ON THE MATSUSHIMA'S THEOREM IN A COMPACT EINSTEIN K-SPACE

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A. Lichnerowicz[1] has proved that the Matsushima's theorem[2] in a compact Kähler-Einstein space holds good in a compact Kählerian space with constant curvature scalar. In the previous paper [4], we have shown that the Matsushima's theorem is valid also in a compact almost-Kähler-Einstein space. The purpose of this paper is to show that it holds equally well in a compact Einstein K-space.

In §1 we shall give definitions and propositions. In §2 we shall give well known identities in a K-space. In §3 we shall prepare some lemmas on contravariant almost-analytic vectors in a K-space. The last §4 will be devoted to the proof of the main theorem.

1. Preliminaries. We consider a $2n$-dimensional almost-Hermitian space X_{2n} which admits an almost complex structure $\varphi_j^{\ i}$[3] and positive definite Riemannian metric tensor g_{ji} satisfying

$$\varphi_r^{\ i}\varphi_j^{\ r} = -\delta_j^i, \tag{1.1}$$

$$g_{rs}\varphi_j^{\ r}\varphi_i^{\ s} = g_{ji}. \tag{1.2}$$

By (1. 1) and (1. 2), we have

$$\varphi_{ji} = -\varphi_{ij}, \quad \nabla_h\varphi_{ji} = -\nabla_h\varphi_{ij} \tag{1.3}$$

where $\varphi_{ji} = \varphi_j^{\ r}g_{ri}$ and ∇_j denotes the operator of Riemannian covariant derivative.

We define the following linear operators

$$O_{i\,h}^{ml} = \frac{1}{2}(\delta_i^m\delta_h^l - \varphi_i^{\ m}\varphi_h^{\ l}), \qquad {}^*O_{i\,h}^{ml} = \frac{1}{2}(\delta_i^m\delta_h^l + \varphi_i^{\ m}\varphi_h^{\ l})$$

and a tensor is called pure (hybrid) in two indices if it is annihilated by

1) A. Lichnerowicz [1]. The number in brackets refers to Bibliography at the end of this paper.
2) Y. Matsushima [2].
3) As to the notations we follow S. Sawaki [3]. Indices run over $1, 2, \ldots, 2n$.

transvection of $^*O(O)$ on these indices. From the definition, we have easily the following

PROPOSITION 1. $^*O_{ih}^{ab}\nabla_j\varphi_{ab} = 0,\ O_{ib}^{ah}\nabla_j\varphi_a{}^b = 0.$

PROPOSITION 2. *For two tensors* T_{ji} *and* S^{ji}, *if* T_{ji} *is pure in* j, i *and* S^{ji} *is hybrid in* j, i *then* $T_{ji}S^{ji}$ *vanishes.*

A vector v^i is called a contravariant almost-analytic vector if its contravariant components satisfy

$$\underset{v}{\mathfrak{L}}\varphi_j{}^i \equiv v^r\nabla_r\varphi_j{}^i - \varphi_j{}^r\nabla_r v^i + \varphi_r{}^i\nabla_j v^r = 0 \tag{1.4}$$

[4]

where $\underset{v}{\mathfrak{L}}$ is the operator of Lie derivative.

From (1.4) we have

$$\nabla_j v^i + \varphi_j{}^a\varphi_b{}^i\nabla_a v^b - v^r(\nabla_r\varphi_j{}^l)\varphi_l{}^i = 0 \tag{1.5}$$

which is equivalent to (1.4).

Lastly multiplying (1.5) by $\frac{1}{2}\varphi_{li}\nabla^k\varphi^{jl}$, we have

$$\frac{1}{2}v^r(\nabla_r\varphi_{jl})\nabla^k\varphi^{jl} + \varphi_{rl}(\nabla^k\varphi_j{}^l)\nabla^j v^r = 0. \tag{1.6}$$

In this place, if $\nabla^j v^r = \nabla^r v^j$, $\varphi_{rl}\nabla^k\varphi_j{}^l$ being anti-symmetric in j, r, we have

$$\varphi_{rl}(\nabla^k\varphi_j{}^l)\nabla^j v^r = 0.$$

Thus from (1.6) we get

$$v^r\nabla_r\varphi_{jl} = 0. \tag{1.7}$$

2. Identities in a K-space. An almost-Hermitian space X_{2n} is called a K-space[5] if it satisfies

$$\nabla_j\varphi_{ih} + \nabla_i\varphi_{jh} = 0 \tag{2.1}$$

from which we have easily

$$\nabla_j\varphi_i{}^j = 0, \tag{2.2}$$

$$^*O_{ji}^{ab}\nabla_a\varphi_{bh} = 0. \tag{2.3}$$

[6]

Hereafter we shall consider only a K-space X_{2n}.

