REMARK ON AUTOMORPHISMS IN CERTAIN COMPACT ALMOST HERMITIAN SPACES

CHEN-JUNG HSU

(Received June 15, 1960)

1. Apte and Lichnerowicz [2] [5]*) have proved that if an affine transformation with respect to the first canonical connection $\overset{1}{\Gamma}_{ji}{}^{h}$ on a compact almost Hermitian manifold V_{2n} preserves the almost complex structure, then it is an automorphism, that is an isometry preserving the almost complex structure.

On the other hand, Tachibana [7] has recently proved that if a transformation on a compact O^*-space preserves the almost complex structure $\varphi_i{}^j$ and moreover the skew symmetric covariant tensor $\hat{K}_{kj} \equiv \overset{1}{K}{}_{kji}{}^{h} \varphi_h{}^i$, where $\overset{1}{K}_{kji}{}^{h}$ denotes the curvature tensor with respect to the first canonical connection $\overset{1}{\Gamma}_{ji}{}^{h}$, then it is an automorphism. By definition, an O^*-space considered by Kotō [3] is an almost Hermitian space on which the tensor $\nabla_k \varphi_{ji}$ is pure. Here ∇ denotes the covariant derivative with respect to the Christoffel symbol $\left\{ {h \atop ji} \right\}$ constructed from the Hermitian metric g_{ij} of V_{2n}. The tensor $\nabla_k \varphi_{ji}$ is called pure if it is pure with respect to any two indices, and $\nabla_k \varphi_{ji}$ is said to be pure with respect to indices k and j when the relation $(\nabla_r \varphi_{ji}) \varphi_k{}^r = (\nabla_k \varphi_{ri}) \varphi_j{}^r$ holds.

Some other special almost Hermitian spaces with additional conditions are considered by several authors. For example, an A-space considered by Apte [1] is by definition a space which fulfils the condition: $\nabla_r \varphi_i{}^r = 0$. K-space considered by Tachibana [6] is a space in which $\nabla_k \varphi_{ji} + \nabla_j \varphi_{ki} = 0$ holds. All these spaces are related each other in the following scheme [3]:

$$\begin{pmatrix}\text{almost}\\ \text{Hermitian}\\ \text{space}\end{pmatrix} \longrightarrow (A\text{-space}) \longrightarrow (O^*\text{-space}) \begin{array}{c} \nearrow \begin{pmatrix}\text{almost}\\ \text{Kählerian}\\ \text{space}\end{pmatrix} \searrow \\ \searrow \quad (K\text{-space}) \quad \nearrow \end{array} \begin{pmatrix}\text{pseudo}\\ \text{Kählerian}\\ \text{space}\end{pmatrix}.$$

Now, let $\underset{v}{\pounds}$ denote the Lie derivative with respect to the infinitesimal transformation v^i, then $\underset{v}{\pounds}\, \varphi_h{}^i = 0$ if v^i preserves the almost complex structure.

*) Numbers in brackets refer to the reference at the end of the paper.

Taking account of

$$(1)\qquad \underset{v}{\pounds}\hat{K}_{kj} = (\underset{v}{\pounds}\overset{1}{K}_{kji}{}^{h})\varphi_h{}^i + \overset{1}{K}_{kji}{}^{h}\underset{v}{\pounds}\varphi_h{}^i$$

and

$$(2)\qquad \overset{1}{\nabla}_h \underset{v}{\pounds}\overset{1}{\Gamma}_{ji}{}^{k} - \overset{1}{\nabla}_j \underset{v}{\pounds}\overset{1}{\Gamma}_{hi}{}^{k} + 2\overset{1}{S}_{hj}{}^{l}\underset{v}{\pounds}\overset{1}{\Gamma}_{li}{}^{k} = \underset{v}{\pounds}\overset{1}{K}_{hji}{}^{k}\ [8],$$

where $\overset{1}{\nabla}$ and $\overset{1}{S}_{hj}{}^{l}$ denote respectively the covariant derivative with respect to $\overset{1}{\Gamma}_{ji}{}^{h}$ and the torsion tensor of $\overset{1}{\Gamma}_{ji}{}^{h}$, it is evident that if the infinitesimal transformation v^i which preserves the almost complex structure is an affine transformation with respect to the first canonical connection $\overset{1}{\Gamma}_{ji}{}^{h}$ (i. e. $\underset{v}{\pounds}\overset{1}{\Gamma}_{ji}{}^{h} = 0$) then it preserves $\hat{K}_{kj}$.

