j + g_{\alpha\beta}\psi^\alpha_i\psi^\beta_j)\xi^i_a\xi^j_p \\ &= 0, \end{aligned}$$

which shows that V_l and V_m at P are orthogonal to each other. Q. E. D.

4. Associated tensor fields.

8°. Suppose M^n be a differentiable manifold with a (ϕ, ψ, g)-structure. If we put

$$(4.1)\qquad \phi_{ij} = g_{ih}\phi^h_j,$$

then, by virtue of $(3.3)_1$, we get

$$(4.2)\qquad \phi_{ij} = \varepsilon\phi_{ji}.$$

Such a tensor will be called as *ε-symmetric* with respect to its indices. Of course, ε-symmetry means symmetry if $\varepsilon = +1$ and skew-symmetry if $\varepsilon = -1$. In the same way, if we put

$$\psi_{ij} = g_{ih}\psi^h_j, \tag{4.3}$$

then, by virtue of $(3.3)_2$, ψ_{ij} is ε'-symmetric, i. e.

$$\psi_{ij} = \varepsilon'\psi_{ji}. \tag{4.4}$$

We can solve (4.1) and (4.3) with respect to ϕ^i_j and ψ^i_j getting

$$\begin{cases} \phi^i_j = g^{ih}\phi_{hj}, \\ \psi^i_j = g^{ih}\psi_{hj}. \end{cases} \tag{4.5}$$

Now, we put

$$\phi^{ij} = \phi^i_h g^{hj},\ \phi^i_j = \phi^{ih} g_{hj}, \tag{4.6}$$

then

$$\begin{aligned} \phi^{ij} &= g^{ik}\phi_{kh} g^{hj} \\ &= \varepsilon g^{ik}\phi_{hk} g^{jh} \\ &= \varepsilon\phi^j_k g^{ki} \end{aligned}$$

$$\therefore\ \phi^{ij} = \varepsilon\phi^{ji}, \tag{4.7}$$

So, ϕ^{ij} is an ε-symmetric contravariant tensor fields. In the same way, if we put

$$\psi^{ij} = \psi^i_h g^{hj},\ \psi^i_j = \psi^{ih} g_{hj}, \tag{4.8}$$

then we get

$$\psi^{ij} = \varepsilon'\psi^{ji}. \tag{4.9}$$

We call four tensor fields ϕ_{ij}, ψ_{ij}, ϕ^{ij}, ψ^{ij} the *associated tensor fields* of the (ϕ, ψ, g)-structure in consideration.

THEOREM 9. *Let ϕ_{ij}, ψ_{ij}, ϕ^{ij}, ψ^{ij} be associated tensor fields of a differentiable manifold with a (ϕ, ψ, g)-structure. Then, the relations*

$$\begin{cases} \phi^{ih}\psi_{hj} = 0, \\ \psi^{ih}\phi_{hj} = 0 \end{cases} \tag{4.10}$$

hold good.

PROOF. By virtue of $(4.6)_2$, (4.3) and (2.3), we see that

$$\begin{aligned} \phi^{ih}\psi_{hj} &= \phi^i_\alpha g^{\alpha h} g_{h\beta}\psi^\beta_j \\ &= \phi^i_\alpha \psi^\alpha_j = 0. \end{aligned}$$

Hence $(4.10)_1$ is proved. In the same way, we can prove $(4.10)_2$.

9°. In the next place, we shall study the converse problem. We assume that M^n be a differentiable manifold with a ε-symmetric tensor field ϕ_{ij} and ε'-symmetric tensor field ψ^{ij} such that

$$\text{(4.11)} \qquad \begin{cases} \text{rank } |\phi_{ij}| = l, \\ \text{rank } |\psi^{ij}| = m, \ (l+m=n) \\ \psi^{ih}\phi_{hj} = 0 \end{cases}$$

hold good. And we shall study if we can find a positive definite Riemannian metric g such that the tensor field

$$\phi^i_j = g^{ih}\phi_{hj}, \ \psi^i_j = \psi^{ih}g_{hj}$$

and g define a (ϕ, ψ, g)-structure over M^n of type (sgn ε, sgn ε') or not.

