CONVEXITY THEOREMS FOR ALLIED FOURIER SERIES

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Recently we proved a number of convexity theorems for Fourier series [2], and they were slightly generalized in [3]. The present paper is a continuation of [2, 3]. Indeed, we shall treat the allied Fourier series analogues.

With respect to Theorems 1, 2 and 6 in [2], the results for allied series will be different a little, while with respect to Theorems 3, 4 and 5 in [2], the results will be almost similar. For the sake of contrast, we shall number the theorems of this paper with the order of theorems in [2].

1. Notations. Let $\psi(t)$ be an odd function integrable in Lebesgue sense in $(0, \pi)$, and periodic of period 2π, and let

$$\psi(t) \sim \sum_{n=1}^{\infty} b_n \sin nt,$$

$$\overline{s}_n^r = \sum_{v=1}^{n} A_{n-v}^r b_v \qquad (-\infty < r < \infty),$$

$$\overline{t}_n^{r+1} = \sum_{v=1}^{n} A_{n-v}^r (v b_v) \qquad (-\infty < r < \infty),$$

$$\overline{\sigma}_n^r = \overline{s}_n^r / A_n^r \qquad (r > -1),$$

where $A_n^r = \begin{pmatrix} r+n \\ n \end{pmatrix}$, $n = 0, 1, 2, \cdots$. We write

$$\Psi_0(t) = \psi_0(t) = \psi(t),$$

$$\Psi_\beta(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-u)^{\beta-1} \psi(u)\, du \quad (\beta > 0),$$

$$\psi_\beta(t) = \Gamma(\beta+1) t^{-\beta} \Psi_\beta(t) \quad (\beta > 0).$$

Similarly, from the function

$$\theta(t) = \frac{2}{\pi} \int_t^{\infty} \frac{\psi(u)}{u}\, du$$

we define $\theta_\beta(t)$ for $\beta \geqq 0$. For the negative value of β, let

$$\theta_\beta(t) = t^{-\beta} \frac{d}{dt} \int_0^t (t-u)^\beta \theta(u)\, du \qquad (-1 < \beta < 0).$$

Further, we understand

1) $t \to 0$ means that $t > 0$ and $t \to 0$,

2) $\psi(t) \to 0(C)$ means that $\psi_\beta(t) \to 0$ as $t \to 0$ for some β.

These notations will be used throughout this paper.

2. Theorems 1, 2 and 6.

THEOREM 1. *Let* $0 \leq \beta$, $-1 \leq c$, $0 < \beta - b \leq \gamma - c$ *and* $-1 < c - b < 2$, (*except the case* $\beta - \gamma = b - c = 0$).

(I) *If* $\Psi_\beta(t) = o(t^\gamma)$, *or more generally*

$$(2.1) \qquad \int_0^t |\Psi_\beta(u)| \, du = o(t^{\gamma+1}) \ as \ t \to 0,$$

and if

$$(2.2) \qquad \bar{s}_n^c - A_n^c \bar{s} = O(n^b) \ as \ n \to \infty$$

holds for some constant $\bar{s}$, *then we have*

$$(2.3) \qquad \bar{s}_n^r - A_n^r \bar{s} = o(n^q), \qquad q = b + (r - c)(\beta - b)/(\gamma - c),$$

as $n \to \infty$, *for* $c < r < \gamma'$, *where*

$$(2.4) \qquad \gamma' = \inf(\gamma, [(b + 2)\gamma - (\beta + 2)c]/(\gamma - c + b - \beta)).$$

(II) *If* (2.1) *holds, and*

$$(2.2)' \qquad \bar{s}_n^c - A_n^c \bar{s} = O^L(n^b) \ as \ n \to \infty,$$

then we have (2.3) *for* $c + 1 \leq r < \gamma'$, *provided that* $c + 1 < \gamma'$.

(N. B. 1) In (I) of this theorem, the range of r, i.e. $c < r < \gamma'$ denotes the common range of $c < r < \gamma$ and $r - q < 2$.

