ON THE EXISTENCE OF RIEMANN METRICS ASSOCIATED WITH A 2-FORM OF RANK $2r$

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1. Introduction. In his paper [3], S.Sasaki proved that if a $(2n+1)$-dimensional manifold M^{2n+1} admits a skew symmetric tensor field ϕ_{ij} of rank $2n$ and a vector field ξ^i such that $\phi_{ij}\xi^j = 0$, then we can find a Riemann metric g_{ij} on M^{2n+1} such that the tensor fields $\phi^i_j = g^{ik}\phi_{kj}$, ξ^i, $\eta_j = g_{jk}\xi^k$, and g_{ij} define a (ϕ, ξ, η, g)-structure on M^{2n+1}, i. e., they satisfy the following relations:

$$\begin{aligned}
\text{rank } |\phi^i_j| &= 2n,\\
\xi^i\eta_i &= 1,\\
\phi^i_j\xi^j &= 0,\\
\phi^i_j\eta_i &= 0,\\
\phi^i_j\phi^j_k &= -\delta^i_k + \xi^i\eta_k,\\
g_{ij}\xi^j &= \eta_i,\\
g_{ij}\phi^i_h\phi^j_k &= g_{hk} - \eta_h\eta_k.
\end{aligned}$$

But, from his proof, it seems difficult to know about the differentiability of the metric tensor g_{ij} defined by him, although it is clearly continuous.

In this paper, we shall give another proof of the existence of such metric with the differentiability of the same class as that of ϕ_{ij} as a corollary of a more general theorem to the effect that for a skew symmetric tensor field ϕ_{ij} of rank $2r$, we can find a Riemann metric g_{ij} with the differentiability of the same class as that of ϕ_{ij} such that the non-zero characteristic values of ϕ_{ij} with respect to this metric are only i and $-i$. There is an analogous theorem in the case of almost complex structures, namely, if an even dimensional manifold M^{2n} admits a skew symmetric tensor field ϕ_{ij} of rank $2n$, then we can find a Riemann metric g_{ij} such that tensors $\phi^i_j = g^{ik}\phi_{kj}$ and g_{ij} define an almost Hermitian structure on M^{2n}. And this theorem also follows as a corollary of the above theorem.

In order to prove the theorem we shall use the representation of $GL(n, R)$ as a product space in [1] (cf. [1], p. 14), i. e., the fact that any regular matrix τ may be written in one and only one way as the product $\tau = \sigma\alpha$ of an orthogonal matrix σ and a positive definite symmetric matrix α, and the factors σ, α of the decomposition of τ are continuous functions of τ. But we must use the fact that σ and α are analytic functions of τ, so we first prove this in §2, and in §3 we shall prove our results.

2. Algebraic preliminaries. In this section we shall prove a theorem which will be needed to prove our results. Let $H(n)$ be the set of all positive definite symmetric matrices of n real variables. Then it is a regularly imbedded analytic submanifold of general linear group of n variables $GL(n, R)$ and its analytic structure is defined by the components $a_{ij}(i \leqq j)$ of its element. And let $O(n)$ be the orthogonal group of n real variables with usual analytic structure. Then we have the following theorem.

THEOREM 1. *If φ is a mapping from the product manifold $O(n) \times H(n)$ to the group $GL(n, R)$ defined by*

$$\varphi(\sigma, \alpha) = \sigma\alpha \qquad \sigma \in O(n),\ \alpha \in H(n),$$

then φ is an analytic mapping, and its differential mapping $d\varphi$ is everywhere onto isomorphism.