First, we introduce an arbitrary positive definite Riemannian metric h over M^n. Then, $\phi_{ih}h^{hk}\phi_{kj}$ is a symmetric tensor field over M^n. Hence all roots of the characteristic equation

$$\text{(4.12)} \qquad |\phi_{ih}h^{hk}\phi_{kj} + \rho h_{ij}| = 0$$

are real. As $|\phi_{ij}|$ is of rank l, 0 is a root with multiplicity m. It is easily seen that all other roots have the opposite sign to ε.

Now, we denote all distinct non-zero roots by $\rho_1, \rho_2, \ldots\ldots, \rho_r$, their multiplicities by $\mu_1, \mu_2, \ldots\ldots, \mu_r$ and the characteristic spaces corresponding to the roots 0 and $\rho_1, \ldots\ldots, \rho_r$ by $V_0, V_1, \ldots\ldots, V_r$ respectively. Then, we see that

$$\dim V_0 = m, \quad \dim V_\lambda = \mu_\lambda, \qquad (\lambda = 1, \ldots\ldots, r)$$

$$\mu_1 + \mu_2 + \cdots\cdots + \mu_r = l.$$

As is well known, all different characteristic spaces are orthogonal to each other with respect to the metric h.

In the same way, we take up the symmetric tensor field $\psi^{ih}h_{hk}\psi^{kj}$ and consider the characteristic equation

$$\text{(4.13)} \qquad |\psi^{ih}h_{hk}\psi^{kj} + \sigma h^{ij}| = 0.$$

Then, it has 0 as a root of multiplicity l. Other roots are all real and they have the opposite sign to ε'. We shall denote all distinct non-zero roots by $\sigma_1, \sigma_2, \ldots \ldots, \sigma_s$, their multiplicities by $\nu_1, \nu_2, \ldots\ldots, \nu_s$ and the characteristic spaces of characteristic covectors corresponding to the roots 0 and $\sigma_1, \sigma_2, \ldots\ldots, \sigma_s$ by W_0^*, $W_1^*, \ldots\ldots, W_s^*$. Then we see that

$$\dim W_0^* = l, \quad \dim W_\lambda^* = \nu_\lambda, \qquad (\lambda = 1, 2, \ldots\ldots, s)$$

$$\nu_1 + \nu_2 + \cdots\cdots + \nu_s = m$$

and $W_0^*, \ldots\ldots, W_s^*$ are orthogonal to each other.

10°. Now, we consider a linear map of the tangent space T_P at a point P of M^n to its dual space T_P^* defined by

$$h: X^i \longrightarrow h_{ij}X^j. \tag{4.14}$$

Then, we get the following

LEMMA 3. *Let hV_λ $(\lambda = 1, \ldots\ldots, r)$ be the image of the vector space V_λ under the map h, then*

$$W_0^* = hV_1 \oplus \cdots\cdots \oplus hV_r. \tag{4.15}$$

PROOF. Suppose that $X \in V_\lambda$ $(\lambda = 1, 2, \ldots\ldots, r$ fixed), then

$$\phi_{ih}h^{hk}\phi_{kj}X^j = -\rho_\lambda h_{ij}X^j.$$

By virtue of the last equation and (2.4) we see that

$$\begin{aligned}&\psi^{ih}h_{hk}\psi^{kj}(-\rho_\lambda h_{jm}X^m)\\ &= \psi^{ih}h_{hk}\psi^{kj}\phi_{jx}h^{\alpha\beta}\phi_{\beta\gamma}X^\gamma = 0.\end{aligned}$$

As $\rho_\lambda \neq 0$, this shows that $hX \in W_0^*$, hence $hV_\lambda \subset W_0^*$. Therefore, $hV_1 \oplus \cdots \cdots \oplus hV_r \subseteqq W_0^*$. However, taking account of the dimensions of W_0^* and $hV_1 \oplus \cdots \cdots \oplus hV_r$, we see that the equality sign holds good. Q. E. D.

LEMMA 4. *The tangent space T_P at any point P of M^n decomposes into the form*

$$T_P = V_1 \oplus \cdots\cdots \oplus V_r \oplus h^{-1}W_1^* \oplus \cdots\cdots \oplus h^{-1}W_s^*, \tag{4.16}$$

and any two of these component spaces are orthogonal to each other.

