A CHARACTERIZATION OF C^*-ALGEBRAS WHOSE CONJUGATE SPACES ARE SEPARABLE

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A well known result by A.Rosenberg [6] shows that the algebra of all completely continuous operators on a separable Hilbert space is characterized by the uniqueness of its irreducible representations. In connecting with this, in 1959, S.Sakai showed in an unpublished paper that such an algebra is also characterized by the conjugate space, that is, a simple C^*-algebra whose conjugate space is separable is necessarily isomorphic to the algebra of all completely continuous operators on a separable Hilbert space.

The purpose of this paper is to determine the structure of all C^*-algebras whose conjugate spaces are separable. The result is the following

THEOREM. *Let* $\boldsymbol{A}$ *be a* C^**-algebra whose conjugate space is separable. Then* $\boldsymbol{A}$ *is a GCR algebra having a composition series* I_ρ *of countable type such that each* $I_{\rho+1}/I_\rho$ *is a separable dual* C^**-algebra, that is, the* $C^*(\infty)$*-sum of the algebra of all completely continuous operators on a separable Hilbert space.*

Since a simple dual C^*-algebra is nothing but the algebra of all completely continuous operators on a suitable Hilbert space, the theorem implies the above Sakai's result.

The proof splits into several lemmas. Let $\boldsymbol{A}$ be a C^*-algebra without assuming a unit. We denote by $\boldsymbol{A}^*$ the conjugate space of $\boldsymbol{A}$. Then, as is well known, the second conjugate space $\boldsymbol{A}^{**}$ of $\boldsymbol{A}$ may be identified with a weakly closed self-adjoint algebra $\widetilde{\boldsymbol{A}}$ on a suitably chosen Hibert space H. Since such an algebra always contains the greatest projection playing as a unit, we may assume without loss of generality that $\widetilde{\boldsymbol{A}}$ contains the identity operator on H, which is denoted by 1. Now the algebra $\widetilde{\boldsymbol{A}}$ has many special properties relating to $\boldsymbol{A}$(cf. [4], [8]) and it will be appropriate to point out some of them before going into discussions. We notice at first that the σ-weak topology of $\widetilde{\boldsymbol{A}}$ is equivalent to $\sigma(\widetilde{\boldsymbol{A}}, \boldsymbol{A}^*)$-topology, and the space $\boldsymbol{A}^*$ is considered to be the space of all σ-weakly continuous linear functionals on $\widetilde{\boldsymbol{A}}$, i. e. $(\widetilde{\boldsymbol{A}})_*$. Thus each bounded linear functional φ on $\boldsymbol{A}$ is canonically considered as a σ-weakly continuous linear functional on $\tilde{A}$, which we denote by $\tilde{\varphi}$. In particular, if φ is a pure state of $A, \tilde{\varphi}$ is a σ-weakly continuous pure state of $\tilde{A}$. Hence there exists the carrier projection e_φ of $\tilde{\varphi}$ which is a minimal projection in $\tilde{A}$. Let π_φ be the irreducible representation induced by φ on the canonical representation space H_φ, then π_φ can be extended to a σ-weakly continuous representation $\tilde{\pi}_\varphi$ of $\tilde{A}$ on H_φ. Clearly $\tilde{\pi}_\varphi$ is an irreducible representation of $\tilde{A}$ on H_φ and it can be shown that this representation is unitarily equivalent to the irreducible representation $\pi_{\tilde{\varphi}}$ of $\tilde{A}$ induced by $\tilde{\varphi}$. On the other hand $\pi_{\tilde{\varphi}} | A$, the restriction of $\pi_{\tilde{\varphi}}$ to A is an irreducible representation of A on $H_{\tilde{\varphi}}$ and this representation is unitarily equivalent to π_φ. We refer to Dixmier [2] for details of discussions concerning the theory of von Neumann algebras.

Now let φ and ψ be pure states of A and e_φ and e_ψ carrier projections of $\tilde{\varphi}$ and $\tilde{\psi}$ respectively. The following lemma is more or less known.

