ON GROUPS OF AUTOMORPHISMS OF FINITE FACTORS

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Introduction. One of the important questions in the theory of the crossed products of rings of operators is the following: *Is the crossed product of a finite factor* $\mathbf{M}$ *also a finite factor for any group* G *of automorphisms of* $\mathbf{M}$? The answer for this question is negative in general ([4]), and some kinds of conditions on G under which the crossed product is a factor have been obtained ([4]). In § 2 we shall deal with this question when G is abelian, and sharpen the results in [2]. In §3 we shall consider the behaviour of the action of G in the crossed product and give a condition on G under which the crossed product is a factor.

1. Throughout this paper, we assume that all W^*-algebras are finite factors with the invariants $C = 1$. An automorphism of a W^*-algebra means a *-automorphism, and a group of outer automorphisms of a W^*-algebra is a group of automorhisms all member of which are outer automorphisms except the unit. The unit of a group will be denoted by e. $R(a_\lambda | \lambda \in \Lambda)$ means the W^*-algebra generated by the family of operators a_λ $(\lambda \in \Lambda)$.

For convenience sake, we shall explain the construction of the crossed product. Let $\mathbf{M}$ be a finite factor with the invariant $C = 1$ on a Hilbert space $\mathbf{H}$ and G a group of automorphisms of $\mathbf{M}$. Let φ be a separating and generating trace vector for $\mathbf{M}$. For each $\sigma \in G$ we define

$$u_\sigma(a\varphi) = a^{\sigma^{-1}}\varphi \qquad \text{for all } a \in \mathbf{M}$$

where a^τ is the image of a by an automorphism τ. Then u_σ can be extended to a unitary operator on $\mathbf{H}$ which will be also denoted by u_σ, and $\sigma \rightarrow u_\sigma$ is a faithful unitary representation of G on $\mathbf{H}$ such that

$$u_\sigma^* a u_\sigma = a^\sigma \qquad \text{for all } a \in \mathbf{M}.$$

Now consider the Hilbert space $\mathbf{H} \otimes l_2(G)$. If we choose the complete orthonormal system $\{\varepsilon_\alpha\}_{\alpha \in G}$ in $l_2(G)$ such as

$$\varepsilon_\alpha(\gamma) = \begin{cases} 1 & \text{if} \quad \gamma = \alpha \\ 0 & \text{otherwise,} \end{cases}$$

each vector of $\mathbf{H} \otimes l_2(G)$ is expressed in the form $\sum \varphi_\alpha \otimes \varepsilon_\alpha$ where $\varphi_\alpha \in \mathbf{H}$

and $\sum_{\alpha\in G}\|\varphi_\alpha\|^2 < \infty$.

For each $a\in \mathbf{M}$ and $\sigma\in G$ we define the operators $a\otimes 1$ and U_σ on $\mathbf{H}\otimes l_2(G)$ by

$$(a\otimes 1)\left(\sum_{\alpha\in G}\varphi_\alpha\otimes\varepsilon_\alpha\right)=\sum_{\alpha\in G}a\varphi_\alpha\otimes\varepsilon_\alpha$$

and

$$U_\sigma\left(\sum_{\alpha\in G}\varphi_\alpha\otimes\varepsilon_\alpha\right)=\sum_{\alpha\in G}u_\sigma\varphi_\alpha\otimes\varepsilon_{\sigma\alpha}$$

for all $\sum_{\alpha\in G}\varphi_\alpha\otimes\varepsilon_\alpha\in\mathbf{H}\otimes l_2(G)$. The set of all operators $a\otimes 1$ $(a\in\mathbf{M})$ will be denoted by $\mathbf{M}\otimes\mathbf{I}$. For $A=a\otimes 1\in\mathbf{M}\otimes\mathbf{I}$ and $\sigma\in G$ we denote $a^\sigma\otimes 1$ by A^σ. Then it is clear that

$$U_\sigma^* A U_\sigma = A^\sigma \qquad \text{for all } A\in\mathbf{M}\otimes\mathbf{I} \text{ and } \sigma\in G.$$