PROOF. Quite analogously to Lemma 3, we can prove that

$$V_0 = h^{-1}W_1^* \oplus \cdots\cdots \oplus h^{-1}W_s^*.$$

As $W_1^*, \ldots\ldots, W_s^*$ are orthogonal to each other, $h^{-1}W_1^*, \ldots\ldots, h^{-1}W_s^*$ are orthogonal to each other too. On the other hand, $V_0 \oplus V_1 \oplus \cdots\cdots \oplus V_r$ is a decomposition of the tangent space at any point of M^n. So our assertion is true. Q. E. D.

Now, suppose $\{U_\alpha\}$ be an open covering of M^n. We take frames $\xi_1, \ldots\ldots, \xi_n$ over U_α such that

$$\begin{array}{ll}\xi_1, \ldots\ldots, \xi_{\mu_1} & \text{span } V_1,\\ \xi_{\mu_1+1}, \ldots\ldots, \xi_{\mu_1+\mu_2} & \text{span } V_2,\\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \xi_{\mu_1+\cdots+\mu_{r-1}+1}, \ldots\ldots, \xi_l & \text{span } V_r,\end{array}$$

$$\xi_{l+1},\ldots\ldots,\xi_{l+\nu_1} \qquad \text{span } h^{-1}W_1^*,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\xi_{l+\nu_1+\cdots+\nu_{s-1}+1},\ldots\ldots,\xi_n \qquad \text{span } h^{-1}W_s^*.$$

As the terminology which is used only in this section, we call such frame an *adapted frame*. Then, with respect to adapted frames, the components of the tensor field h have the form:

$$(4.17) \qquad h = \begin{pmatrix} h_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & h_r & & & & \\ & & & h_{r+1} & & & \\ & & & & \ddots & & \\ 0 & & & & & & h_{r+s} \end{pmatrix},$$

where $h_1,\ldots\ldots,h_r,\ h_{r+1},\ldots\ldots,h_{r+s}$ are matrices of order $\mu_1,\ldots\ldots,\mu_r$ and $\nu_1,\ldots\ldots,\nu_s$ respectively.

11°. To find the form of components of ϕ_{ij} with respect to adapted frames, we consider the linear map $X\to\widetilde{X}$ of the tangent space defined by

$$(4.18) \qquad \widetilde{X}^i = h^{ih}\phi_{hj}X^j.$$

LEMMA 5. *If $X\in V_\lambda$ $(\lambda=1,\ldots\ldots,r)$, then $\widetilde{X}\in V_\lambda$.*

PROOF. By virtue of

$$(4.19) \qquad \phi_{ih}h^{hk}\phi_{kj}X^j = -\rho h_{ij}X^j,$$

we can easily verify that

$$\begin{aligned}(\phi_{ih}h^{hk}\phi_{kj} &+ \rho_\lambda h_{ij})h^{jl}\phi_{lm}X^m \\ &= \phi_{ih}h^{hk}(-\rho_\lambda h_{km}X^m)+\rho_\lambda\phi_{im}X^m \\ &= 0.\end{aligned}$$

Q. E. D.

LEMMA 6. *If $X\in V_\lambda$ $(\lambda=1,\ldots\ldots,r)$, then*

$$(4.20) \qquad \widetilde{\widetilde{X}}^i = -\rho_\lambda X^i.$$

PROOF. By virtue of (4. 19), we see that

$$\begin{aligned}\widetilde{\widetilde{X}}^i &= h^{ih}\phi_{hj}(h^{jl}\phi_{lm}X^m) \\ &= h^{ih}(-\rho_\lambda h_{hm}X^m) \\ &= -\rho_\lambda X^i.\end{aligned}$$