Let $R_{kji}{}^h$ and $R_{ji} = R_{rji}{}^r$ be Riemannian and Ricci tensor respectively and put

4) S. Tachibana [5].
5) S. Tachibana [5].
6) S. Sawaki [3].

(2. 4) $$R^*_{ji} = \frac{1}{2}\varphi^{ab}R_{abri}\varphi_j^{\ r},\ R^*_j{}^i = R^*_{jr}g^{ri}$$

Applying the Ricci's identity to $\varphi_i^{\ h}$, we get

$$\nabla_k\nabla_j\varphi_i^{\ h} - \nabla_j\nabla_k\varphi_i^{\ h} = R_{kjr}{}^h\varphi_i^{\ r} - R_{kji}{}^r\varphi_r^{\ h}.$$

Transvecting the last equation with g^{ji} and using (2. 2) and the Bianchi's identity, we have

(2. 5) $$\nabla^r\nabla_j\varphi_r^{\ h} = \frac{1}{2}\varphi^{pq}R_{pqj}{}^h + R_j^{\ r}\varphi_r^{\ h}$$

or using (2. 1)

(2. 6) $$\nabla^r\nabla_r\varphi_{jh} = -\frac{1}{2}\varphi^{pq}R_{pqjh} - R_j^{\ r}\varphi_{rh}.$$

If we notice the anti-symmetry w. r. t. j and h in (2. 6), we find that

$$R_j^{\ r}\varphi_{rh} + R_h^{\ r}\varphi_{rj} = 0$$

from which we have

$$O_{jh}^{ab}R_{ab} = 0,$$

i. e. R_{jh} is hybrid in j, h.

On the other hand, in a K-space we know that

(2. 7) $$R^*_{ji} = R^*_{ij}, \qquad (\nabla_j\varphi_{ab})\nabla_i\varphi^{ab} = R_{ji} - R^*_{ji}$$ [7]

and transvecting (2. 5) with $\varphi_h^{\ k}$ we get

(2. 8) $$\varphi_h^{\ k}\nabla^r\nabla_j\varphi_r^{\ h} = R^*_j{}^k - R_j^{\ k}.$$

Since by (2. 3), $(\nabla_j\varphi_{ab})\nabla_i\varphi^{ab}$ is hybrid in j, i, from the last equation of (2. 7). it follows that R^*_{ji} is also hybrid in j, i.

In this place, since (2. 6) can be written as

$$\nabla^r\nabla_r\varphi_{jh} = R^*_{rh}\varphi_j^{\ r} - R_j^{\ r}\varphi_{rh},$$

we see that $\nabla^r\nabla_r\varphi_{ji}$ is hybrid in j, i.
Again by the Ricci's identity

$$\varphi^{sh}\nabla_s\nabla_h\varphi_{kt} = \frac{1}{2}\varphi^{sh}(\nabla_s\nabla_h\varphi_{kt} - \nabla_h\nabla_s\varphi_{kt})$$

$$= \frac{1}{2}\varphi^{sh}(-R_{shk}{}^a\varphi_{at} - R_{sht}{}^a\varphi_{ka})$$

7) S. Tachibana [5].

$$= - R^*_{tk} + R^*_{kt}$$

and therefore by (2.7) we have

$$\varphi^{sh}\nabla_s\nabla_h\varphi_{kt} = 0. \tag{2.9}$$

Moreover, making use of (2.7) and Proposition 2, we have

$$\begin{aligned}\nabla^j(R_{ji} - R^*_{ji}) &= \nabla^j(\nabla_j\varphi_{ab}\cdot\nabla_i\varphi^{ab}) \\ &= (\nabla^j\nabla_j\varphi_{ab})\nabla_i\varphi^{ab} + (\nabla_j\varphi_{ab})\nabla^j\nabla_i\varphi^{ab} \\ &= (\nabla_j\varphi_{ab})\nabla^j\nabla_i\varphi^{ab}\end{aligned}$$