In order to prove this theorem we begin with the following

LEMMA 1. *Let α be an element of $H(n)$, then the characteristic values of the linear mapping defined by* $\mathrm{ad}(\alpha): \mathfrak{gl}(n, R) \to \mathfrak{gl}(n, R)$ $(a \to \alpha\, a\, \alpha^{-1})$ *are all positive numbers.*

PROOF. Since $\alpha \in H(n)$, its characteristic values are all positive numbers, and we denote them by $\lambda_i (i = 1, \ldots\ldots, n)$. By virtue of a classical result (cf. [1], p.12), we can find an orhogonal matrix ν such that $\nu\alpha\nu^{-1}$ is a diagonal matrix δ, and the coefficients of the diagonal of δ are λ_i's. Since $\mathrm{ad}(\alpha) = \mathrm{ad}(\nu)^{-1}\mathrm{ad}(\delta)\ \mathrm{ad}(\nu)$, the characteristic values of $\mathrm{ad}(\alpha)$ are equal to those of $\mathrm{ad}(\delta)$. Hence, by direct calculation, we get that the characteristic values of $\mathrm{ad}(\alpha)$ are equal to n^2 numbers $\lambda_i\lambda_j^{-1}(i, j = 1, \ldots\ldots, n)$. So the characteristic values of $\mathrm{ad}(\alpha)$ are all positive numbers.

COROLLARY. *Let A be a skew symmetric matrix of n variables and $\alpha \in H(n)$. If $A\alpha$ is a symmetric matrix, then $A = 0$.*

In fact, if $A\alpha$ is a symmetric matrix we have

$$A\alpha = {}^t\alpha{}^tA = -\alpha\ A,$$

which shows

$$\mathrm{ad}\,(\alpha)A = -\ A.$$

So we get $A = 0$ by virtue of the above lemma.

Now, using the above corollary, we shall give a proof of Theorem 1.

PROOF OF THEOREM 1. Since $O(n)$ and $H(n)$ are analytic submanifolds of $GL(n, R)$ and the product of two elements of $GL(n, R)$ is an analytic mapping, φ is an analytic mapping. In order to prove the latter part, observing that the both dimensions of $O(n) \times H(n)$ and $GL(n, R)$ are equal to n^2, we only have to prove that the differential mapping $d\varphi$ is univalent. Let x be a tangent vector of $O(n) \times H(n)$ at (σ, α) which is defined by a parametrized curve $(\sigma\exp(tA), \alpha + tB)$ where A is a skew symmetric matrix and B is a symmetric matrix. Then $d\varphi(x)$ is given by

$$d\varphi(x) = \lim_{t\to 0} \frac{\sigma \exp(tA)(\alpha + tB) - \sigma\alpha}{t} = \sigma A\alpha + \sigma B.$$

So if $d\varphi(x) = 0$, we have $\sigma A\alpha + \sigma B = 0$, and therefore

$$A\alpha = -B.$$

Hence $A\alpha$ is a symmetric matrix. By virtue of the above corollary, $A = 0$ and so $B = 0$ which shows that $d\varphi$ is univalent.

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Now we consider a continuous mapping $\psi : GL(n, R) \to O(n) \times H(n)$ given by the decomposition $\tau = \sigma\alpha$ where $\tau \in GL(n, R)$, $\sigma \in O(n)$ and $\alpha \in H(n)$ stated in the Proposition 1, p. 14, [1]. Clearly ψ is the inverse mapping of φ. So from the Theorem 1 and the Proposition 3, p. 80, [1], ψ is an analytic mapping. Summarizing these, we get the following theorem.

THEOREM 2. *Any regular matrix τ (of real coefficients) may be written in one and only one way as the product $\tau = \sigma\alpha$ of an orthogonal matrix σ and a positive definite symmetric matrix α. And the mapping from $GL(n, R)$ to $O(n) \times H(n)$ defined by this decomposition gives an analytic homeomorphism of these two manifolds with respect to the usual analytic structures.*

