ON THE DIFFERENTIAL GEOMETRY OF TANGENT BUNDLES OF RIEMANNIAN MANIFOLDS II

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1. Introduction. Let M^n be an n-dimensional Riemannian manifold and $T(M^n)$ be its tangent bundle. We can introduce to $T(M^n)$ a natural Riemannian metric from the Riemannian metric of M^n.[1]

Now, let us denote by $T_1(M^n)$ the set of all unit tangent vectors of M^n. As we can reduce the structural group of $T(M^n)$ to $O(n)$, $T_1(M^n)$ may be regarded as a sphere bundle. We shall call it the tangent sphere bundle of M^n. As $T_1(M^n)$ is a submanifold of $T(M^n)$, it has a Riemannian metric naturally induced from that of $T(M^n)$. In this paper I shall study on the differential geometry of this $(2n-1)$-dimensional Riemannian manifold $T_1(M^n)$ regarding it as a submanifold of $T(M^n)$, because it is rather simple analytically.

2. The Riemannian metric and the connection of $T_1(M^n)$. Let U be a coordinate neighborhood of M^n with coordinates x^i such that $U \times E^n$ is diffeomorphic with $\pi^{-1}(U)$, where E^n is the vector space which is the standard fibre of $T(M^n)$ and π is the natural projection of $T(M^n)$ onto M^n. If we denote the components of tangent vector of M^n at $x^i \in U$ with respect to the natural frame $\dfrac{\partial}{\partial x^i}$ by v^i, then the ordered set of variables (x^i, v^i) can be regarded as local coordinates of $\pi^{-1}(U)$ which is an open subset of $T(M^n)$.

Suppose the Riemannian metric of M^n is given in U by the quadratic form

$$ds^2 = g_{jk}(x)dx^j dx^k. \tag{2. 1}$$

Then the Riemannian metric of $T(M^n)$ is given in $\pi^{-1}(U)$ by the quadratic form

$$d\sigma^2 = g_{jk}(x)dx^j dx^k + g_{jk}(x)Dv^j Dv^k,\text{[2]} \tag{2. 2}$$

where Dv^j means the covariant differential of v^j, i.e.

$$Dv^j = dv^j + \begin{Bmatrix} j \\ lm \end{Bmatrix} v^l dx^m. \tag{2. 3}$$

The components of the fundamental metric tensor of $T(M^n)$ in $\pi^{-1}(U)$ can be

1) cf. S. Sasaki, On the differential geometry of tangent bundles of Riemannian manifolds, Tôhoku Math. J. 10 (1958) pp. 338-354. This paper will be cited as I.

2) Throughout this paper, we use the same notation as in the paper I.

given by

$$(2.4)\qquad \begin{cases} G_{jk} = g_{jk} + g_{\beta\gamma} \begin{Bmatrix} \beta \\ \mu j \end{Bmatrix} \begin{Bmatrix} \gamma \\ \nu k \end{Bmatrix} v^\mu v^\nu, \\ G_{j\ n+k} = [\lambda j,\ k] v^\lambda, \\ G_{n+j\ n+k} = g_{jk}. \end{cases}$$

The geometrical meaning of the metric (2.2) is as follows: Let (x^i, v^i) and $(x^i + dx^i,\ v^i + dv^i)$ be indefinitely nearby points in $\pi^{-1}(U)$. In $U \subset M^n$, we consider the tangent vector $v^i + dv^i$ of M^n at the point $x^i + dx^i$ and translate it parallelly to the point x^i by Levi-Civita's parallelism. If we denote the angle between the tangent vector thus obtained and the tangent vector v^i at x^i by $d\theta$ and the length of the vector v^i by v, then

$$(2.5)\qquad d\sigma^2 = ds^2 + v^2 d\theta^2.$$

From (2. 4), we can easily see that the length of the horizontal component $(dx^i, -\begin{Bmatrix} i \\ jk \end{Bmatrix} v^j dx^k)$ of the vector $(dx^i,\ dv^i)$ is ds^2 and the length of the vertical component $(0, Dv^i)$ of the vector (dx^i, dv^i) is $v^2 d\theta^2$. So (2. 5) is nothing but the local Phythagorean theorem.