because $\nabla_i\varphi^{ab}$ is pure in a,b and $\nabla^j\nabla_j\varphi_{ab}$ is hybrid in a, b. By the Ricci's identity and the Bianchi's identity the last equation turns to

$$\begin{aligned}\nabla^j(R_{ji} - R^*_{ji}) &= \nabla_j\varphi_{ab}(\nabla_i\nabla^j\varphi^{ab} + R^j{}_{is}{}^a\varphi^{sb} + R^j{}_{is}{}^b\varphi^{as}) \\ &= (\nabla_j\varphi_{ab})\nabla_i\nabla^j\varphi^{ab} + 2(\nabla_j\varphi_{ab})R^j{}_{is}{}^a\varphi^{sb} \\ &= \frac{1}{2}\nabla_i(\nabla_j\varphi_{ab}\cdot\nabla^j\varphi^{ab}) - (\nabla_b\varphi_{ja})R^{ja}{}_{is}\varphi^{sb},\end{aligned}$$

from which we have

$$\nabla^j R_{ji} - \nabla^j R^*_{ji} = \frac{1}{2}\nabla_i(R - R^*) - (\nabla_b\varphi_{ja})R^{ja}{}_{is}\varphi^{sb}, \qquad \text{i. e.}$$

$$(\nabla_b\varphi_{ja})R^{ja}{}_{is}\varphi^{sb} = \nabla^j R^*_{ji} - \frac{1}{2}\nabla_i R^* - \left(\nabla^j R_{ji} - \frac{1}{2}\nabla_i R\right), \tag{2.10}$$

where $R = R_{ji}g^{ji}$ and $R^* = R^*_{ji}g^{ji}$.
In general, since $\nabla^j R_{ji} = \frac{1}{2}\nabla_i R$ [8], from (2.10) we obtain

$$(\nabla_b\varphi_{ja})R^{ja}{}_{is}\varphi^{sb} = \nabla^j R^*_{ji} - \frac{1}{2}\nabla_i R^*. \tag{2.11}$$

And by the Bianchi's identity the left hand side of (2.11) can be written as

$$\begin{aligned}(\nabla_b\varphi_{ja})R^{ja}{}_{is}\varphi^{sb} &= \nabla_b\varphi_{ja}(- R^j{}_{is}{}^a - R^j{}_s{}^a{}_i)\varphi^{sb} \\ &= \nabla_b\varphi_{ja}(- R_s{}^{aj}{}_i - R^j{}_s{}^a{}_i)\varphi^{sb}.\end{aligned}$$

But as we have by virtue of (2.3)

$$(\nabla_b\varphi_{ja})\varphi^{sb} = (\nabla^s\varphi_{ba})\varphi_j{}^b, \quad (\nabla_b\varphi_{ja})\varphi^{sb} = (\nabla^s\varphi_{jb})\varphi_a{}^b,$$

the above equation becomes

8) K. Yano and S. Bochner [7], p. 19.

$$(\nabla_b\varphi_{ja})R^{ja}{}_{is}\varphi^{sb} = -(\nabla_b\varphi^s{}_a)R_s{}^a{}_i{}^j\varphi_j{}^b - (\nabla_j\varphi^s{}_b)R^j{}_{si}{}^a\varphi_a{}^b, \qquad \text{i. e.}$$

(2. 12) $$3(\nabla_b\varphi_{ja})R^{ja}{}_{is}\varphi^{sb} = 0.$$

Consequently from (2. 11) we have

(2. 13) $$\nabla^j R^*{}_{ji} = \frac{1}{2}\nabla_i R^*.$$

And by the Bianchi's identity and (2. 1)

$$\begin{aligned}\varphi_h{}^r(\nabla^k\varphi^{ji})R_{kjir} &= -\varphi_h{}^r(\nabla^k\varphi^{ji})(R_{kirj}+R_{krji})\\ &= -\varphi_h{}^r(\nabla^k\varphi^{ij})R_{kijr} - \varphi_h{}^r(\nabla^j\varphi^{ik})R_{jikr}, \qquad \text{i. e.}\end{aligned}$$