The crossed product of $\mathbf{M}$ by G, denoted by $(\mathbf{M}, G)$ is the W^*-algebra on $\mathbf{H}\otimes l_2(G)$ generated by the set of all finite linear combinations $\sum_i A_i U_{\alpha_i}$ $(A_i\in\mathbf{M}\otimes\mathbf{I},\ \alpha_i\in G)$, and $(\mathbf{M}, G)$ is of finite type. It is noted that each element $A\in(\mathbf{M}, G)$ is uniquely expressed in the form

$$A=\sum\nolimits'_{\alpha\in G}A_\alpha U_\alpha$$

where $A_\alpha\in\mathbf{M}\otimes\mathbf{I}$ and $\sum'$ is taken in the sense of the metrical convergence, and $\varphi\otimes\varepsilon_e$ is a separating and generating vector for the crossed product $(\mathbf{M}, G)$. The crossed product defined above seems to depend on the choice of the representation of G on $\mathbf{H}$, but it is shown that the crossed product is uniquely determined by $\mathbf{M}$ and G within unitary equivalence. For the details of the theory of crossed products see [4].

2. First we shall prove the following Theorem.

THEOREM 1. *Let* $\mathbf{M}$ *be a finite factor with the invariant* $C=1$ *on a Hilbert space* $\mathbf{H}$ *and* G *an abelian group of automorphisms of* $\mathbf{M}$. *Let* $\mathbf{P}$ *be the fixed algebra of* G *in* $\mathbf{M}$[1]. *Then the crossed product* $(\mathbf{M}, G)$ *is a factor if and only if there are no* $a\in\mathbf{P}$ $(a\neq\lambda 1$, λ *is a scalar) and* $\sigma\in G$ $(\sigma\neq e)$ *such as*

$$xa = ax^\sigma \qquad \text{for all } x\in\mathbf{M}.$$

PROOF. Necessity. Suppose that there exist an $a\in\mathbf{P}u(a\neq\lambda 1)$ and a $\sigma\in G$ $(\sigma\neq e)$ such as $xa=ax^\sigma$ for all $x\in\mathbf{M}$. Since $a\in\mathbf{P}$, $u_\alpha^* a u_\alpha = a$ for all $\alpha\in G$. Thus for any $\sum'_{\alpha\in G}X_\alpha U_\alpha\in(\mathbf{M}, G)$,

1) The fixed algebra of G in $\mathbf{M}$ is the subalgebra of $\mathbf{M}$ composed of all elements $a\in\mathbf{M}$ such that $a^\alpha=a$ for all $\alpha\in G$ ([3: Definition 2]).

$$(AU_\sigma^*)\left(\sum_{\alpha\in G}' X_\alpha U_\alpha\right) = \sum_{\alpha\in G}' AX_\alpha^\sigma U_{\sigma^{-1}\alpha} = \sum_{\alpha\in G}' X_\alpha AU_{\alpha\sigma^{-1}}$$

$$= \sum_{\alpha\in G}' X_\alpha AU_\alpha U_\sigma^* = \sum_{\alpha\in G}' X_\alpha U_\alpha AU_\sigma^* = \left(\sum_{\alpha\in G}' X_\alpha U_\alpha\right)(AU_\sigma^*)$$

where $A = a\otimes 1$. Hence $AU_\sigma^* \in (\mathbf{M}, G)\cap(\mathbf{M}, G)'$. On the other hand,

$$AU_\sigma^*(\varphi\otimes\varepsilon_e) = au_\sigma^*\varphi\otimes\varepsilon_{\sigma^{-1}}.$$

where φ is a separating and generating trace vector for $\mathbf{M}$, and AU_σ^* is not the scalar multiple of the identity operator on $\mathbf{H}\otimes l_2(G)$.