Now, let us denote the natural projection $T_1(M^n) \to M^n$ by π_1. Then $\pi_1^{-1}(U)$ is given, as an $(2n-1)$-dimensional submanifold of $\pi^{-1}(U)$, by

$$(2.6)\qquad g_{jk}(x) v^j v^k = 1.$$

Hence, the Riemannian metric of $T_1(M^n)$ naturally induced from that of $T(M^n)$ is given geometrically by

$$(2.7)\qquad d\sigma^2 = ds^2 + d\theta^2.$$

The covariant components of the normal vector to $T_1(M^n)$ at $(x^i, v^i) \in \pi_1^{-1}(U)$ is easily seen to be given by

$$(2.8)\qquad ([\lambda i, \mu] v^\lambda v^\mu,\ g_{ik} v^k).$$

The contravariant components of the last vector is easily calculated by means of

$$(2.9)\qquad \begin{cases} G^{jk} = g^{jk}, \\ G^{j\,n+k} = -\begin{Bmatrix} k \\ \mu l \end{Bmatrix} g^{jl} v^\mu, \\ G^{n+j\,n+k} = g^{jk} + g^{\beta\gamma} \begin{Bmatrix} j \\ \mu\beta \end{Bmatrix} \begin{Bmatrix} k \\ \nu\gamma \end{Bmatrix} v^\mu v^\nu, \end{cases}$$

and we get $(0,\ v^i)$ as the components of the unit normal vector of $T_1(M^n)$ at the point $(x^i,\ v^i) \in T_1(M^n)$.

Tangent vectors of $T_1(M^n)$ at the point $(x^i, v^i) \in T_1(M^n)$ are perpendicular to the normal vector $(0, v^i)$. So the necessary and sufficient condition that a tangent vector ξ of $T(M^n)$ at a point (x^i, v^i) of $T_1(M^n)$ be a tangent vector of $T_1(M^n)$ is that its components (ξ^i, ξ^{n+i}) satisfy the equation

$$g_{ij} v^i \left(\xi^{n+j} + \left\{ {j \atop hk} \right\} \xi^h v^k\right) = 0. \tag{2. 10}$$

The lift of a tangent vector ξ^i of M^n at a point $x^i \in U$ to (x^i, v^i) of $\pi_1^{-1}(U)$ is given by $\left(\xi^i, -\left\{ {i \atop jk} \right\} \xi^j v^k\right)$. So it is a tangent vector of $T_1(M^n)$ at (x^i, v^i). Hence we see that the tangent space of $T_1(M^n)$ at $(x^i, v^i) \in T_1(M^n)$ is a direct sum of the tangent $(n-1)$-space of the fibre

$$g_{jk}(x) v^j v^k = 1 \qquad x^i \text{ fixed}$$

and the horizontal n-space of $T(M^n)$ at (x^i, v^i).

In $T(M^n)$ every fibre is orthogonal to the horizontal n-space at every point of it. So in $T_1(M^n)$, every tangent $(n-1)$-space of a fibre is orthogonal to the horizontal n-spaces through the point. Hence $T_1(M^n)$ may be considered to have a connection defined by the restriction of horizontal n-spaces to points on $T_1(M^n)$ and we may speak of the lift of any tangent vector of M^n and the lift of any curve of M^n to $T_1(M^n)$.

Especially, we may speak of the GF-vector field and the geodesic flow in $T_1(M^n)$, because the former is nothing but the set of the lifts of unit tangent vectors v^i at the point $x^i \in M^n$ to the point (x^i, v^i) of $T_1(M^n)$ and the latter is the one parameter group of transformations generated by the trajectories of the GF-vector field.

3. Isometries and Killing vector fields in $T_1(M^n)$. Let f be a diffeomorphism of M^n onto itself. We have proved in the former paper I that the extension $\bar{f}$ of f to $T(M^n)$ is an isometry of $T(M^n)$ if and only if f is an isometry of M^n. If we restrict $\bar{f}$ to $T_1(M^n)$, then we get the following theorem:

THEOREM 1. *Suppose f is an isometry of a Riemannian manifold M^n, then the extended mapping $\bar{f}$ of f induces an isometry of the tangent sphere bundle $T_1(M^n)$.*

COROLLARY. *If a Riemannian manifold M^n admits an r-parameter Lie group of isometries, then the tangent sphere bundle $T_1(M^n)$ admits an r-parameter group of isometries too.*

Now, we shall give some theorems about Killing vector field.