(2. 14) $$3\varphi_h{}^r(\nabla^k\varphi^{ji})R_{kjir} = 0.$$

Thus multiplying

$$\nabla_k\nabla_j\varphi_{ih} - \nabla_j\nabla_k\varphi_{ih} = -R_{kji}{}^r\varphi_{rh} - R_{kjh}{}^r\varphi_{ir}$$

by $\nabla^k\varphi^{ji}$, we have

(2. 15) $$\begin{aligned}2(\nabla^k\varphi^{ji})\nabla_k\nabla_j\varphi_{ih} &= -\nabla^k\varphi^{ji}(R_{kji}{}^r\varphi_{rh} + R_{kjh}{}^r\varphi_{ir})\\ &= -(\nabla^k\varphi^{ji})R_{kji}{}^r\varphi_{rh} + (\nabla_i\varphi^{kj})R_{kjhr}\varphi^{ri}\\ &= 0\end{aligned}$$

because of (2. 1) (2. 12) and (2. 14).

On the other hand, taking account of (2. 1) and (2. 7), we get

$$\begin{aligned}\frac{1}{2}\nabla_k(R-R^*) &= (\nabla^j\varphi^{rs})\nabla_k\nabla_j\varphi_{rs}\\ &= \nabla^j\varphi^{rs}(\nabla_j\nabla_k\varphi_{rs} - R_{kjr}{}^t\varphi_{ts} - R_{kjs}{}^t\varphi_{rt}) = 0\end{aligned}$$

because of (2. 1), (2. 12), (2. 15) and $(\nabla^j\varphi^{rs})\varphi^t{}_s = (\nabla^s\varphi^{rt})\varphi^j{}_s$.
That is, we see that in a K-space

(2. 16) $$R - R^* = \text{constant.}$$ [9]

For the Nijenhuis tensor, by (2. 1),

$$N_{jih} = \varphi_j{}^l(\nabla_l\varphi_{ih} - \nabla_i\varphi_{lh}) - \varphi_i{}^l(\nabla_l\varphi_{jh} - \nabla_j\varphi_{lh})$$

becomes

(2. 17) $$N_{jih} = 4\varphi_j{}^l\nabla_l\varphi_{ih}.$$

Finally, for any vector v_i we have

(2. 18) $$\varphi_l{}^i\varphi^{ab}\nabla_a\nabla_b v_i = \frac{1}{2}\varphi_l{}^i\varphi^{ab}(\nabla_a\nabla_b v_i - \nabla_b\nabla_a v_i)$$

9) S. Tachibana [6].

$$= -\frac{1}{2}\varphi_l{}^i\varphi^{ab}R_{abi}{}^s v_s = -v_s R^*{}_l{}^s.$$

3. Contravariant almost-analytic vectors in a K-space. In a K-space, we know the following lemma.

LEMMA 3. 1.[10] *In a compact K-space, a necessary and sufficient condition that a contravariant vector v^i be almost-analytic is that it satisfies*

(i) $\nabla^l\nabla_l v^i + R_r{}^i v^r = 0$ (ii) $N_{rlk}\nabla^r v^l + 2v^r(R_{rk} - R^*{}_{rk}) = 0.$

In general, even if v^k is almost-analytic, $\tilde{v}^k = \varphi_r{}^k v^r$ is not necessarily almost-analytic. Suppose that for a contravariant almost-analytic vector v^k in a K-space, $\tilde{v}^k$ is also almost-analytic, then we have from (1. 5)

$$\nabla_j\tilde{v}^k + \varphi_j{}^r\varphi_l{}^k\nabla_r\tilde{v}^l - \tilde{v}^r(\nabla_r\varphi_j{}^l)\varphi_l{}^k = 0$$

or using (2. 3)

(3. 1) $$(\nabla_j v^a)\varphi_a{}^k - \varphi_j{}^r\nabla_r v^k + v^a(2\nabla_j\varphi_a{}^k - \nabla_a\varphi_j{}^k) = 0.$$

Transvecting (3. 1) with $\varphi_k{}^i$ it follows that

(3. 2) $$\nabla_j v^i + \varphi_j{}^r\varphi_k{}^i\nabla_r v^k - \varphi_k{}^i v^a(2\nabla_j\varphi_a{}^k - \nabla_a\varphi_j{}^k) = 0.$$