3. The existence theorem. Let M^n be an n-dimensional differentiable manifold of class $C^k (k = 2, 3, \ldots\ldots, \infty)$, and let ϕ_{ij} be a skew symmetric tensor field of class $C^l (l \leqq k - 1)$ of rank $2r$, i. e., $\phi = \frac{1}{z} \phi_{ij} dx^i \wedge dx^j$ is a 2-form such that $\phi^r \neq 0$ and $\phi^{r+1} = 0$. We can find a Riemann metric on M^n of class C^{k-1}, say h_{ij}. Since $\operatorname{rank}|\phi_{ij}| = 2r$, if we set $D_P = \{X^j; X^j \in T_P(M^n), \phi_{ij}X^j = 0\}$ for $P \in M^n$, then the mapping $P \to D_P$ defines an $(n-2r)$-dimensional distribution D with differentiability of class C^l. Hence, for any point of M^n, we can find its open neighborhood U and a field of orthonormal frames $P' \mathfrak{e}_1 \ldots\ldots \mathfrak{e}_{2r} \xi_{2r+1} \ldots\ldots \xi_n$, $P' \in U$, of class C^l defined on U such that their last $(n - 2r)$-vectors $\xi_{2r+1}, \ldots\ldots, \xi_n$ belong to the distribution D. Let $\mathfrak{U} = \{U\}$ be a covering of M^n by such neighborhoods U's with the field of orthonormal frames satisfying the above conditions. Then, if $U \cap V \neq 0$, the transformation of the field of orthonormal frames on U to that on V in $U \cap V$ is of the following form

$$\gamma_{UV} = \begin{pmatrix} \gamma' & 0 \\ 0 & \gamma'' \end{pmatrix}, \tag{1}$$

where $\gamma' \in O(2r)$ and $\gamma'' \in O(n-2r)$. Now we begin by stating the following lemma which follows immediately from the fact that $\gamma_{UV} \in O(n)$ without proof.

LEMMA 2. *We suppose that on each U a tensor field of degree 2 is* *given, and we denote the matrix whose components are the components of this tensor field relative to the frames on U by α_U. Then the set $\{\alpha_U,\ U \in \mathfrak{U}\}$ defines a tensor field globally defined on M^n if and only if $\gamma_{UV}\alpha_U{}^t\gamma_{UV} = \alpha_V$ on $U \cap V$ for all pairs $U,\ V \in \mathfrak{U}$ such that $U \cap V \neq 0$.*

REMARK. Since $\gamma_{UV} \in O(n)$, we can consider the tensor given by the set $\{\alpha_U\}$ satisfying the above conditions as a tensor of type $(2, 0)$, $(1, 1)$ or $(0, 2)$.

Next, let ϕ_U be a matrix whose components are the components of the tensor ϕ_{ij} relative to the frame on U. Then ϕ_U is of the form

$$\phi_U = \begin{pmatrix} \phi'_U & 0 \\ 0 & 0 \end{pmatrix}, \tag{2}$$

where ϕ'_U is a regular $(2r, 2r)$ skew symmetric matrix. Then, by virtue of the Theorem 2, we can write

$$\phi'_U = \alpha'_U \cdot \beta'_U,$$

where $\alpha'_U \in O(2r)$, $\beta'_U \in H(2r)$, and the components of α'_U and β'_U are analytic functions of ϕ'_U. So, if we set

$$\alpha_U = \begin{pmatrix} \alpha'_U & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \beta_U = \begin{pmatrix} \beta'_U & 0 \\ 0 & E_{n-2r} \end{pmatrix}, \tag{3}$$

where E_{n-2r} denotes a unit matrix of $(n-2r)$-variables, α_U and β_U define two tensor fields of class C^l on U. Since α'_U is skew symmetric, we have

$$-\alpha'_U\beta'_U = \beta'_U{}^t\alpha'_U,$$

and so, using the fact $\alpha'_U \in O(2r)$, we get

$$\beta'_U = -\alpha'^2_U \cdot {}^t\alpha'_U\beta'_U\alpha'_U.$$

On the other hand $-\alpha'^2_U \in O(2r)$, ${}^t\alpha'_U\beta'_U\alpha'_U \in H(2r)$, therefore by the uniqueness of this decomposition we get

$$\alpha'^2_U = -E_{2r},\ \alpha'_U\beta'_U = \beta'_U\alpha'_U.$$

and the non-zero characteristic roots of α_U are only i and $-i$. Next, if we consider the relation between ϕ_U and ϕ_V, from (1) and (2), we get

$$\gamma'\phi'_U{}^t\gamma' = \phi'_V.$$

And so

$$\alpha'_V\beta'_V = \gamma'\alpha'_U\beta'_U{}^t\gamma' = \gamma'\alpha'_U{}^t\gamma' \cdot \gamma'\beta'_U{}^t\gamma'.$$