REMARK 1. In Theorem 1 (and also in Theorem 2 below), if $\beta \geq \gamma$ (except the case $\beta - \gamma = b - c = 0$) then we have steadily $q - r > 0$ and so in view of (2.3) $\bar{s}$ may be quite arbitrary. In particular, we then may put $\bar{s} = 0$. On the contrary, if $\beta < \gamma$ then $q - r < 0$ occurs in the range $c < r < \gamma'$, and for such r there exists

$$\lim_{t \to 0} \theta_r(t) \left[= \lim_{t \to 0} \theta_\beta(t) \right],$$

which should be equal to the constant $\bar{s}$. For details, see the proof of Theorem 1.

Concerning Theorem 6 below it is slightly different. If $\beta = \gamma$ then $\bar{s}$ cannot be arbitrary.

(N. B. 2) In the exceptional case $\beta - \gamma = b - c = 0$, Theorem 1 (and also Theorem 2 below) is true if and only if $\theta(t) \to \bar{s}(C)$, in addition to the assumptions. Also cf. the proof of Theorem 1.

THEOREM 2. *Let* $0 \leq b$, $0 < \beta - b \leq \gamma - c$ *and* $-1 < c - b < 2$, (*except*

the case $\beta - \gamma = b - c = 0$). If (2. 1) holds, and if $\Psi_b(t) = O(t^c)$, or more generally

(2. 5) $$\int_0^t (|\Psi_b(u)| - \Psi_b(u))\,du = O(t^{c+1}) \; as \; t \to 0,$$

then we have the conclusion in (I) of Theorem 1, $\bar{s}$ having the same meaning as in Remark 1.

THEOREM 6. Let $0 \leqq b, 0 < \beta - b \leqq \gamma - c$ and $-1 < c - b < 2, [b(\gamma + 1)$ $< (c + 1)\beta]$. If

$$\bar{s}_n^\gamma - A_n^\gamma \bar{s} = o(n^\beta) \; as \; n \to \infty,$$

for some constant $\bar{s}$, and if $\Psi_b(t) = O(t^c)$, or more generally (2. 5) holds, then we have (2. 3) for $c < r < \gamma$.

(N. B. 3) In this theorem, the restriction $r - q < 2$ is superfluous. When $b = 0$ and $c = -1$, Theorem 1 can be slightly modified as follows.

THEOREM 1^0. If $0 < \beta < \gamma + 1$, and $\Psi_\beta(t) = O(t^\gamma)$ or more generally

(2. 1)' $$\int_0^t |\Psi_\beta(u)|\,du = O(t^{\gamma+1}),$$

then we have

$$\bar{s}_n^r - A_n^r \bar{s} = o(n^{(r+1)\beta/(\gamma+1)})$$

for

$$-1 \leqq r < \inf\,(\gamma, [2(\gamma + 1) + \beta]/(\gamma + 1 - \beta)),$$

where $\bar{s} = 0$ when $\beta \geqq \gamma$, and $\bar{s} = \lim \theta_\beta(t)$ when $\beta < \gamma$.

From Theorem 2 follows the following

THEOREM 2^0. If $0 \leqq \beta, -1 < \gamma < \beta + 2, \beta \neq \gamma$, and if $\Psi_\beta(t) = o(t^\gamma)$, or more generally

(2. 1) $$\int_0^t |\Psi_\beta(u)|\,du = o(t^{\gamma+1}),$$

then we have

$$\bar{s}_n^{\gamma+\eta} - A_n^{\gamma+\eta} s = o(n^{\beta+\eta}), \; \eta > 0,$$

where $\bar{s} = 0$ when $\beta > \gamma$, and $\bar{s} = \lim \theta_\beta(t)$ when $\beta < \gamma$.

(N. B. 4) This theorem is not true when $\beta = \gamma$. Cf. Lemma 7 in the next article.

Similarly as in the paper [2], making $r = q = \alpha$ $(c < \alpha < \gamma)$ we have the following summability theorems from the above theorems respectively.