From (1. 5) and (3. 2) we have

$$2\varphi_k{}^i v^a(\nabla_j\varphi_a{}^k - \nabla_a\varphi_j{}^k) = 0, \qquad \text{i. e.}$$

$$v^r\nabla_j\varphi_{rk} = 0,$$

or

$$v^r\nabla_r\varphi_{jk} = 0.$$

Thus we have

LEMMA 3. 2. *When a contravariant vector v^k in a K-space is almost-analytic, a necessary and sufficient condition that $\tilde{v}^k$ be almost-analytic is that it satisfies*

$$v^r\nabla_r\varphi_{jk} = 0.$$

4. A generalization of the Matsushima's theorem.

THEOREM. *In a compact Einstein K-space $X_{2n}(R \neq 0)$, any contravariant almost-analytic vector v^i is decomposed in the form*

$$v^i = p^i + \varphi_r{}^i q^r$$

where p^i and q^i are both Killing vectors and $\varphi_r{}^i q^r$ is a gradient vector. The decomposition stated above is unique.

10) S. Tachibana [5].

PROOF. Let v^i be a contravariant almost-analytic vector in a compact Einstein K-space, then from Lemma 3.1 we have

$$\nabla^l\nabla_l v^i + \frac{R}{2n} v^i = 0. \tag{4.1}$$

From this equation, we can easily deduce

$$\nabla^l\nabla_l\nabla_r v^r + \frac{R}{n}\nabla_r v^r = 0 \tag{4.2}$$

and

$$\nabla^l\nabla_l\nabla^i\nabla_r v^r + \frac{R}{2n}\nabla^i\nabla_r v^r = 0. \tag{4.3}$$

If we put

$$p^h = v^h + \frac{n}{R}\eta^h \tag{4.4}$$

where $\eta^h = \nabla^h\nabla_r v^r$, then by (4.2) we have

$$\nabla_h p^h = \nabla_h v^h + \frac{n}{R}\nabla_h\nabla^h\nabla_r v^r = 0 \tag{4.5}$$

and by (4. 1) and (4. 3) we have

$$\nabla^l\nabla_l p^i + \frac{R}{2n} p^i = 0. \tag{4.6}$$

But since (4. 5) and (4. 6) is a necessary and sufficient condition that p^i in a compact Einstein space be a Killing vector,[11] it follows that p^i is a Killing vector.

Next, to prove that η^i is almost-analytic, putting

$$-P_{jk} = \nabla_j\eta_k + \varphi_j{}^r\varphi_{lk}\nabla_r\eta^l - \eta^r(\nabla_r\varphi_j{}^l)\varphi_{lk}$$

and writing out the square of P_{jk}, we get

$$\frac{1}{2}P_{jk}P^{jk}$$
$$= (\nabla_j\eta_k)\nabla^j\eta^k + \varphi_j{}^r\varphi_{lk}(\nabla^j\eta^k)\nabla_r\eta^l - 2\eta^r\varphi_{lk}(\nabla^j\eta^k)\nabla_r\varphi_j{}^l + \frac{1}{2}\eta^a\eta^r(\nabla_r\varphi_{jb})\nabla_a\varphi^{jb}.$$

Consequently, we have

$$\frac{1}{2}P_{jk}P^{jk} + \nabla^j(P_{jk}\eta^k) = \frac{1}{2}P_{jk}P^{jk} + (\nabla^j P_{jk})\eta^k + P_{jk}\nabla^j\eta^k \tag{4.7}$$

11) K. Yano and S. Bochner [7], p. 56.

$$= \eta^k\left[\nabla^j P_{jk} + \frac{1}{2}\eta^r(\nabla_r\varphi_{jb})\nabla_k\varphi^{jb} + \varphi_{bj}(\nabla_k\varphi_a{}^j)\nabla^a\eta^b\right]$$

and therefore by virtue of Green's theorem we have

$$(4.8)\quad \int_{X_{2n}}\left[\eta^k\left\{\nabla^j P_{jk} + \frac{1}{2}\eta^r(\nabla_r\varphi_{jb})\nabla_k\varphi^{jb} + \varphi_{bj}(\nabla_k\varphi_a{}^j)\nabla^a\eta^b\right\} - \frac{1}{2}P_{jk}P^{jk}\right]d\sigma = 0,$$

where $d\sigma$ means the volume element of the space X_{2n}.