Sufficiency. Suppose that the condition in Theorem 1 is satisfied. If $\sum_{\alpha\in G}' X_\alpha U_\alpha$ is contained in the center of $(\mathbf{M}, G)$,

$$\left(\sum_{\alpha\in G}' X_\alpha U_\alpha\right)(AU_\sigma) = (AU_\sigma)\left(\sum_{\alpha\in G}' X_\alpha U_\alpha\right)$$

for all $A = a\otimes 1 \in \mathbf{M}\otimes\mathbf{I}$ and $\sigma\in G$. Then

$$\left(\sum_{\alpha\in G}' X_\alpha U_\alpha\right)(AU_\sigma) = \sum_{\alpha\in G}' X_\alpha A^{\alpha^{-1}} U_{\alpha\sigma}$$

and

$$(AU_\sigma)\left(\sum_{\alpha\in G}' X_\alpha U_\alpha\right) = \sum_{\alpha\in G}' AX_\alpha^{\sigma^{-1}} U_{\alpha\sigma}.$$

Thus we have $ax_\alpha^{\sigma^{-1}} = xa_\alpha^{\alpha^{-1}}$ for each $\alpha\in G$ where $X_\alpha = x_\alpha\otimes 1$. Take $\sigma = e$, and we have

$$ax_\alpha = x_\alpha a^{\alpha^{-1}} \qquad \text{for all } a\in\mathbf{M}.$$

Hence by the assumption $x_\alpha = \lambda_\alpha 1$ for all $\alpha\in G$, $\alpha\neq e$ where λ_α are scalars, and so $a = a^{\alpha^{-1}}$ for all $\alpha\in G$, $\alpha\neq e$ if $\lambda_\alpha\neq 0$ for $\alpha\neq e$, which contradicts to the arbitrariness of $a\in\mathbf{M}$. From the relation $ax_\alpha = x_\alpha a^{\alpha^{-1}}$ for all $a\in\mathbf{M}$, x_e is the scalar multiple of the identity operator on $\mathbf{H}$, and $(\mathbf{M}, G)$ is a factor.

As a corollary of Theorem 1, we obtain the slight improvement of the example in [4]:

COROLLARY 1. *Let $\mathbf{M}$ be a finite factor with the invariant $C = 1$ on a Hilbert space and α a non-trivial automorphism of $\mathbf{M}$. Let G be a cyclic group generated by α. Then α is outer if the crossed product $(\mathbf{M}, G)$ is a factor. In particular, when the order of α is 2 or 3, $(\mathbf{M}, G)$ is a factor if and only if G is outer.*

PROOF. Suppose that $(\mathbf{M}, G)$ is a factor and α is inner. Then there exists a unitary operator $u\in\mathbf{M}$ such that $u^*au = a^\alpha$ for all $a\in\mathbf{M}$. Since $(u)^{\alpha^n} = (u^*)^n u(u)^n = u$ for all $n = 0, \pm 1, \pm 2, \dots$, u is contained in the fixed algebra of G in $\mathbf{M}$. Moreover we have

$$au = ua^{\alpha} \qquad \text{for all } a \in \mathbf{M}.$$

Thus, by Theorem 1, $(\mathbf{M}, G)$ is not a factor which is a contradiction, and α is an outer automorphism of $\mathbf{M}$.

To prove the second part of our assertion, it is sufficient to show the "only if" part because the "if part" is known ([1], [4]). Suppose that $(\mathbf{M}, G)$ is a factor and $\alpha^3 = e$. α is an outer automorphism of $\mathbf{M}$ as shown above, and since $\alpha^2 = \alpha^{-1}$, α^2 is also an outer automorphism of $\mathbf{M}$. Thus G is outer. If $\alpha^2 = e$, it is obvious that G is outer.

The case where $\alpha^2 = e$ is nothing but the example in [3].

By virtue of Corollary 1 we can prove the following Theorem which is closely related to [2] and sharpens the results in [2].

THEOREM 2. *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be finite factors with the invariants* $C = 1$, *and let* G *and* H *be groups of outer automorphisms of* $\mathbf{M}$ *and* $\mathbf{N}$ *respectively. Then* $G \times H$[2] *is a group of outer automorphisms of* $\mathbf{M} \otimes \mathbf{N}$.