Thus if one concerns only with O^*-space, Tachibana's theorem can be seen as a generalization of the theorem of Apte and Lichnerowicz, but in the former a rather strong restriction on the space is made. Concerning to this fact we have the following theorem which is situated in the middle stage between the above two theorems:

THEOREM 1. *On compact A-space, if a transformation preserving the almost complex structure preserves moreover both* $\hat{K}_{kj}$ *and the torsion tensor of the first canonical connection, then it is an automorphism.*

COROLLARY. *On compact A-space if a transformation preserving the almost complex structure also preserves both the curvature tensor and the torsion tensor of the first canonical connection, then it is an automorphism.*

For the proof of Theorem 1 we first prove the following:

LEMMA 1. *If an infinitesimal transformation* v^i *preserving the almost complex structure on an almost Hermitian space preserves moreover the torsion tensor of the first canonical connection, then the following relation holds*

$$g^{rs}\nabla_r\nabla_s v^h + R_r{}^h v^r = 0,$$

where the tensor $R_r{}^h$ *is obtained from the curvature tensor* $R_{tji}{}^{h}$ *of* $\left\{ {h \atop ji} \right\}$, *that is* $R_t^h = R_{tl}\, g^{lh}$ *and* R_{tl} *is the Ricci tensor.*

PROOF. It is well known that the first canonical connection can be expressed as follows [4] [8]:

$$(3)\qquad \overset{1}{\Gamma}_{ji}{}^{h} = \left\{ {h \atop ji} \right\} + \overset{1}{T}_{ji}{}^{h},$$

where

$$\overset{1}{T}_{ji}{}^{h} = -\frac{1}{2}\varphi_r{}^h \nabla_j \varphi_i{}^r. \tag{4}$$

As the infinitesimal transformation preserves the torsion tensor, from (3) we have

$$\underset{v}{\pounds}(\overset{1}{T}_{ji}{}^{h} - \overset{1}{T}_{ij}{}^{h}) = 0. \tag{5}$$

It is known that [8]

$$\underset{v}{\pounds}\nabla_k \varphi_r{}^s - \nabla_k \underset{v}{\pounds}\varphi_r{}^s = t_{kp}{}^s \varphi_r{}^p - t_{kr}{}^p \varphi_p{}^s, \tag{6}$$

where

$$t_{ji}{}^h \equiv \underset{v}{\pounds}\left\{ {h \atop ji} \right\} = \nabla_j \nabla_i v^h + R_{tji}{}^h v^t. \tag{7}$$

Take account of $\underset{v}{\pounds}\varphi_i{}^h = 0$, we have from (6) the following:

$$\underset{v}{\pounds}\nabla_k \varphi_r{}^s = t_{kp}{}^s \varphi_r{}^p - t_{kr}{}^p \varphi_p{}^s. \tag{8}$$

From (4) and (8) we have

$$\underset{v}{\pounds}\overset{1}{T}_{ji}{}^{h} = -\frac{1}{2}(t_{ji}{}^h + t_{jp}{}^r \varphi_i{}^p \varphi_r{}^h). \tag{9}$$

Put (9) in (5) and then note that $\varphi_r{}^h$ is non-singular, we have

$$t_{jp}{}^r \varphi_i{}^p = t_{ip}{}^r \varphi_j{}^p, \tag{10}$$

from which it follows that

$$-t_{jl}{}^r = t_{ip}{}^r \varphi_j{}^p \varphi_l{}^i. \tag{11}$$

Multiply g^{jl} and then contract, we have

$$-g^{jl} t_{jl}{}^r = t_{ip}{}^r \varphi_j{}^p \varphi_l{}^i g^{jl} = t_{il}{}^r g^{li}, \tag{12}$$

from which we have

$$g^{ji} t_{ji}{}^h = 0. \tag{13}$$

From (13) and (7) we have

$$g^{ji} \nabla_j \nabla_i v^h + R_t{}^h v^t = 0. \tag{14}$$

It is proved by Tachibana [7] that in a compact A-space, if an analytic vector v^i (i.e. an infinitesimal transformation which preserves the almost complex structure) satisfies $\underset{v}{\pounds}\hat{K}_{kj} = 0$, then v^i is volume-preserving, that is

$$\nabla_t v^t = 0. \tag{15}$$

On the other hand it is well known [8] that in order for a vector v^i to generate a one-parameter group of motions in a compact orientable Riemannian manifold, it is necessary and sufficient that v^i satisfy (14) and (15).