Q. E. D.

Now, we put components of the tensor ϕ with respect to adapted frames in the form

$$\begin{pmatrix} \phi_{11} & \cdots\cdots & \phi_{1r+s} \\ \vdots & & \vdots \\ \phi_{r+s1} & \cdots\cdots & \phi_{r+s\,r+s} \end{pmatrix},$$

where $\phi_{uv}(u, v = 1,\ldots\ldots, r)$ is a (μ_u, μ_v)-matrix, $\phi_{r+u,v}(u = 1,\ldots\ldots, s,\ v = 1,\ldots\ldots, r)$ is a (ν_u, μ_v)-matrix, $\phi_{u,r+v}(u = 1,\ldots\ldots, r, v = 1,\ldots\ldots, s)$ is a (μ_u, ν_v)-matrix, and $\phi_{r+u,r+v}(u, v = 1,\ldots\ldots, s)$ is a (ν_u, ν_v)-matrix. Then, we see that

$$h^{-1}\phi = \begin{pmatrix} h_1^{-1}\phi_{11} & \cdots\cdots & h_1^{-1}\phi_{1r+s} \\ \vdots & & \vdots \\ h_{r+s}^{-1}\phi_{r+s1} & \cdots\cdots & h_{r+s}^{-1}\phi_{r+s\,r+s} \end{pmatrix}.$$

If we assume that $X \in V_1$, then its components with respect to adapted frames are of the form $(X_1, 0,\ldots\ldots, 0)$, where X_1 is a vector with μ_1 components. So, in this case the vector $\widetilde{X} = h^{-1}\ \phi X$ has components $(h_1^{-1}\phi_{11}X_1,\ h_2^{-1}\phi_{21}X_1,\ldots\ldots, h_{r+s}^{-1}\phi_{r+s1}X_1)$. However, as $\widetilde{X} \in V_1$, we see that

$$h^{-1}\phi_{\lambda 1}X_1 = 0, \qquad (\lambda = 2,\ldots\ldots, r + s).$$

Since X_1 is an arbitrary vector of V_1, we get

$$\phi_{21} = \phi_{31} = \cdots\cdots = \phi_{r+s1} = 0.$$

In the same way, by considering vectors of $V_2,\ldots\ldots, V_r$, we get

$$\begin{cases} \phi_{uv} = 0 & (u \neq v\ ;\ u, v = 1,\ldots\ldots, r), \\ \phi_{r+u,v} = 0 & (u = 1,\ldots\ldots, s,\ v = 1,\ldots\ldots, r). \end{cases}$$

LEMMA 7. *If* $Y \in h^{-1}W_\lambda^*\ (\lambda = 1,\ldots\ldots, s)$, *then*

$$\phi_{ij}Y^j = 0. \tag{4.21}$$

PROOF. By assumption $Y^* = hY \in W_j^*$, so

$$\psi^{ih}h_{hk}\psi^{kj}Y_j^* = -\sigma_\lambda h^{ij}Y_j^* = -\sigma_1 Y^i.$$

By virtue of this and $(4.11)_3$, we see that

$$-\sigma_\lambda\phi_{ij}Y^j = \phi_{ij}\psi^{j\alpha}h_{\alpha\beta}\psi^{\beta\gamma}Y_\gamma^* = 0,$$

which is to be proved. Q. E. D.

Now, suppose $Y \in h^{-1}W_1^* \oplus \cdots\cdots \oplus h^{-1}W_s^*$, then the components of Y with respect to an adapted frame have the form $(0,\ldots\ldots, 0,\ Y_{r+1},\ldots\ldots, Y_{r+s})$, where $Y_{r+u}(u = 1,\ldots\ldots, s)$ are vectors with ν_u components. Hence, (4.21) shows us that

$$\phi_{u\, r+v} = 0, \quad \phi_{r+u\, r+v} = 0,$$

where $u = 1, \ldots\ldots, r$, and $v = 1, \ldots\ldots, s$. Consequently, we see that ϕ has the form

$$\phi = \left(\begin{array}{cccc|c} \phi_1 & & & 0 & \\ & \phi_2 & & & \\ & & \ddots & & 0 \\ 0 & & & \phi_r & \\ \hline & & 0 & & 0 \end{array}\right) \tag{4.22}$$

with respect to adapted frames, where ϕ_λ $(\lambda = 1, \ldots\ldots, r)$ is a ε-symmetric μ_λ-matrix.