PROOF. Let $(\alpha, \beta) \in G \times H$ be an arbitrary element which is different from the unit (e, e) of $G \times H$, and let $\mathfrak{G}_{(\alpha,\beta)}$ be a cyclic group generated by (α, β). Then it is sufficient to show that the crossed product $(\mathbf{M}\otimes\mathbf{N}, \mathfrak{G}_{(\alpha,\beta)})$ is a factor by Corollary 1. By [2: Theorem 1] and [3: Theorem] we have 1

$$\begin{aligned}(\mathbf{M}\otimes\mathbf{N}, \mathfrak{G}_{(\alpha,\beta)}) \cap (\mathbf{M}\otimes\mathbf{N}, \mathfrak{G}_{(\alpha,\beta)})' &\subseteqq (\mathbf{M}\otimes\mathbf{N}, G \times H) \cap (\mathbf{M}\otimes \mathbf{N}, \{(e,e)\})' \\ &= (\mathbf{M}, G) \otimes (\mathbf{N}, H) \cap ((\mathbf{M}, \{e\}) \otimes (\mathbf{N}, \{e\}))' \\ &= ((\mathbf{M}, G) \cap (\mathbf{M}, \{e\})') \otimes ((\mathbf{N}, H) \cap (\mathbf{N}, \{e\})').\end{aligned}$$

On the other hand by [4: Theorem 3], $(\mathbf{M}, G) \cap (\mathbf{M}, \{e\})'$ (resp. $(\mathbf{N}, H) \cap (\mathbf{N}, \{e\})'$) coincides with the center of $(\mathbf{M}, \{e\})$ (resp. $(\mathbf{N}, \{e\})$), because G (resp. H) is outer. Thus $(\mathbf{M}\otimes\mathbf{N}, \mathfrak{G}_{(\alpha,\beta)})$ is a factor, and the proof is completed.

REMARK. Theorem 2 holds when $\mathbf{M}$ and $\mathbf{N}$ are semi-finite factors. A sketch of the proof is as follows. Let $\mathbf{M}$ be a standard factor on a Hilbert space $\mathbf{H}$ and G a group of automorphisms of $\mathbf{M}$. Then Lemmas 1 and 2 in [4] remain true, and so we can define the crossed product as the same way as in the case of finite factor, and Lemma 5 and Theorem 3 in [4] are also true[3]. Hence we can easily seen that Corollary 1 is valid and the same computations as the proof of Theorem 2 are available.

3. Let $\mathbf{M}$ be a finite factor with the invariant $C = 1$ on a Hilbert space $\mathbf{H}$ and G a group of automorphisms of $\mathbf{M}$. Let $\mathbf{P}$ be the fixed algebra of G in

2) For the definition of $G \times H$, see Lemma 2 in [2].
3) These facts were pointed out by N. Suzuki when he published the paper [4].

$\mathbf{M}$. Then $(\mathbf{P}, G)$ means the W^*-subalgebra of the crossed product $(\mathbf{M}, G)$ generated by all finite linear combinations $\sum_i A_i U_{\alpha_i}$ where $A_i = a_i \otimes 1$, $\alpha_i \in G$. It is easily seen that each element in $(\mathbf{P}, G)$ can be expressed uniquely in the form $\sum_{\alpha\in G} A_\alpha U_\alpha$ where $A_\alpha = a_\alpha \otimes 1$, $a_\alpha \in \mathbf{P}$. The set of all operators $a \otimes 1$ on $\mathbf{H} \otimes l_2(G)$ such as $a \in \mathbf{P}$ will be denoted by $\mathbf{P} \otimes \mathbf{I}$.

LEMMA 1. *If G is abelian*

$$(\mathbf{P}, G) = (\mathbf{M}, G) \cap R(U_\alpha \,|\, \alpha \in G)'.$$

PROOF. We first recall that $a \in \mathbf{P}$ if and only if $a u_\alpha = u_\alpha a$ for all $\alpha \in G$. Let $\sum'_{\alpha\in G} A_\alpha U_\alpha$ be an element in $(\mathbf{P}, G)$. Then for each $\sigma \in G$ we have

$$\begin{aligned} U_\sigma\left(\sum\nolimits'_{\alpha\in G} A_\alpha U_\alpha\right) &= \sum\nolimits'_{\alpha\in G} U_\sigma A_\alpha U_\alpha = \sum\nolimits'_{\alpha\in G} A_\alpha U_{\sigma\alpha} \\ &= \sum\nolimits'_{\alpha\in G} A_\alpha U_{\alpha\sigma} = \left(\sum\nolimits'_{\alpha\in G} A_\alpha U_\alpha\right) U_\sigma, \end{aligned}$$

and so

$$(\mathbf{P}, G) \subseteqq (\mathbf{M}, G) \cap R(U_\alpha \,|\, \alpha \in G)'.$$