Hence, by virtue of the uniqueness of this decomposition, we have

$$\alpha'_V = \gamma'\alpha'_U{}^t\gamma', \qquad \beta'_V = \gamma'\beta'_U{}^t\gamma',$$

which show together with (1) and (3) that

$$\alpha_V = \gamma_{UV}\alpha_U{}^t\gamma_{UV}, \qquad \beta_V = \gamma_{UV}\beta_U{}^t\gamma_{UV}$$

hold good. Therefore, from the Lemma 2, the sets $\{\alpha_U,\ U \in \mathfrak{U}\}$ and $\{\beta_U,\ U \in \mathfrak{U}\}$ define two global tensor fields of class C^l. Let ϕ^i_j and g_{ij} be the tensor fields defined by $\{\alpha_U, U \in \mathfrak{U}\}$ and $\{\beta_U, U \in \mathfrak{U}\}$, respectively. Then, since β'_U, and so β_U is a positive definite matrix, g_{ij} defines a Riemann metric on M^n of class C^l, and the non-zero characteristic roots of ϕ_{ij} with respect to g_{ij}, i. e., those of $g^{ij}\phi_{jk} = \phi^i_k$ are only i and $-i$.

Summarizing these, we get the following

THEOREM 3. *If an n-dimensional manifold M^n of class $C^k (k = 2, 3, \ldots, \infty)$ admits a skew symmetric tensor field ϕ_{ij} of class $C^l (l \leqq k-1)$ of rank $2r$, then we can find a Riemann metric g_{ij} of class C^l on M^n such that the non-zero characteristic roots of ϕ_{ij} with respect to g_{ij} are only i and $-i$.*

COROLLARY 1. *If a $2n$-dimensional manifold M^{2n} of class C^k admits a skew symmetric tensor field ϕ_{ij} of class $C^l (l \leqq k-1)$ of rank $2n$, then we can find a Riemann metric g_{ij} of class C^l on M^{2n} such that the tensor fields $g^{ik}\phi_{kj}$ and g_{ij} give an almost Hermitian structure on M^{2n}.*

COROLLARY 2. *If a $(2n+1)$-dimensional orientable manifold M^{2n+1} of class C^k admits a skew symmetric tensor field ϕ_{ij} of class C^l $(l \leqq k-1)$ of rank $2n$, then we can find a Riemann metric g_{ij} of class C^l on M^{2n+1} and a vector field ξ^i of class C^l such that the tensor fields $g^{ik}\phi_{kj}$, ξ^i, $g_{jk}\xi^k$ and g_{ij} give a (ϕ, ξ, η, g)-structure on M^{2n+1}.*

In fact, let g_{ij} be a Riemann metric associated with ϕ_{ij} given by the Theorem 3. Then, by virtue of S.Ishihara's result (cf. [2], p. 450), if we set

$$w^i = \frac{1}{\sqrt{g}} \varepsilon^{i_1 \cdots i_{2n} i} \phi_{i_1 i_2} \cdots \phi_{i_{2n-1} i_{2n}},$$

where $g = |g_{ij}|$ and $\varepsilon^{i_1 \cdots i_{2n+1}}$ is equal to $+1$ if $(i_1 \ldots\ldots i_{2n+1})$ is an even permutation; equal to -1 if $(i_1 \ldots\ldots i_{2n+1})$ is an odd permutation; equal to 0 otherwise, the vector field w^i is everywhere non-zero and $\phi_{ij}w^j = 0$. So, if we put

$$\xi^i = w^i / \sqrt{g_{kl} w^k w^l},$$

we can easily verify that $g^{ik}\phi_{kj}$, ξ^i, $g_{jk}\xi^k$ and g_{ij} give a (ϕ, ξ, η, g)-structure on M^{2n+1}.

BIBLIOGRAPHY

[1] C. CHEVALLEY, Theory of Lie groups. Princeton University Press, 1940.

[2] S. ISHIHARA, On a tensor field ϕ^i_h satisfiyng $\phi^p = \pm I$, Tôhoku Math. J. 13(1961)443-454.

[3] S. SASAKI, On differentiable manifolds with certain structures which are closely related to almost contact structures I, Tôhoku Math. J. 12(1960) 459-476.

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