In this place, by using (2.1), (2.2), (2.7), (2.8), (2.17) and (2.18), we have

$$\begin{aligned}\nabla^j P_{jk} &= -\nabla^j\nabla_j\eta_k - \varphi_j{}^r\varphi_{lk}\nabla^j\nabla_r\eta^l + \eta^r(\nabla^j\nabla_r\varphi_j{}^l)\varphi_{lk} - \varphi_j{}^r(\nabla_r\eta^l)\nabla^j\varphi_{lk}\\ &\quad + \nabla^j\eta^r(\nabla_r\varphi_j{}^l)\varphi_{lk} + \eta^r(\nabla_r\varphi_j{}^l)\nabla^j\varphi_{lk}\\ &= -\nabla^j\nabla_j\eta_k - \eta^s R^*{}_{ks} + \eta^r(R^*{}_{kr} - R_{kr}) - \varphi_j{}^r(\nabla_r\eta^l)\nabla^j\varphi_{lk}\\ &\quad + \nabla^j\eta^r(\nabla_r\varphi_j{}^l)\varphi_{lk} + \eta^r(\nabla_r\varphi_j{}^l)\nabla^j\varphi_{lk}\end{aligned}$$

and hence

$$\begin{aligned}&\nabla^j P_{jk} + \frac{1}{2}\eta^r(\nabla_r\varphi_{jb})\nabla_k\varphi^{jb} + \varphi_{bj}(\nabla_k\varphi_a{}^j)\nabla^a\eta^b\\ &= -\nabla^j\nabla_j\eta_k - \eta^r R_{rk} + 3\varphi_r{}^j(\nabla^r\eta^l)\nabla_j\varphi_{lk} + \frac{3}{2}\eta^r(\nabla_r\varphi_{jb})\nabla_k\varphi^{jb}\\ &= -\nabla^j\nabla_j\eta_k - \eta^r R_{rk} + \frac{3}{4}N_{rlk}\nabla^r\eta^l + \frac{3}{2}\eta^r(R_{rk} - R^*{}_{rk}).\end{aligned}$$

Thus (4.8) turns to

$$(4.9)\quad \int_{X_{2n}}\left[\eta^k\left\{-\nabla^l\nabla_l\eta_k - \frac{R}{2n}\eta_k + \frac{3}{4}N_{rlk}\nabla^r\eta^l + \frac{3}{2}\eta^r\left(\frac{R}{2n}g_{rk} - R^*{}_{rk}\right)\right\}\right.$$
$$\left. - \frac{1}{2}P_{jk}P^{jk}\right]d\sigma = 0.$$

Substituting $\eta^k = \frac{R}{n}p^k - \frac{R}{n}v^k$ in (4.9) and using (4.3), we have

$$(4.10)\quad \int_{X_{2n}}\left[\frac{3R}{4n}\eta^k\left\{N_{rlk}\nabla^r(p^l - v^l) + 2(p^r - v^r)\left(\frac{R}{2n}g_{rk} - R^*{}_{rk}\right)\right\}\right.$$
$$\left. - \frac{1}{2}P_{jk}P^{jk}\right]d\sigma = 0.$$

On the other hand v^k being almost-analytic, from Lemma 3.1 we have

$$N_{rlk}\nabla^r v^l + 2v^r\left(\frac{R}{2n}g_{rk} - R^*{}_{rk}\right) = 0.$$

Hence (4.10) becomes

$$(4.11)\quad \int_{X_{2n}} \left[\frac{3R}{4n}\, \eta^k \left\{ N_{rlk}\nabla^r p^l + 2p^r\left(\frac{R}{2n}\, g_{rk} - R^*{}_{rk}\right)\right\} - \frac{1}{2} P_{jk}P^{jk} \right] d\sigma = 0.$$

Furthermore (4. 11) can be written as

$$(4.12)\quad \int_{X_{2n}} \left[\frac{3R}{4n}\, \nabla^k \left\{ (\nabla_l v^l) N_{rlk}\nabla^r p^l + 2(\nabla_l v^l) p^r\left(\frac{R}{2n}\, g_{rk} - R^*{}_{rk}\right)\right\} - \frac{1}{2} P_{jk}P^{jk} \right] d\sigma = 0.$$