To find components of ψ with respect to adapted frames, we consider the linear map

$$\widetilde{Y}_j^* = \psi^{ih} g_{hj} Y_i^*. \tag{4.23}$$

Then, we can easily prove lemmas analogous to Lemmas 5, 6 and 7. Making use of these facts, we can similarly prove that the components of ψ with respect to adapted frames have the following form:

$$\psi = \left(\begin{array}{c|cccc} 0 & & & 0 & \\ \hline & \psi^1 & & & \\ 0 & & \psi^2 & & 0 \\ & & & \ddots & \\ & 0 & & & \psi^s \end{array}\right), \tag{4.24}$$

where $\psi^\lambda (\lambda = 1, \ldots\ldots, s)$ is a ε'-symmetric ν_λ-matrix.

12°. Now, making use of the decomposition of the tangent spaces stated in Lemma 4, let us introduce a new Riemannian metric g over M^n by

$$g = \begin{pmatrix} g_1 & & & & & 0 \\ & \ddots & & & & \\ & & g_r & & & \\ & & & g_{r+1} & & \\ & & & & \ddots & \\ 0 & & & & & g_{r+s} \end{pmatrix}, \tag{4.25}$$

where we have put

$$\begin{cases} g_u = \sqrt{-\varepsilon \rho_u}\, h_u & (u = 1, \ldots\ldots, r), \\ g_{r+v} = h_{r+v} / \sqrt{-\varepsilon' \sigma_v} & (v = 1, \ldots\ldots, s). \end{cases} \tag{4.26}$$

As characteristic roots and characteristic spaces are independent upon the choice of coordinate neighborhoods, the Riemannian metrics defined thus for every coordinate neighborhood of the covering $\{U_\alpha\}$ constitute a single globally defined Riemannian metric over M^n.

We put

$$\phi^i_j = g^{ih}\phi_{hj}. \tag{4.27}$$

Then, ϕ^i_j is a globally defined tensor field over M^n too. By virtue of (4.22) and (4.25), we see that ϕ^i_j has components of the form

$$(\phi^i_j) = \left(\begin{array}{ccc|c} g_1^{-1}\phi_1 & & 0 & 0 \\ & \ddots & & \\ 0 & & g_r^{-1}\phi_r & \\ \hline & 0 & & 0 \end{array}\right) \tag{4.28}$$

with respect to adapted frames. We now define a modified linear map of (4.18) by

$$\hat{X}^i = g^{ih}\phi_{hj}X^j. \tag{4.29}$$

If we assume $X \in V_1$, then with respect to adapted frames, we see that

$$\begin{aligned} \hat{X}^u &= g^{uv}\phi_{vw}X^w \qquad (u, v, w = 1, \ldots\ldots, \mu_1) \\ &= \frac{1}{\sqrt{-\varepsilon\rho_1}}\tilde{X}^u. \end{aligned}$$

Hence, we have

$$\hat{\hat{X}}^u = \frac{1}{-\varepsilon\rho_1}\tilde{\tilde{X}}^u = \varepsilon X^u.$$

This shows that

$$(g_1^{-1}\phi_1)^2 = \varepsilon.$$

Similar formulas hold good for matrices $g_\lambda^{-1}\phi_\lambda (\lambda = 1, \ldots\ldots, r)$ too. Hence, we see that

$$(\phi^i_j\phi^j_k) = \left(\begin{array}{ccc|c} \varepsilon & & 0 & \\ & \ddots & & 0 \\ 0 & & \varepsilon & \\ \hline & 0 & & 0 \end{array}\right). \tag{4.30}$$

Finally, we put

$$\psi_j^i = \psi^{ih} g_{hj}, \tag{4.31}$$

then, ψ_j^i is a globally defined tensor field over M^n. And, in the same way as above, we can prove that (ψ_j^i) and $(\psi_j^i \psi_k^j)$ have the forms

$$(\psi_j^i) = \left(\begin{array}{c|ccc} 0 & & 0 & \\ \hline & \psi^1 g_{r+1} & & 0 \\ 0 & & \ddots & \\ & 0 & & \psi^s g_{r+s} \end{array}\right) \tag{4.32}$$

and

$$(\psi_j^i \psi_k^j) = \left(\begin{array}{c|ccc} 0 & & 0 & \\ \hline & \varepsilon' & & 0 \\ 0 & & \ddots & \\ & 0 & & \varepsilon' \end{array}\right) \tag{4.33}$$

with respect to adapted frames.