COROLLARY 1. *Let* $0 < \beta < \gamma$ *and* $0 < \delta < 1$. *If* (2. 1) *holds, and if either of the two conditions*

$$\begin{cases} b_n = O(n^{-(1-\delta)}), \\ b_n = O_L(n^{-(1-\delta)}), \ \beta \geqq \gamma(1 - \delta), \end{cases}$$

is satisfied, then there exists $\bar{s} = \lim \theta_\alpha(t)$, *and*

$$\bar{\sigma}_n^\alpha \to \bar{s}, \qquad \alpha = (\beta - \gamma + \gamma\delta)/(\gamma - \beta + \delta).$$

The existence of $\lim \theta_\alpha(t)$ will be shown later in §4.

COROLLARY 2. *Let* $0 < \beta < \gamma$ *and* $0 < \delta < 1$. *If* (2. 1) *holds, and*

$$\psi(t) = O_L(t^{-\delta}),$$

then there exists $\bar{s} = \lim \theta_\alpha(t)$, *and*

$$\bar{\sigma}_n^\alpha \to \bar{s}, \qquad \alpha = \beta\delta/(\gamma - \beta + \delta).$$

COROLLARY 6. *Let* $0 < \beta < \gamma$ *and* $0 < \delta < 1$. *If*

$$\bar{s}_n^\gamma - A_n^\gamma \bar{s} = o(n^\beta) \quad and \quad \psi(t) = O_L(t^{-\delta}),$$

then we have actually $\bar{s} = \lim \theta_\alpha(t)$, *and*

$$\bar{\sigma}_n^\alpha \to \bar{s}, \ \alpha = \beta\delta/(\gamma - \beta + \delta).$$

(N. B. 5) In Corollaries 2 and 6, we could show that more generally

if $\gamma - \beta < \beta\delta$ *then* $\bar{s} = \lim \theta_{\alpha'}(t), \ 0 < \alpha' = \alpha - (\beta - \alpha)/\beta < \alpha,$
and if $\gamma - \beta \geqq \beta\delta$ *then* $\bar{s} = \lim \theta_0(t) = \lim \theta(t),$ *in place of* $\bar{s} = \lim \theta_\alpha(t).$

COROLLARY 1⁰. *If* $0 < \beta < \gamma$, *and* (2. 1)′ *holds, then there exists* $\bar{s} = \lim \theta_\alpha(t)$, *and*

$$\bar{\sigma}_n^\alpha \to \bar{s}, \qquad \alpha = \beta/(\gamma - \beta + 1).$$

When $\beta = 0$, clearly this corollary is true. Similarly, Corollaries 1 and 2 are true when $\beta = 0$.

3. Preliminary lemmas. In order to prove the theorems we need a number of lemmas.

LEMMA 1. *Let* $0 \leqq \beta, \ -1 \leqq c, \ 0 < \beta - b \leqq \gamma - c$ *and* $-1 < c - b < 2$. (I) *If* (2. 1) *holds, i.e.*

(3. 1) $$\int_0^t |\Psi_\beta(u)| \, du = o(t^{\gamma+1}),$$

and $\bar{t}_n{}^{c+1} = O(n^{b+1})$, *then we have*

(3. 2) $\qquad\qquad \bar{t}_n{}^{r+1} = o(n^{q+1}), \qquad q = b + (r - c)(\beta - b)/(\gamma - c),$

for $c < r < \gamma'$, *where*

(3. 3) $\qquad \gamma' = \inf\,(\gamma, [(b + 2)\gamma - (\beta + 2)c]/(\gamma - c + b - \beta)).$

(II) *If* (3. 1) *holds, and* $\bar{t}_n{}^{c+1} = O_L(n^{b+1})$, *then we have* (3. 2) *for* $c + 1 \leqq r < \gamma'$, *provided that* $c + 1 < \gamma'$.

LEMMA 2. *Let* $0 \leqq b, 0 < \beta - b \leqq \gamma - c$ *and* $-1 < c - b < 2$. *If* (3. 1) *holds, and if* $\Psi_b(t) = O(t^c)$, *or more generally*

(3. 4) $\qquad\qquad \displaystyle\int_0^t (|\Psi_b(u)| - \Psi_b(u))\,du = O(t^{c+1}),$

then we have (3. 2) *for* $c < r < \gamma'$, γ' *being defined by* (3. 3).