Thus Theorem 1 follows from these two facts and Lemma 1.

2. Apte and Lichnerowicz [2] have also proved that if the homogeneous holonomy group of a compact almost Hermitian manifold with respect to the second canonical connection is irreducible, then an affine transformation with respect to the second canonical connection preserving the almost complex structure is an automorphism.

On the other hand it is known that the first, second and third canonical connection of an almost Hermitian space can be expressed respectively as follows [8]:

$$\overset{a}{\Gamma}{}_{ji}{}^{h} = \left\{ {h \atop ji} \right\} + \overset{a}{T}{}_{ji}{}^{h}, \ (a = 1, 2. 3) \tag{16}$$

where

$$\overset{1}{T}{}_{ji}{}^{h} = -\frac{1}{2}(\nabla_j \varphi_{il})\, \varphi^{lh}, \tag{17}$$

$$\overset{2}{T}{}_{ji}{}^{h} = -\frac{1}{2}(\nabla_j \varphi_{il} + \nabla_i \varphi_{jl} + \nabla_l \varphi_{ji})\, \varphi^{lh}, \tag{18}$$

$$\overset{3}{T}{}_{ji}{}^{h} = -\frac{1}{2}(\nabla_j \varphi_{il} - \nabla_i \varphi_{jl} - \nabla_l \varphi_{ji})\, \varphi^{lh}, \tag{19}$$

From these expressions we have immediately the following:

LEMMA 2. *For all the three canonical connections of an almost Hermitian manifold to coincide, it is necessary and sufficient that the considered almost Hermitian space reduces to a K-space.*

Noting this lemma and the theorem of Apte and Lichnerowicz stated in §1, it is evident that the theorem of Apte and Lichnerowicz in §2 holds for K-space without the restriction that the homogeneous holonomy group is irreducible. Now a question arises if this theorem may hold without the same restriction for some wider space. In relation to this point, we have the following:

THEOREM 2. *In compact O^*-space, if an affine transformation with respect to the second canonical connection preserves the almost complex structure, then it is an automorphism.*

PROOF. It is proved by Kotō [3] that in an O^*-space if an infinitesimal transformation v^i preserves the almost complex structure, then the relation (14) holds.

Taking account of this fact, it is evident that for the proof of Theorem 2 we have only to prove that v^i is volume-preserving.

As v^i preserves the torsion tensor of the second canonical connection, we have

$$(20)\qquad \underset{v}{£}\,(\overset{2}{T}_{ji}{}^{h} - \overset{2}{T}_{ij}{}^{h}) = 0.$$

Substitute (18) in (20), we have

$$(21)\qquad \underset{v}{£}\,(\varphi^{lh}\nabla_l\varphi_{ji}) = 0.$$

Since v^i is an affine transformation with respect to the second canonical connection we have $\underset{v}{£}\,\overset{2}{\Gamma}_{ji}{}^{h} = 0$, therefore we have from (16), (18) and (21) the following:

$$(22)\qquad t_{ji}{}^{h} - \frac{1}{2}\underset{v}{£}\,(\varphi^{lh}\nabla_j\varphi_{il}) - \frac{1}{2}\underset{v}{£}\,(\varphi^{lh}\nabla_i\varphi_{jl}) = 0.$$

Taking account of $\underset{v}{£}\varphi_i{}^j = 0$ and $\varphi^{lh}\nabla_j\varphi_{il} = \varphi_l{}^h\nabla_j\varphi_i{}^l$, we have from (22) the following:

$$(23)\qquad t_{ji}{}^{h} - \frac{1}{2}\varphi_l{}^h\underset{v}{£}\,(\nabla_j\varphi_i{}^l) - \frac{1}{2}\varphi_l{}^h\underset{v}{£}\,(\nabla_i\varphi_j{}^l) = 0.$$

Substitute (8) in (23) and then note that $t_{ji}{}^h$ is symmetric with respect to j,i we have

$$(24)\qquad (t_{jp}{}^l\varphi_i{}^p + t_{ip}{}^l\varphi_j{}^p)\varphi_l{}^h = 0.$$