THEOREM 2. *In order that the extension $\left(\xi^i, \frac{\partial \xi^i}{\partial x^j} v^j\right)$ in $T(M^n)$ of a vector field ξ^i of a Riemannian manifold M^n is always tangent to $T_1(M^n)$ at every point of $T_1(M^n)$, it is necessary and sufficient that $\xi^i(x)$ be a Killing* *vector field of* M^n.

PROOF. As the covariant components of the unit normal vector of $T_1(M^n)$ at $(x^i, v^i) \in T_1(M^n)$ are given by (2. 8), the condition in order that $(\xi^i, \frac{\partial \xi^i}{\partial x^j} v^j)$ is tangent to $T_1(M^n)$ at $(x^i, v^i) \in T_1(M^n)$ is written down as

$$[\lambda i,\mu] v^\lambda v^\mu \xi^i + g_{ik} v^k \frac{\partial \xi^i}{\partial x^j} v^j = 0, \tag{3. 1}$$

provided that

$$g_{jk}(x) v^j v^k = 1. \tag{3. 2}$$

The equation (3. 1) can be transformed easily to

$$\xi_{k,j}\ v^j v^k = 0. \tag{3. 3}$$

The equation (3. 3) holds for every (x^i, v^i) such that (3. 2) is true. So we can deduce easily

$$\xi_{j,k} + \xi_{k,j} = 0,$$

which is to be proved.

Now, let us consider an m-dimensional Riemannian manifold M^m and a regular submanifold M^{m-1} of it. We assume that V be a coordinate neighborhood of M^m at a point of M^{m-1}, $x^A(A, B, C = 1, 2, \cdots, m)$ are coordinates in V and

$$x^A = x^A(u^1, \cdots, u^{m-1}) \qquad (u^1, \cdots, u^{m-1}) \in D$$

are local parametric equations of M^{m-1} in V. We denote the fundamental tensor of M^m by G_{AB}.

Suppose ξ be a vector field of M^m such that at every point on M^{m-1} the vector of the field is tangent to M^{m-1}. We denote by ξ^A the components of the given vector field. Then, in D, there exist functions $\xi^a(a, b, c = 1, 2, \cdots, m-1)$ such that

$$\xi^A = X^A_a \xi^a, \tag{3. 4}$$

where we have put

$$X^A_a = \frac{\partial x^A}{\partial u^a}.$$

LEMMA 1. *Suppose that* ξ^i *be a Killing vector field of* M^m *such that at every point of a regular submanifold* M^{m-1} *the vector of the field is tangent to* M^{m-1}. *Then* ξ^i *restricted to* M^{m-1} *is a Killing vector field of* M^{m-1}.

PROOF. It is sufficient to show that ξ^a is a Killing vector field of M^{m-1}. If we contract $G_{AB} X^B_b$ to both sides of (3. 4), we get

(3. 5) $$G_{AB}\xi^A X^B_b = G_{AB}X^A_a X^B_b \xi^a = g_{ab}\xi^a,$$

where g_{ab}'s are components of the fundamental metric tensor of M^{m-1}. Hence we get

(3. 6) $$\xi_b = \xi_B X^B_b.$$

Differentiating both sides of the last equation covariantly with respect to the Christoffel's symbols of M^{m-1}, we get

$$\xi_{b;c} = \xi_{B,C}X^B_b X^C_c + \xi_B X^B_{b,c}.$$

Putting the Gauss' equation

(3. 7) $$X^A_{b,c} = \Omega_{bc}N^A$$

(N^A denotes the unit normal of M^{m-1}) into the last equation we get

$$\xi_{b;c} = \xi_{B,C}X^B_b X^C_c + \Omega_{bc}\xi_B N^B = \xi_{B,C}X^B_b X^C_c,$$

as ξ^A is orthogonal to N^A by assumotion. Therefore, we see that

$$\xi_{b;c} + \xi_{c;b} = (\xi_{B,C} + \xi_{C,B})\, X^B_b X^C_c = 0,$$

because ξ^A is a Killing vector field of M^m. Hence ξ^a is a Killing vector field of M^{m-1}. Q.E.D.

Combining Theorem 2 and the last Lemma in which M^m and M^{m-1} are replaced by $T(M^n)$ and $T_1(M^n)$ we can easily see that the following theorem is true.