LEMMA 1. *The necessary and sufficient condition that φ and ψ induce unitarily equivalent representations of A is that e_φ and e_ψ are equivalent projections in $\tilde{A}$.*

PROOF. Suppose that φ and ψ induce unitarily equivalent representations of A. The $\tilde{\varphi}$ and $\tilde{\psi}$ induce unitarily equivalent representations of $\tilde{A}$. The canonical representation $\pi_{\tilde{\varphi}}$ is unitarily equivalent to the representation of $\tilde{A}$ as a ring of left multiplication operators on the Hilbert space $\tilde{A}e_\varphi$ endowed with the inner product $(xe_\varphi, ye_\varphi) = \lambda$ such as $e_\varphi y^* x e_\varphi = \lambda e_\varphi$. Therefore there exists an isometric operator U from $\tilde{A}e_\varphi$ to $\tilde{A}e_\psi$ such as

$$UxU^{-1}[ye_\psi] = xye_\psi \text{ for all } x, y \in \tilde{A}.$$

Let $z(e_\varphi)$ and $z(e_\psi)$ be central envelopes of e_φ and e_ψ in $\tilde{A}$, then both projections $z(e_\varphi)$, $z(e_\psi)$ are minimal in the center of $\tilde{A}$ and either $z(e_\varphi)z(e_\psi) = 0$ or $z(e_\varphi) = z(e_\psi)$ holds. If $z(e_\varphi)z(e_\psi) = 0$, we have

$$Uz(e_\varphi)U^{-1}[xe_\psi] = (ze_\varphi)xe_\psi = 0 \text{ for all } x \in \tilde{A}$$

and on the other hand

$$z(e_\varphi)xe_\varphi = xe_\varphi \qquad \text{for all } x \in \tilde{A},$$

a contradiction. Hence $z(e_\varphi) = z(e_\psi)$ and the comparability theorem shows that $e_\varphi \sim e_\psi$ in $\tilde{A}$.

Conversely suppose that $e_\varphi \sim e_\psi$ in $\tilde{A}$ and let v be a partially isometric operator in $\tilde{A}$ such as $v^*v = e_\varphi$, $vv^* = e_\psi$. Define the map

$$U : \tilde{A}e_\varphi \to \tilde{A}e_\psi \qquad \text{by } U(ye_\varphi) = yv^*e_\psi.$$

Clearly U is an onto map and since e_φ is a minimal projection in $\widetilde{A}$, we get

$$\begin{aligned}\|U(ye_\varphi)\|^2 &= \|yv^*e_\psi\|^2 = \|ye_\varphi v^*\|^2 \\ &= \|ve_\varphi y^*ye_\varphi v^*\| = \|e_\varphi y^*ye_\varphi\| \ \|ve_\varphi v^*\| \\ &= \|e_\varphi y^*ye_\varphi\| = \|ye_\varphi\|^2.\end{aligned}$$

Since we have $|ye_\varphi| = \|ye_\varphi\|$ where $|\cdot|$ means the norm in the Hilbert space $\widetilde{A}e_\varphi$, the above equality shows that U is an isometry. Moreover, for arbitrary elements $x, y \in \widetilde{A}$, we have

$$UxU^{-1}(ye_\psi) = U(xyve_\varphi) = xye_\psi.$$

Hence $\pi_{\tilde{\varphi}}$ is unitarily equivalent to $\pi_{\tilde{\psi}}$, and this implies the unitary equivalence of irreducible representations of A induced by φ and ψ.

An easy conclusion of Lemma 1 is the following

COROLLARY (Glimm-Kadison [5 : Corollary 9]). *If φ and ψ induce disjoint irreducible representations of A, then $\|\varphi - \psi\| = 2$. The converse does not necessarily hold.*

PROOF. In this case, e_φ is orthogonal to e_ψ (in fact $z(e_\varphi)z(e_\psi) = 0$). Hence $\|e_\varphi - e_\psi\| = 1$ and

$$2 \geqq \|\varphi - \psi\| = \|\tilde{\varphi} - \tilde{\psi}\| \geqq |(\tilde{\varphi} - \tilde{\psi})(e_\varphi - e_\psi)| = 2.$$

We assume, for the rest of the discussions, that the conjugate space A^* of A is separable. Then we have

LEMMA 2. *A is a GCR algebra containing a non-zero minimal projection.*

PROOF. Let B be a maximal commutative self-adjoint subalgebra of A and Ω the spectrum of B. For each $\omega \in \Omega$ we denote by φ_ω the associated pure state of B. Let ω_1 and ω_2 be an arbitrary pair of distinct points in Ω, then one easily verifies that $\|\varphi_{\omega_1} - \varphi_{\omega_2}\|_B = 2$ where $\|\cdot\|_B$ means the functional norm of $\varphi_{\omega_1} - \varphi_{\omega_2}$ on B. Denoting by φ_1 and φ_2 the extensions of φ_{ω_1} and φ_{ω_2} to the states of A we get the same relation, $\|\varphi_1 - \varphi_2\| = 2$. Therefore, since A^* is separable, the family of states of A, $\{\varphi_\alpha | \omega_\alpha \in \Omega\}$ defined by the above way is at most countable and Ω is at most a countable set. Because the latter is a locally compact Hausdorff space, one sees that there exists at least one isolated point ω_0 in Ω. Let $e(\omega)$ be the caracteristic function of $\{\omega_0\}$. Identifying B with the space of all continuous functions on Ω vanishing at infinity we may assume that e belongs to B. Clearly e is a minimal projection of B. We assert that e is also minimal in A. In fact, for arbitrary self-adjoint element $x \in A$ and $y \in B$ we have,