Combining (4.30) and (4.33), we see that

$$\varepsilon \phi_j^i \phi_k^j + \varepsilon' \psi_j^i \psi_k^j = \delta_k^i \tag{4.34}$$

holds good with respect to adapted frames. However, as (4.34) is a tensor equation, it does hold for any frame, especially for natural frames. We can easily prove, by virtue of (4.28) and (4.32) that

$$\phi_j^i \psi_k^j = 0, \quad \psi_j^i \phi_k^j = 0. \tag{4.35}$$

As the ranks of $|\phi_j^i|$, $|\psi_j^i|$ are l and m respectively, we see that our ϕ_j^i and ψ_j^i give a (ϕ, ψ)-structure of type (sgn ε, sgn ε') to M^n.

We can easily prove that

$$\begin{aligned} \phi_{ij} &= g_{ih}\phi_j^h = \varepsilon g_{jh}\phi_i^h \\ \psi^{ij} &= \psi_h^i g^{hj} = \varepsilon' \psi_h^j g^{hi} \end{aligned} \tag{4.36}$$

and

$$g_{ij}\phi_h^i \psi_k^j = 0.$$

Finally, as we have shown in the proof of Theorem 6, the relation

$$g_{ij}\phi_h^i \phi_k^j + g_{ij}\psi_h^i \psi_k^j = g_{hk}$$

follows from (4.35) and (4.36). Consequently, we get the following

THEOREM 10. *Suppose M^n be a differentiable manifold and there exist ε-symmetric tensor field ϕ_{ij} and ε'-symmetric tensor field ψ^{ij} such that*

$$\text{rank } |\phi_{ij}| = l, \quad \text{rank } |\psi^{ij}| = m, \quad l + m = n$$

and

$$\psi^{ih}\phi_{hj} = 0$$

hold good. Then, we can find a positive definite Riemannian metric g such that the tensor fields

$$\phi^i_j = g^{ih}\phi_{hj}, \quad \psi^i_j = \psi^{ih}g_{hj}$$

and g define a (ϕ, ψ, g)*-structure over* M^n *of type* (sgn ε, sgn ε').

COROLLARY. *Suppose* M^n *be a differentiable manifold and there exists* ε*-symmetric tensor field* ϕ_{ij} *over* M^n *such that*

$$\text{rank } |\phi_{ij}| = l, \quad l \leqq n.$$

Then we can find a symmetric tensor field ψ^{ij} *such that*

$$\text{rank } |\psi^{ij}| = m, \quad m = n - l$$

and a positive definite Riemannian metric g so that

$$\phi^i_j = g^{ih}\phi_{hj}, \quad \psi^i_j = \psi^{ih}g_{hj}$$

and g define a (ϕ, ψ, g)*-structure of type* (sgn ε, $+$).

PROOF. By assumption, at any point P of M^n, the set of vectors such that

$$\phi_{ij}\xi^j = 0$$

span an m-dimensional vector subspace $V_m(P)$ of the tangent space T_P at P. Now, suppose h be a positive definite Riemannian metric h over M^n and take m orthonormal vectors ξ^i_p with respect to the metric h and put

$$\psi^{ij}(P) = \sum_p \xi^i_p \xi^j_p.$$

When P moves over M^n, it is evident that ψ^{ij}'s constitute a globally defined symmetric tensor field over M^n. We can easily verify that

$$\text{rank } |\psi^{ij}| = m,$$

$$\psi^{ih}\phi_{hj} = 0.$$

Hence, by virtue of Theorem 10, we can conclude that our assertion is true.

Q. E. D.

5. The structure groups of the tangent bundles of manifolds with (ϕ, ψ, g)-structures.

13°. The structure group of the tangent bundle $T(M^n)$ of a differentiable manifold M^n is in general the general linear group $GL(n)$. However, we can prove the following

THEOREM 11. *Suppose M^n be a differentiable manifold with (ϕ, ψ, g)-structure of type* (sgn ε, sgn ε'). *Then, the structure group of the tangent bundle $T(M^n)$ is reducible to the following one*:

(i) $\quad U(l/2) \times U(m/2) \quad$ if $\varepsilon = \varepsilon' = -1$,

(ii) $\quad U(l/2) \times O(m_1) \times O(m_2) \quad$ if $\varepsilon = -1$, $\varepsilon' = +1$,

(iii) $\quad O(l_1) \times O(l_2) \times O(m_1) \times O(m_2) \quad$ if $\varepsilon = \varepsilon' = +1$,

where $l_1(l_2)$ is the number of negative (positive) roots of the characteristic equation of ϕ^i_j and $m_1(m_2)$ is that of ψ^i_j.