THEOREM 3. *The extension of any Killing vector field of a Riemannian manifold M^n in $T(M^n)$ induces a Killing vector field of $T_1(M^n)$.*

This theorem is a particular case of Theorem 1 when f is an infinitesimal isometry of M^n.

THEOREM 4. *In order that the extension $(\frac{\partial \xi_i}{\partial x^j} v^j, \xi_i)$ in $T(M^n)$ of a covariant vector field ξ_i of M is orthogonal to the geodesic flow of $T(M^n)$ at every point of $T(M^n)$ is that ξ_i's are covariant components of a Killing vector field.*

PROOF. The condition of orthogonality of the extended vector field $(\frac{\partial \xi_i}{\partial x^j} v^j, \xi_i)$ and the geodesic flow is easily seen to be

$$\xi_{i;j}v^i v^j = 0.$$

As v^i's are arbitrary we get

$$\xi_{i;j} + \xi_{j;i} = 0.$$ Q. E. D.

4. The geodesic flow of M^n in the tangent sphere bundle.

THEOREM 5. *Every lift of any geodesic of a Riemannian manifold* M^n *in the tangent sphere bundle* $T_1(M^n)$ *is a geodesic of* $T_1(M^n)$. *Especially, every trajectory of the geodesic flow in* $T_1(M^n)$ *is a geodesic of* $T_1(M^n)$.

PROOF. Every lift of any geodesic of M^n in $T_1(M^n)$ is also a lift of the geodesic of M^n in $T(M^n)$. As we have proved it in I, the latter is a geodesic of $T(M^n)$. Hence, it is also a geodesic of $T_1(M^n)$ as a submanifold with induced metric from $T(M^n)$.

Now, we shall prove the exact generalization of the Poincaré's theorem on the incompressibility of the geodesic flow. We begin with a lemma.

LEMMA 2. *Let* M^m *be a Riemannian manifold and* M^{m-1} *be a submanifold of it. Suppose* ξ^i *be a vector field of* M^m *such that the vector of the field is tangent to* M^{m-1} *at every point on* M^{m-1}. *Then, in order that the vector field be an incompressible vector field of* M^{m-1}, *it is necessary and sufficient that the equation*

$$\xi^A{}_{,A} - \xi_{B,C}N^B N^C = 0 \tag{4. 1}$$

holds good.

PROOF. Using the same notation as in §3 we have (3. 4), from which we get

$$\xi^a = g^{ab}\ G_{AB}\xi^A X^B_b. \tag{4. 2}$$

Differentiating both sides of the last equation covariantly, we get

$$\xi^a{}_{,a} = g^{ab}G_{AB}(\xi^A{}_{,C}X^C_a X^B_b + \xi^A X^B{}_{a,b}).$$

The right hand side of the last equation can be transformed, by virtue of the Gauss' equation, to

$$= g^{ab}G_{AB}(\xi^A{}_{,C}X^C_a X^B_b + \xi^A N^B \Omega_{ab}).$$

Hence, we get

$$\begin{aligned}\xi^a{}_{,a} &= g^{ab}\xi_{B,C}X^B_b X^C_a \\ &= (G^{BC} - N^B N^C)\xi_{B,C} \\ &= \xi^A{}_{,A} - \xi_{B,C}N^B N^C.\end{aligned}$$

Q. E. D.

THEOREM 6. *The geodesic flow of the tangent sphere bundle* $T_1(M^n)$ *of a Riemannian manifold* M^n *is incompressible.*

PROOF. We have proved in the former paper I that the GF-vector field

$$\xi^i = v^i, \qquad \xi^{n+i} = -\left\{ {i \atop jk} \right\} v^j v^k \tag{4. 3}$$

in $T(M^n)$ is incompressible. These components define GF-vector field in $T_1(M^n)$ if $(x^i, v^i) \in T_1(M^n)$. Hence, by Lemma 2, it is sufficient to show that

$$\xi_{B,C}N^B N^C = 0, \tag{4. 4}$$

where $N^A(0, v^i)$ are components of the unit normal vector at $(x^i, v^i) \in T_1(M^n)$. Now,

$$\begin{aligned}\xi_{B,C}N^B N^C &= \xi_{n+j,n+k}v^j v^k \\ &= \left(\frac{\partial \xi_{n+j}}{\partial v^k} - \begin{Bmatrix} A \\ n+j \ n+k \end{Bmatrix}^* \xi_A\right) v^j v^k,\end{aligned}$$

where $\begin{Bmatrix} A \\ n+j \ n+k \end{Bmatrix}^*$'s are components of Christoffel's symbols of $T(M^n)$. As

$$\xi_{n+j} = 0, \qquad \begin{Bmatrix} A \\ n+j \ n+k \end{Bmatrix}^* = 0,$$

we see that (4. 4) is true. Hence, our assertion is true. Q. E. D.