Conversely, if we take an arbitrary element $\sum'_{\alpha\in G} A_\alpha U_\alpha$ in $(\mathbf{M}, G) \cap R(U_\alpha \,|\, \alpha \in G)'$,

$$U_\sigma\left(\sum\nolimits'_{\alpha\in G} A_\alpha U_\alpha\right) = \left(\sum\nolimits'_{\alpha\in G} A_\alpha U_\alpha\right) U_\sigma \text{ for all } \sigma \in G.$$

Thus

$$\sum\nolimits'_{\alpha\in G} A_\alpha^{\sigma^{-1}} U_{\alpha\sigma} = \sum\nolimits'_{\alpha\in G} A_\alpha U_{\alpha\sigma} \text{ for all } \sigma \in G,$$

hence we have $a_\alpha = a_\alpha^{\sigma^{-1}}$ for each $\alpha \in G$ where $A_\alpha = a_\alpha \otimes 1 \in \mathbf{M} \otimes \mathbf{I}$. Since $\sigma \in G$ is arbitrary, $a_\alpha \in \mathbf{P}$ for all $\alpha \in G$. This proves that $\sum'_{\alpha\in G} A_\alpha U_\alpha \in (\mathbf{P}, G)$ and

$$(\mathbf{P}, G) \supseteqq (\mathbf{M}, G) \cap R(U_\alpha \,|\, \alpha \in G)',$$

So we have $(\mathbf{P}, G) = (\mathbf{M}, G) \cap R(U_\alpha \,|\, \alpha \in G)'$.

As an immediate consequence of Lemma 1, we have the following result.

COROLLARY 2. *If G is ergodic and abelian, $R(U_\alpha \,|\, \alpha \in G)$ is a maximal abelian W^*-subalgebra of the factor $(\mathbf{M}, G)$.*

In fact, the ergodicity of G leads to $\mathbf{P} = \{\lambda 1\}$, and $(\mathbf{P}, G) = R(U_\alpha \,|\, \alpha \in G)$. Thus, by Lemma 1,

$$R(U_\alpha \,|\, \alpha \in G) = (\mathbf{M}, G) \cap R(U_\alpha \,|\, \alpha \in G)'.$$

This proves Corollary 2 since $R(U_\alpha | \alpha \in G)$ is abelian.

Next lemma is a non-abelian analogue of Lemma 1.

LEMMA 2. *Assume that G satisfies the condition: every non-trivial conjugate class of G it infinite, that is for every $\alpha \in G$ other than the identity, the class $\{\sigma\alpha\sigma^{-1} | \sigma \in G\}$ is infinite. Then we have*

$$(\mathbf{M}, G) \cap R(U_\alpha | \alpha \in G)' = \mathbf{P} \otimes \mathbf{I}.$$

PROOF. Let $A = \sum'_{\alpha \in G} A_\alpha U_\alpha$ be an arbitrary element in $(\mathbf{M}, G) \cap R(U_\alpha | \alpha \in G)'$. Then

$$AU_\sigma = U_\sigma A \qquad \text{for all } \sigma \in G.$$

Since $AU_\sigma = \sum'_{\alpha \in G} A_\alpha U_{\alpha\sigma} = \sum'_{\alpha \in G} A_{\sigma\alpha\sigma^{-1}} U_{\sigma\alpha}$ and $U_\sigma A = \sum'_{\alpha \in G} A_\alpha^{\sigma^{-1}} U_{\sigma\alpha}$, we have

$$(*) \qquad a_\alpha^{\sigma^{-1}} = a_{\sigma\alpha\sigma^{-1}} \qquad \text{for all } \sigma \in G \text{ and } \alpha \in G,$$