PROOF. We shall prove only the case (ii). The other cases can be proved quite analogously.

We take sufficiently fine open covering $\{U_\alpha\}$ of M^n and determine in every U_α suitable frames. To do so, we take first a unit vector field ξ^i_1 over U_α contained in D_l and put

$$\xi^i_{1^*} = \phi^i_j \xi^j_1, \quad 1^* = l' + 1, \quad l' = l/2. \tag{5.1}$$

Then, we can easily see that $\xi^i_{1^*}$ is a unit vector field orthogonal to ξ^i_1 and contained in D_l. Secondly, if we take a unit vector field ξ^i_2 orthogonal to ξ^i_1 and $\xi^i_{1^*}$ and contained in D_l, then

$$\xi^i_{2^*} = \phi^i_j \xi^j_2, \quad 2^* = l' + 2 \tag{5.2}$$

is a unit vector field orthogonal to ξ^i_1, ξ^i_2, $\xi^i_{1^*}$ and contained in D_l. Continuing this process, we can find orthonormal vector fields $\xi^i_\lambda(\lambda = 1, \ldots\ldots, l')$ and

$$\xi^i_{\lambda^*} = \phi^i_j \xi^j_\lambda, \quad \lambda^* = l' + \lambda \tag{5.3}$$

so that they span D_l in U_α. By virtue of (2.5) we can solve (5.3) as follows:

$$\xi^i_\lambda = -\phi^i_j \xi^j_{\lambda^*}. \tag{5.4}$$

Next, we consider the characteristic equation

$$|\psi^i_j + \sigma\delta^j_i| = 0. \tag{5.5}$$

Then, the characteristic roots are 0, -1 or $+1$. The characteristic space corresponding to 0 is the vector space V_l of the distribution D_l. We denote the multiplicities of the roots -1 and $+1$ by m_1 and m_2 and the characteristic spaces corresponding to -1 and $+1$ by W_- and W_+. Then, as is easily seen, V_l, W_- and W_+ are orthogonal to each other.

Now, we take orthonormal frames $(\xi^i_\lambda, \xi^i_{\lambda^*}, \xi^i_M, \xi^i_S)$ over U_α so that ξ^i_λ, $\xi^i_{\lambda^*}$

are related by (5.3) and span V_l, $\xi^i_M (M = l+1, \ldots\ldots, l+m_1)$ span W_- and ξ^i_S $(S = l + m_1 + 1, \ldots\ldots, n)$ span W_+. Then, we can easily see that g, ϕ and ψ have the following forms with respect to such frames:

$$g = \begin{pmatrix} E_{l'} & & & 0 \\ & E_{l'} & & \\ & & E_{m_1} & \\ 0 & & & E_{m_2} \end{pmatrix}, \tag{5.6}$$

$$\phi = \left(\begin{array}{cc|c} 0 & -E_{l'} & \\ E_{l'} & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right), \tag{5.7}$$

$$\psi = \left(\begin{array}{c|cc} 0 & \multicolumn{2}{c}{0} \\ \hline & -E_{m_1} & 0 \\ 0 & 0 & E_{m_2} \end{array}\right). \tag{5.8}$$

Suppose $U_\alpha \cap U_\beta$ is not empty and $\bar{\xi}^i_\lambda$, $\bar{\xi}^i_{\lambda^*}$, $\bar{\xi}^i_M$, $\bar{\xi}^i_S$ be vector fields over U_β defined in the same way as above, then over $U_\alpha \cap U_\beta$ we get

$$\begin{cases} \xi^i_\mu = a^\lambda_\mu \bar{\xi}^i_\lambda + b^\lambda_\mu \bar{\xi}^i_{\lambda^*}, \\ \xi^i_{\mu^*} = c^\lambda_\mu \bar{\xi}^i_\lambda + d^\lambda_\mu \bar{\xi}^i_{\lambda^*}, \end{cases}$$

$$\begin{cases} \xi^i_N = u^M_N \bar{\xi}^i_M, \\ \xi^i_T = u^S_T \bar{\xi}^i_S, \end{cases}$$

where $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ and (U^M_N) and (U^S_T) are orthogonal matrices. However, by virtue of (5.3), we can easily see that $d = a$, $c = -b$. Hence, the structure group of $T(M^n)$ in consideration is reducible to $U(l') \times O(m_1) \times O(m_2)$.