5. Geodesics on the tangeut sphere bundle. We shall give here the differential equation of geodesics of $T_1(M^n)$.

Using the same notaiton as in §3, let $u^a(\sigma)$ be a differentiable curve of M^{m-1}. Then we get

$$\begin{aligned}\frac{dx^A}{d\sigma} &= X^A_a \frac{du^a}{d\sigma}, \\ \frac{D}{d\sigma}\left(\frac{dx^A}{d\sigma}\right) &= \Omega_{ab}\frac{du^a}{d\sigma}\frac{du^b}{d\sigma}N^A + X^A_a \frac{D}{d\sigma}\left(\frac{du^a}{d\sigma}\right).\end{aligned}$$

Hence, the differential equation of geodesics in M^{m-1} is given by

$$\frac{D}{d\sigma}\left(\frac{dx^A}{d\sigma}\right) = \Omega_{ab}\frac{du^a}{d\sigma}\frac{du^b}{d\sigma}N^A. \tag{5. 1}$$

Replacing M^m and M^{m-1} by $T(M^n)$ and $T_1(M^n)$ and noticing that the left hand side of the last equation is given by

$$\frac{d^2x^i}{d\sigma^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{dx^j}{d\sigma}\frac{dx^k}{d\sigma} - R^i_{j\lambda\mu}\frac{dx^j}{d\sigma}v^\lambda \frac{Dv^\mu}{d\sigma} = 0, \qquad \frac{D^2v^i}{d\sigma^2} = 0,$$

we see that the differential equation of geodesics of $T_1(M^n)$ is of the following form:

$$\frac{d^2x^i}{d\sigma^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{dx^j}{d\sigma}\frac{dx^k}{d\sigma} = R^i_{j\lambda\mu}\frac{dx^j}{d\sigma}v^\lambda \frac{Dv^\mu}{d\sigma}, \qquad \frac{D^2v^i}{d\sigma^2} = \rho v^i, \tag{5. 2}$$

because the components of the unit normal of $T_1(M^n)$ is $(0,v^i)$.

Next, we shall state some elementary theorems on closed geodesics of M^n

and $T_1(M^n)$.

Let g be a closed geodesic of M^n and P_0 be a point of g. We translate the tangent vector space $E^n_{P_0}$ at P_0 parallelly along g, then we get an orthogonal transformation T of $E^n_{P_0}$ onto itself which sends every vector at $E^n_{P_0}$ to its image by Levi-Civita's parallelism along g. We call T the orthogonal transformation associated with g.

THEOREM 7. *Let g be a directed closed geodesic of a Riemannian manifold M^n and P_0 be a point on g. If the orthogonal transformation T associated to g fixes a vector other than the tangent vector of g, then $T_1(M^n)$ has a continuous family of closed geodesics with the same length as g.*

PROOF. We denote by v_0 a unit vector which is invariant under T. Then, the vector field $v_s(0 \leqq s \leqq L$, L is the length of $g)$ parallel to v_0 along g defines a geodesic of $T_1(M^n)$ with the same length as g. If we denote the unit tangent vector on g at the point s by u_s, then $u_s \cos\alpha + v_s \sin\alpha$ is a parallel field of vectors along g. So it defines also a geodesic of $T_1(M^n)$ with the same length as g for every value α. Q. E. D.