$$exey = exeye = eyexe = yexe$$

because $eye = \lambda e$ for some complex number λ. It follows, since $\boldsymbol{B}$ is a maximal commutative self-adjoint subalgebra of $\boldsymbol{A}$, $exe \in \boldsymbol{B}$ whence $exe = \lambda e$ for some real number λ, which implies the minimality of e in $\boldsymbol{A}$. Therefore we see that $\boldsymbol{A}$ contains a non-zero *CCR*, hence *GCR* ideal.

Now let K be the largest *GCR* ideal in $\boldsymbol{A}$ and assume that $\boldsymbol{A} \supsetneqq K$. Then $\boldsymbol{A}/K$ has no non-zero *GCR* ideals (cf. Kaplansky [7: Theosem 7.5]). On the other hand, as is well known the conjugate space $(\boldsymbol{A}/K)^*$ of $\boldsymbol{A}/K$ is isometric to K^0, the polar of K in $\boldsymbol{A}^*$ and this implies the separability of $(\boldsymbol{A}/K)^*$. Then the above arguments show that the C^*-algebra $\boldsymbol{A}/K$ contains a non-zero *GCR* ideal, a contradiction. Hence $\boldsymbol{A} = K$. This completes the proof.

LEMMA 3. *$\widetilde{\boldsymbol{A}}$ is the direct sum of a countable number of factors of type I.*

PROOF. Take an arbitrary central projection z in $\widetilde{\boldsymbol{A}}$. As in the last paragraph of the proof of Lemma 2, $(\boldsymbol{A}z)^*$ is a separable space. Hence $\boldsymbol{A}z$ contains a minimal projection e which is also minimal in $\widetilde{\boldsymbol{A}}z$ because $\boldsymbol{A}z$ is σ-weakly dense in $\widetilde{\boldsymbol{A}}z$. We have, $e \leqq z$. It follows that $\widetilde{\boldsymbol{A}}$ is the direct sum of factors of type I, i.e. $\widetilde{\boldsymbol{A}} = \sum_{\alpha} \widetilde{\boldsymbol{A}}z_\alpha$ where z_α is a minimal central projection. Let φ_α be a pure state of $\boldsymbol{A}$ such as $e_{\varphi\alpha} \leqq z_\alpha$. Then we see that $\|\varphi_\alpha - \varphi_\beta\| = 2$ for every pair of $\alpha, \beta (\alpha \neq \beta)$ and the separability of $\boldsymbol{A}^*$ implies that the index set $\{\alpha\}$ is (at most) countable. That is, $\widetilde{\boldsymbol{A}} = \sum_{n=1}^{\infty} \widetilde{\boldsymbol{A}}z_n$.

It will be worth to notice that considering Lemma 1, [5: Corollary 4] and the property of $\widetilde{\boldsymbol{A}}$ we may assume, without loss of generality, that $\widetilde{\boldsymbol{A}}$ acts on a Hilbert space H such that, for each n, $\widetilde{\boldsymbol{A}}z_n = \mathfrak{B}(z_nH)$ the algebra of all bounded operators on z_nH. Since $\boldsymbol{A}z_n$ is σ-weakly dense in $\widetilde{\boldsymbol{A}}z_n$, $\boldsymbol{A}z_n$ acts irreducibly on z_nH.

LEMMA 4. *If $\boldsymbol{A}$ is CCR, then it is a dual C^*-algebra.*

PROOF. Let $\{e_\alpha\}$ be the maximal family of orthogonal minimal projections in $\boldsymbol{A}$ and put

$$\boldsymbol{B} = \{a \in \boldsymbol{A} \mid ae_\alpha = e_\alpha a = 0 \text{ for every } e_\alpha\}.$$

Suppose $\boldsymbol{B} \neq \{0\}$, then $\boldsymbol{B}$ is a C^*-subalgebra of $\boldsymbol{A}$ and since $\boldsymbol{B}^*$ is isometric to the factor space $\boldsymbol{A}^*/\boldsymbol{B}^0$ of $\boldsymbol{A}^*$ by the polar of $\boldsymbol{B}$ in $\boldsymbol{A}^*$ the space $\boldsymbol{B}^*$ is separable. Therefore $\boldsymbol{B}$ has a non-zero minimal projection e. Moreover, for each element $a \in \boldsymbol{A}$, we have $eae \in \boldsymbol{B}$ which implies the minimality of e in $\boldsymbol{A}$, however this contradicts the maximality of the family $\{e_\alpha\}$.