where $A_\alpha = a_\alpha \otimes 1 \in \mathbf{M} \otimes \mathbf{I}$. Suppose that $a_{\alpha_0} \neq 0$ for an $\alpha_0 \in G$, $\alpha_0 \neq e$. Let φ be a separating and generating trace vector for $\mathbf{M}$. Then, by our hypothesis the conjugate class $\{\sigma\alpha_0\sigma^{-1} | \sigma \in G\}$ is infinite. As $\|a\varphi\| = \|a^\sigma\varphi\|$ for all $a \in \mathbf{M}$ and $\sigma \in G$, we have by (*)

$$\|a_{\sigma\alpha_0\sigma^{-1}}\varphi\| = \|a_{\alpha_0}^{\sigma^{-1}}\varphi\| = \|a_{\alpha_0}\varphi\| \text{ for all } \sigma \in G.$$

Thus we have

$$\sum\nolimits'_{\alpha \in G} \|a_\alpha\varphi\|^2 = \infty.$$

which is a contradiction. Hence $a_\alpha = 0$ for all $\alpha \in G$, $\alpha \neq e$, and $A = A_e \in \mathbf{P} \otimes \mathbf{I}$ because, again by (*) $a_e^{\sigma^{-1}} = a_e$ for all $\sigma \in G$, and so $(\mathbf{M}, G) \cap R(U_\alpha | \alpha \in G)' \subseteqq \mathbf{P} \otimes \mathbf{I}$. On the other hand, it is obvious that $\mathbf{P} \otimes \mathbf{I} \subseteqq (\mathbf{M}, G) \cap R(U_\alpha | \alpha \in G)'$ since $au_\alpha = u_\alpha a$ for all $a \in \mathbf{P}$ and $\alpha \in G$. Therefore we have

$$(\mathbf{M}, G) \cap R(U_\alpha | \alpha \in G)' = \mathrm{P} \otimes \mathbf{I}.$$

By Lemma 2 we have the following theorem.

THEOREM 3. *Let $\mathbf{M}$ be a finite factor with the invariant $C = 1$ and G a group of automorphisms of $\mathbf{M}$ whose non-trivial conjugate classes are all infinite. Then $(\mathbf{M}, G)$ is a factor.*

PROOF. Let $A = \sum'_{\alpha \in G} A_\alpha U_\alpha$ be an arbitrary element in the center of $(\mathbf{M}, G)$. Since $(\mathbf{M}, G) \cap (\mathbf{M}, G)' \subseteqq (\mathbf{M}, G) \cap R(U_\alpha | \alpha \in G)'$, $A = A_e \in \mathbf{P} \otimes \mathbf{I}$, where $\mathbf{P}$ is the fixed algebra of G in $\mathbf{M}$ by Lemma 2. Moreover A commutes with all $x \otimes 1 \in \mathbf{M} \otimes \mathbf{I}$, and so we have $a_e x = x a_e$ for all $x \in \mathbf{M}$ where $A_e = a_e \otimes 1$. Thus $a_e \in \mathbf{M} \cap \mathbf{M}'$. Hence A is the scalar multiple of the identity operator on

$\mathbf{H} \otimes l_2(G)$, and $(\mathbf{M}, G)$ is a factor.

REMARK. Theorem 3 can be slightly generalized as follows. Assume that G has a subgroup G_0 such that for every element $\alpha \in G$ other than the identity, the set $\{\sigma\alpha\sigma^{-1} | \sigma \in G_0\}$ is infinite. Then the commutant of $(\mathbf{M}, G_0)$ in the crossed product $(\mathbf{M}, G)$ is the scalar multiples of the identity operator on $\mathbf{H} \otimes l_2(G)$, where $(\mathbf{M}, G_0)$ is a subalgebra of $(\mathbf{M}, G)$ composed of all $A = \sum'_{\alpha \in G_0} A_\alpha U_\alpha \in (\mathbf{M}, G)$. In particular $(\mathbf{M}, G)$ is a factor.

This is a non-commutative version of Lemma 3 in [5].

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