As $\varphi_l{}^h$ is non singular, from (24) we have

$$(25)\qquad t_{jp}{}^l\varphi_i{}^p + t_{ip}{}^l\varphi_j{}^p = 0.$$

Contracting with respect to l and j, we have

$$(26)\qquad t_{pr}{}^r\varphi_i{}^p + t_{ip}{}^l\varphi_l{}^p = 0,$$

that is

$$(27)\qquad t_i \equiv t_{ip}{}^l\varphi_l{}^p = -\,t_{pr}{}^r\varphi_i{}^p.$$

On the other hand it is proved by Tachibana [7] that in an A-space an analytic vector v^i satisfies the following:

$$(28)\qquad t_i = t_{pr}{}^r\varphi_i{}^p.$$

From (27) and (28), it follows that

$$(29)\qquad t_{pr}{}^r = \partial_p(\nabla_t v^t) = 0,$$

therefore $\nabla_t v^t =$ const. Since the considered space is compact and orientable it follows moreover from Green's theorem that $\nabla_t v^t = 0$.

3. Finally, concerning to the third canonical connection we have the following:

THEOREM 3. *In an almost Hermitian space an affine transformation with respect to the third canonical connection is an affine motion with respect to the Hermitian metric* g_{ij}, *that is the affine transformation with respect to* $\left\{ {h \atop ji} \right\}$. *Conversely, if an affine motion preserves the torsion tensor of the third canonical connection, then it is an affine transformation with respect to the third canonical connection.*

PROOF. If the infinitesimal transformation v^i preserves the torsion tensor of the third canonical connection, then we have

$$\text{(30)} \qquad \underset{v}{\pounds}\,(\overset{3}{T}_{ji}{}^{h} - \overset{3}{T}_{ij}{}^{h}) = 0.$$

Substitute (19) in (30), we have

$$\text{(31)} \qquad \underset{v}{\pounds}\,(\varphi^{lh}\nabla_j\varphi_{il}) - \underset{v}{\pounds}\,(\varphi^{lh}\nabla_i\varphi_{jl}) - \underset{v}{\pounds}\,(\varphi^{lh}\nabla_l\varphi_{ji}) = 0.$$

On the other hand, from (16) and (19) we have

$$\text{(32)} \qquad \underset{v}{\pounds}\,\overset{3}{\Gamma}_{ji}{}^{h} = t_{ji}{}^{h} - \frac{1}{2}\{\underset{v}{\pounds}(\varphi^{lh}\nabla_j\varphi_{il}) - \underset{v}{\pounds}\,(\varphi^{lh}\nabla_i\varphi_{jl}) - \underset{v}{\pounds}\,(\varphi^{lh}\nabla_l\varphi_{ji})\}.$$

From (31) and (32) we have

$$\text{(33)} \qquad \underset{v}{\pounds}\,\overset{3}{\Gamma}_{ji}{}^{h} = t_{ji}{}^{h},$$

from which the Theorem 3 follows, for an affine transformation preserves the torsion tensor.

As a one-parametric group of affine motions in a compact orientable Riemannian space is a group of motions, it follows immediately from Theorem 3 the following:

COROLLARY 1. *In a compact almost Hermitian space, an affine transformation with respect to the third canonical connection is an isometry.*

COROLLARY 2. *In a compact almost Hermitian space, if an affine transformation with respect to the third canonical connection preserves the almost complex structure, then it is an automorphism.*

In concluding, I wish to express my hearty thanks to Prof. S. Sasaki for his kind guidance and valuable suggestions.

REFERENCES

[1] M. APTE, Sur certaines variétés hermitiques, C. R. Paris, 240 (1954).
[2] M. APTE AND A. LICHNEROWICZ, Sur les transformations affines d'une variété presque-

hermitienne compacte, C.R.Paris, 242 (1956).

[3] S. KOTŌ, Remarks on almost Kählerian spaces, to appear in Journ. of the Math. Soc. of Japan.

[4] A. LICHNEROWICZ, Théorie globale des connexions et des groupes d'holonomie, Ed. Cremonese, Rome (1955).

[5] A. LICHNEROWICZ, Géométrie des groupes de transformations, Dunod, Paris (1958).

[6] S. TACHIBANA, On almost-analytic vectors in certain almost-Hermitian manifolds, Tôhoku Math. Journ., 11 (1959), 351-363.

[7] S. TACHIBANA, On automorphisms of certain compact almost-Hermitian spaces, Tôhoku Math. Journ. 13 (1961),

[8] K. YANO, The Theory of Lie derivatives and its applications, Amsterdam (1957).

TÔHOKU UNIVERSITY AND NATIONAL TAIWAN UNIVERSITY (FORMOSA).