Put $f = \sum_{\alpha} e_\alpha$ and consider the central envelope z of f in $\widetilde{\boldsymbol{A}}$. Since the set

$\{a \in \boldsymbol{A} \mid e_\alpha a = ae_\alpha = 0\} = \{0\}$, the map: $\boldsymbol{A} \to \boldsymbol{A}z$ is one-to-one. We assert that $z = 1$. Suppose, on the contrary, that $z \neq 1$ then there exists an integer n such as $z_n \leqq 1 - z$. Let φ be a pure state on $\boldsymbol{A}$ and assume that the carrier projection of φ is contained in z_n. We consider the pure state ψ of $\boldsymbol{A}z$ defined by $\psi(az) = \varphi(a)$ for $a \in \boldsymbol{A}$ and denote by $\widehat{\psi}$ the pure state extension of ψ to $\widetilde{\boldsymbol{A}}z$. Since $\widehat{\psi}$ is multiplicative on the center of $\widetilde{\boldsymbol{A}}z$ there exists an integer m with $\widehat{\psi}(z_m) = 1$. Thus $\widehat{\psi}$ is considered to be a pure state of $\widetilde{\boldsymbol{A}}z_m$ and

$$\widehat{\psi} \mid \boldsymbol{A}z_m = \psi \mid \boldsymbol{A}z = \psi \neq 0.$$

On the other hand, as $\boldsymbol{A}z_m$ acts irreducibly on $z_m H$ and $\boldsymbol{A}$ is CCR we see that $\boldsymbol{A}z_m = C(z_m H)$, the algebra of all completely continuous operatators on $z_m H$, that is, $\boldsymbol{A}z_m$ is an ideal of $\widetilde{\boldsymbol{A}}z_m$. Hence $\widehat{\psi} \mid \boldsymbol{A}z_m$ is a pure state of $\boldsymbol{A}z_m$. Therefore by Dixmier [1 : Theorem 3] $\widehat{\psi}$ is σ-weakly continuous on $\widetilde{\boldsymbol{A}}z_m$, hence on the whole algebra $\widetilde{\boldsymbol{A}}z$. Thus the canonical irreducible representation $\pi_{\widehat{\psi}}$ induced by $\widehat{\psi}$ is σ-weakly continuous, so that $\pi_{\widehat{\psi}} \mid \boldsymbol{A}z$, the restriction of $\pi_{\widehat{\psi}}$ to $\boldsymbol{A}z$, is an irreducible representation of $\boldsymbol{A}z$. Hence $\pi_{\widehat{\psi}} \mid \boldsymbol{A}z$ is unitarily equivalent to the representation π_ψ induced by ψ and the former is the composition of the maps $\boldsymbol{A}z \to \boldsymbol{A}z_m$ and $\boldsymbol{A}z_m \to \pi_{\widehat{\psi}}(\boldsymbol{A}z)$. It follows that $\pi_\psi^{-1}(0) = \{x \in \boldsymbol{A}z \mid xz_m = 0\}$. Therefore the transposition of $\pi_\psi^{-1}(0)$ by the isomorphism $\boldsymbol{A}z \to \boldsymbol{A}$ coincides with the set $\{a \in \boldsymbol{A} \mid az_m = 0\}$. However the set is

$$\begin{aligned} &\{a \in \boldsymbol{A} \mid az \in \pi_\psi^{-1}(0)\} \\ &= \{a \in \boldsymbol{A} \mid \psi(b^* zazcz) = 0 \text{ for every } b, c \in \boldsymbol{A}\} \\ &= \{a \in \boldsymbol{A} \mid \varphi(b^* ac) = 0 \text{ for every } b, c \in \boldsymbol{A}\} \\ &= \pi_\varphi^{-1}(0) = \{a \in \boldsymbol{A} \mid az_n = 0\}. \end{aligned}$$

Thus,

$$\{a \in \boldsymbol{A} \mid az_m = 0\} = \{a \in \boldsymbol{A} \mid az_n = 0\} \text{ and } n \neq m.$$

Since $\boldsymbol{A}$ is CCR, the above relation induces the unitary equivalence between the irreducible representations, $\boldsymbol{A} \to \boldsymbol{A}z_m$ on $z_m H$ and $\boldsymbol{A} \to \boldsymbol{A}z_n$ on $z_n H$ but this is a contradiction. Hence $z = 1$.