COROLLARY. *Let g be a closed geodesic of a two dimensional orientable Riemannian manifold M^2. Then every lift of g is a geodesic of $T_1(M^2)$ with the same length as g.*

THEOREM 8. *Let g be a closed geodesic of a Riemannian manifold M^n and g^* be the closed geodesic of $T_1(M^n)$ determined by unit tangent vectors of g. If g is of minimum type, then g^* is also of minimum type.*

PROOF. Suppose C^* be an arbitrary closed curve near g^* and denote its projection $\pi_1 c^*$ by c. Then, denoting the length of curves g, g^* etc. by J_g, J_{g^*} etc., we get by virtue of (2. 2)

$$J_{C^*} \geqq J_C,$$
$$J_{g^*} = J_g,$$

as g^* is horizontal. By assumption g is a closed geodesic of minimum type, so

$$J_C \geqq J_g.$$

Hence, we see that

$$J_{C^*} \geqq J_{g^*}\,. \qquad \text{Q. E. D.}$$

5. The contact structure of the tangent sphere bundle. We define in $T_1(M^n)$ a differential 1-form ω by

$$\omega = g_{ij} v^j dx^i. \tag{5. 1}$$

Then we can easily see that

(5. 2) $$d\omega = g_{ij}Dv^j \wedge dx^i$$

and

(5. 3) $$\omega \wedge (d\omega)^{n-1} \neq 0.$$

Hence, the tangent sphere bundle $T_1(M^n)$ of M^n is a $(2n-1)$-dimensional Riemannian manifold with contact structure.

THEOREM 9. *The associated vector field of the contact structure ω of the tangent sphere bundle $T_1(M^n)$ of a Riemannian manifold M^n is the GF-vector field in $T_1(M^n)$.*

PROOF. Let U be a coordinate neighboorhood of M^n with coordinates x^i, such that $U \times E^n$ is diffeomorphic with $\pi^{-1}(U)$, where π is the natural projection $T(M^n) \to M^n$. Then (x^i, v^i) can be taken as local coordinates of $\pi^{-1}(U)$ and hence (x^i, v^a) $(a, b, c = 1,\cdots, n-1)$ can be taken as local coordinates of $T_1(M^n)$.

Now, solving

$$g_{ij}v^j Dv^i = 0$$

with respect to dv^n, we get

(5. 4) $$dv^n = -\frac{1}{v^n}\left(\left\{ {i \atop hk} \right\} v^h v_i \, dx^k + dv^a v_a\right).$$

Putting the last equation into (5. 2), we get after an easy calculation the following equation:

$$d\omega = S_{jk}dx^j \wedge dx^k + 2S_{n+a\ j}dv^a \wedge dx^j,$$

where we have put

(5. 5) $$S_{jk} = \frac{1}{2v_n}\left[\left\{ {i \atop hj} \right\} v^h (v_n g_{ik} - g_{nk} v_i) - \left\{ {i \atop hk} \right\} v^h (v_n g_{ij} - g_{nj} v_i)\right],$$

$$S_{n+aj} = \frac{1}{2v_n}(v_n g_{aj} - g_{nj} v_a).$$

Hence, if we put

(5. 5) $$S_{jn+a} = - S_{n+aj}, \quad S_{n+a\ n+b} = 0, \quad x^{n+a} = v^a,$$

then we can write

(5. 6) $$d\omega = S_{\lambda\mu} dx^\lambda \wedge dx^\mu. \ (\lambda, \mu = 1, 2,\cdots, 2n-1)$$

The associated vector field of the contact structure is given as a set of solutions of the equations

$$S_{\lambda\mu}\, X^{\mu} = 0, \tag{5. 7}$$

that is

$$S_{jk}X^{k} + S_{jn+c}X^{n+c} = 0, \quad S_{n+bk}X^{k} = 0.$$

As the rank of the matrix $||S_{\lambda\mu}||$ is $2n-2$, the last equations have only a set of independent solutions. We can easily verify that

$$X^{i} = v^{i}, \qquad X^{n+a} = -\left\{ {a \atop hk} \right\} v^{h}v^{k}. \tag{5. 8}$$

As we can see from (5. 4), this vector field in $T_1(M^n)$ has $2n$-th components X^{2n} which is given by

$$X^{2n} = -\frac{1}{v^{n}}\left(\left\{ {i \atop hk} \right\} X^{h}v^{k}v_{i} + v_{a}X^{n+a}\right).$$

Putting (5. 8) into the right hand side of the last equation, we get

$$X^{2n} = -\left\{ {n \atop hk} \right\} v^{h}v^{k}.$$

Hence, the associated direction to the contact structure of $T_1(M^n)$ has components $\left(v^{i}, -\left\{ {i \atop hk} \right\} v^{h}v^{k}\right)$ in $T(M^n)$. Therefore it is nothing but the GF-vector field of $T_1(M^n)$.

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