In fact, taking account of (2. 17), we have

$$\begin{aligned} \nabla^k(N_{rlk}\nabla^r p^l) &= 4\nabla^k(\nabla^r p^l \cdot \varphi_k{}^t \nabla_t \varphi_{rl}) \\ &= 4\{(\nabla^k\nabla^r p^l)\varphi_k{}^t\nabla_t\varphi_{rl} + (\nabla^r p^l)\varphi_k{}^t\nabla^k\nabla_t\varphi_{rl}\} \\ &= 4(\nabla^k\nabla^r p^l)\varphi_k{}^t\nabla_t\varphi_{rl} \end{aligned}$$

because of (2. 9).
Here by (2. 1), (2. 3) and (2. 12), we have

$$\begin{aligned} 4(\nabla^k\nabla^r p^l)\varphi_k{}^t\nabla_t\varphi_{rl} &= 4\varphi_l{}^t(\nabla_k\varphi_{rt})\nabla^k\nabla^r p^l \\ &= 2\varphi_l{}^t(\nabla_k\varphi_{rt})(\nabla^k\nabla^r p^l - \nabla^r\nabla^k p^l) \\ &= 2\varphi_l{}^t(\nabla_k\varphi_{rt})R^{kr}{}_s{}^l p^s \\ &= 0. \end{aligned}$$

Consequently, we have

$$\nabla^k(N_{rlk}\nabla^r p^l) = 0.$$

On the other hand

$$\nabla^k\left\{p^r\left(\frac{R}{2n}\, g_{rk} - R^*{}_{rk}\right)\right\} = \nabla^k p^r\left(\frac{R}{2n}\, g_{rk} - R^*{}_{rk}\right) - p^r\nabla^k R^*{}_{rk}$$

vanishes.
Because since $\nabla^k p^r$ is anti-symmetric in k, r and $\frac{R}{2n}\, g_{rk} - R^*{}_{rk}$ is symmetric in k, r, the first term of the right hand side vanishes.
For the second term, from (2. 13) and (2. 16) we have

$$2\nabla^k R^*{}_{rk} = \nabla_r R^* = \nabla_r R = 0.$$

Thus again by Green's theorem from (4. 12), we have

$$\int_{X_{2n}} \frac{1}{2} P_{jk}P^{jk} d\sigma = 0$$

from which we have $P_{jk} = 0$, that is, we see that η^t is a contravariant almost-analytic vector.

Next, we shall show that $\tilde{\eta}^{i} = \varphi_{r}{}^{i}\eta^{r}$ is also almost-analytic. Since η^{i} is almost-analytic and $\nabla^{j}\eta^{r} = \nabla^{r}\eta^{j}$, from (1. 7) we have

$$\eta^{r}\nabla_{r}\varphi_{jl} = 0 \tag{4. 13}$$

which shows by virtue of Lemma 3. 2 that $\tilde{\eta}^{i}$ is also almost-analytic.

Accordingly if we put

$$q^{h} = \frac{n}{R}\varphi_{a}{}^{h}\eta^{a}, \tag{4. 14}$$

then q^{h} is a contravariant almost-analytic vector and a Killing vector. In fact $\nabla_{h}q^{h} = \frac{n}{R}\varphi_{a}{}^{h}\nabla_{h}\eta^{a} = 0$ and

$$\nabla^{l}\nabla_{l}q^{h} + \frac{R}{2n}q^{h} = 0.$$

From (4. 4) and (4.14) we have

$$v^{h} = p^{h} + \varphi_{r}{}^{h}q^{r}, \qquad \varphi_{r}{}^{h}q^{r} = -\frac{n}{R}\eta^{h}.$$

Finally we shall prove that such a decomposition is unique.
If we have

$$v^{h} = p^{h} + \varphi_{r}{}^{h}q^{r}, \qquad v^{h} = {}'p^{h} + \varphi_{r}{}^{h\prime}q^{r}$$

where $\varphi_{r}{}^{h}q^{r}$ and $\varphi_{r}{}^{h\prime}q^{r}$ are both gradient vectors, then

$$p^{h} - {}'p^{h} = \varphi_{r}{}^{h}({}'q^{r} - q^{r}). \tag{4. 15}$$

Since the left hand side of (4. 15) is a Killing vector and the right hand side is a gradient vector, we have

$$\nabla_{i}\xi^{h} = 0$$

where $\xi^{h} = p^{h} - {}'p^{h}$.
Hence by the Ricci's identity we have

$$\nabla_{j}\nabla_{i}\xi^{h} - \nabla_{i}\nabla_{j}\xi^{h} = R_{jis}{}^{h}\xi^{s} = 0$$

from which we get

$$R_{is}\xi^{s} = \frac{R}{2n}\xi_{i} = 0.$$

Thus we have $p^{h} = {}'p^{h}$ and $q^{h} = {}'q^{h}$. q. e. d.

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