Now let $P_n = \{a \in \boldsymbol{A} \mid az_n = 0\}$. Then it is easily seen that the structure space $\Omega(\boldsymbol{A})$ which is the set of all primitive ideals in $\boldsymbol{A}$ with hull-kernel topology is nothing but the set $\{P_n \mid n = 1, 2, \cdots\}$. Take an arbitrary point P_k, then clearly $\{P_k\}$ is a closed set in $\Omega(\boldsymbol{A})$. We assert that $\{P_k\}$ is an open set in $\Omega(\boldsymbol{A})$. In fact, we can find a minimal projection e in $\boldsymbol{A}$ such as $e \in \boldsymbol{A} \cap \boldsymbol{A}z_k$ because $z =$ central envelope of $(f = \sum_\alpha e_\alpha) = 1$. We have, $e \in \bigcap_{n \neq k} P_n$ and $e \notin P_k$, which implies that the set $\{P_n \mid n \neq k\}$ is closed in $\Omega(\boldsymbol{A})$. It follows that $\Omega(\boldsymbol{A})$ is a discrete space and $\boldsymbol{A}$ is isomorphic to the $C^*(\infty)$-sum of $\{\boldsymbol{A}/P_n \mid n = 1,2,\cdots\}$.

Since A/P_n is isomorphic to the algebra of all completely continuous operators on z_nH, $\boldsymbol{A}$ becomes a dual C^*-algebra (cf. [7]).

REMARK. From the assumption that $\boldsymbol{A}^*$ is separable, one can not deduce the conclusion that $\boldsymbol{A}$ is a *CCR* algebra. For example, let $\boldsymbol{A}$ be the C^*-algebra generated by the identity operator and the algebra of all completely continuous operators on an infinitely dimensional Hilbert space. Then $\boldsymbol{A}$ is not *CCR* whereas $\boldsymbol{A}^*$ is a separable space.

Now we proceed the proof of the theorem. Recall that a composition series of a C^*-algebra $\boldsymbol{A}$ means a well-ordered ascending series of closed two-sided ideals I_ρ, beginning with 0 and ending with $\boldsymbol{A}$ and such that for any limit ordinal λ, I_λ is, the closure of the union of the preceding I,s. By Lemma 2, $\boldsymbol{A}$ has a composition series I_ρ such that each $I_{\rho+1}/I_\rho$ is a *CCR*-algebra. Let $\widetilde{I_\rho}$ be the σ-weak closure of I_ρ in $\boldsymbol{A}$, then $\widetilde{I_\rho}$ is a σ-weakly closed ideal in $\widetilde{\boldsymbol{A}}$. Hence there exists a central projection z_ρ such as $\widetilde{I_\rho} = \widetilde{\boldsymbol{A}}z_\rho$. Then, one easily sees that $(z_{\lambda+1} - z_\lambda)\,(z_{\rho+1} - z_\rho) = 0$ if $\lambda \neq \rho$. Therefore, by Lemma 3, the index set $\{\rho\}$ is at most countable. On the other hand,

$$(I_{\rho+1}/I_\rho)^* \cong I_\rho^\circ \text{ in } I_{\rho+1}^*$$

and $I_{\rho+1}^*$ is a separable space. Hence the subspace I_ρ°, and $(I_{\rho+1}/I_\rho)^*$ is a separable one. Thus, by Lemma 4, we get the conclusion of the theorem.

It will be worth to notice that the converse of the theorem is also true. For, with the notations above we have

$$1 = \sup_\rho (z_{\rho+1} - z_\rho), \text{ and get the identity } \widetilde{\boldsymbol{A}} = \sum_\rho \widetilde{\boldsymbol{A}}(z_{\rho+1} - z_\rho).$$

Hence,

$$\widetilde{\boldsymbol{A}}_* = \sum_\rho \oplus (\widetilde{\boldsymbol{A}}(z_{\rho+1} - z_\rho))_* \qquad (l^1\text{-sum}).$$

On the other hand, we have $(I_{\rho+1}/I_\rho)^* \cong (\widetilde{\boldsymbol{A}}(z_{\rho+1} - z_\rho))_*$. Since the conjugate space of any separable dual C^*-algebra is also separable, one concludes that $\widetilde{\boldsymbol{A}}_*$, hence $\boldsymbol{A}^*$ is a separable space.

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