14. Converse to Theorem 11, we can prove the following

THEOREM 12. *Suppose M^n be a differentiable manifold such that the structure group of the tangent bundle $T(M^n)$ is reducible to*

(i) $U(l') \times U(m')$, $\quad 2(l' + m') = n$, *or*

(ii) $U(l') \times O(m_1) \times O(m_2)$, $\quad 2l' + m_1 + m_2 = n$, *or*

(iii) $O(l_1) \times O(l_2) \times O(m_1) \times O(m_2)$, $\quad l_1 + l_2 + m_1 + m_2 = n$.

Then, we can introduce (ϕ, ψ, g)-structure over M^n of type $(-,-)$, $(-,+)$ or $(+,+)$ according as the structure group is of type (i), (ii) *or* (iii).

PROOF. We shall prove only the case (ii). The other cases can be proved quite analogously. Let $\{U_\alpha\}$ be sufficiently fine open covering of M^n by coordinate neighborhoods. Then, in every U_α, we can take frames ξ^i_λ, $\xi^i_{\lambda^*}(\lambda = 1,\ldots\ldots, l'$, $\lambda^* = l' + \lambda)$, $\xi^i_M(M = l+1,\ldots\ldots, l+m_1)$, $\xi^i_S(S = l+m_1+1,\ldots\ldots, n)$ so that the transformation of frames of U_α and of U_β over non-empty $U_\alpha \cap U_\beta$ is given by an orthogonal transformation of the form

$$\begin{cases} \xi^i_\mu = a^\lambda_\mu \bar{\xi}^i_\lambda + b^\lambda_\mu \bar{\xi}^i_{\lambda^*}, \\ \xi^i_{\mu^*} = -\, b^\lambda_\mu \bar{\xi}^i_\lambda + a^\lambda_\mu \bar{\xi}^i_{\lambda^*}, \end{cases}$$

$$\begin{cases} \xi^i_N = u^M_N \bar{\xi}^i_M, \\ \xi^i_T = u^S_T \bar{\xi}^i_S. \end{cases}$$

We denote the inverse matrix of (ξ^i_α) by (η^α_j) and define over every U_α tensor fields by

$$g_{ij} = \sum_{\alpha=1}^{n} \eta^\alpha_i \eta^\alpha_j,$$

$$\phi^i_j = -\, \xi^i_\lambda \eta^{\lambda^*}_j + \xi^i_{\lambda^*} \eta^\lambda_j,$$

$$\psi^i_j = -\, \xi^i_M \eta^M_j + \xi^i_S \eta^S_j,$$

then all g_{ij}'s, all ϕ^i_j's and all ψ^i_j's corresponding to U_α's constitute global tensor fields g, ϕ and ψ respectively. This can be easily proved by virtue of the relations

$$\begin{cases} \bar{\eta}^\lambda_j = a^\lambda_\mu \eta^\mu_j - b^\lambda_\mu \eta^{\mu^*}_j, \\ \bar{\eta}^{\lambda^*}_j = b^\lambda_\mu \eta^\mu_i + a^\lambda_\mu \eta^{\mu^*}_i, \end{cases}$$

$$\begin{cases} \bar{\eta}^M_j = u^M_N \eta^N_j, \\ \bar{\eta}^S_j = u^S_T \eta^T_j. \end{cases}$$

As g, ϕ, ψ have components of the form (5. 5), (5. 6) and (5. 7) with respect to our frames in consideration, we can easily verify that (2. 3), (2. 5), (3. 3) ($\varepsilon = -1$, $\varepsilon' = 1$) hold good. However, these equations are all tensor equations. Hence, they all hold good for any frames, especially for natural frames. Consequently, our Theorem is proved. Q. E. D.

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