LEMMA 3. *Let* $0 \leqq b, 0 < \beta - b \leqq \gamma - c$ *and* $-1 < c - b < 2$. *If* $\bar{t}_n{}^{\gamma+1} = o(n^{\beta+1})$, *and if* $\Psi_b(t) = O(t^c)$, *or more generally* (3. 4) *holds, then we have* (3. 2) *for* $c < r < \gamma$.

LEMMA 4 *If* $0 < \beta < \gamma + 1$, *and* (2. 1)' *holds, i.e.*

(3. 1)' $\qquad\qquad \displaystyle\int_0^t |\Psi_\beta(u)|\,du = O(t^{\gamma+1}),$

then we have

$$\bar{t}_n{}^{r+1} = o(n^{q+1}), \qquad q = (r + 1)\beta/(\gamma + 1),$$

for

$$-1 \leqq r < \inf\,(\gamma,\ [2(\gamma + 1) + \beta]/(\gamma + 1 - \beta)).$$

PROOFS OF LEMMAS 1 and 2. We sketch the proofs. They run quite analogously as those of Theorems 1 and 2 in the paper [2], the common principal part of which is essentially based on Lemma 1 in there, in particular on the use of the kernel

(3. 5) $\qquad \chi_n^r(t) = \dfrac{1}{n^q m^k} \displaystyle\sum_{\nu_1=1}^m \sum_{\nu_2=1}^m \cdots \sum_{\nu_k=1}^m D_{n+\nu_1+\nu_2+\,\ldots\,+\nu_k}^r(t),$

where

$$D_n^r(t) = \frac{1}{2}\,A_n^r + \sum_{\nu=1}^n A_{n-\nu}^r \cos\,\nu t,$$

and $m = m(n) < (2k)^{-1}n$, k being a sufficiently large integer.

From the definition for $\bar{t}_n{}^{r+1}$ in §1 we have

(3. 6) $$\bar{t}_n^{r+1} = -\frac{2}{\pi}\int_0^\pi \psi(t)\,\frac{d}{dt}\,D_n^r(t)\,dt.$$

In order to prove the present Lemmas 1, 2, as a matter of fact, it is sufficient to employ the identity

$$\frac{\bar{t}_n^{r+1}}{n^{q+1}} = \frac{1}{n^{q+1}m^k}\sum_{\nu_1=1}^m\sum_{\nu_2=1}^m\cdots\sum_{\nu_k=1}^m \bar{t}_{n+\nu_1+\nu_2+\ldots+\nu_k}^{r+1}$$

$$-\frac{1}{n^{q+1}m^k}\sum_{\nu_1=1}^m\sum_{\nu_2=1}^m\cdots\sum_{\nu_k=1}^m (\bar{t}_{n+\nu_1+\nu_2+\ldots+\nu_k}^{r+1} - \bar{t}_n^{r+1}),$$

and its analogue in which $n + \nu_1 + \nu_2 + \ldots + \nu_k$ is replaced by $n - \nu_1 - \nu_2 - \cdots - \nu_k$. The second term of the right hand side can be treated quite similarly as in [2]. The first term is

$$-\frac{2}{\pi n}\int_0^\pi \psi(t)\,\frac{d}{dt}\,\chi_n^r(t)\,dt,$$

where $\chi_n^r(t)$ denotes, by (3. 6), the same expression as in (3. 5).

Thus, the only difference between the method of the proofs of Lemmas 1, 2 and that of Theorems 1, 2 in [2] exists in using the kernel

$$\frac{d}{dt}\,(n^{-1}\chi_n^r(t))$$

in place of $\chi_n^r(t)$. Here, we do not reproduce the argument since these kernels have almost all the same properties.

The difference between these two kernels, which is analogous to that between the Fejér kernel and its conjugate, well interprets the reason why we may restrict to be $r - q < 2$, cf. (N. B. 1), in the present lemmas, in place of the restriction $r - q < 1$ in Theorems 1, 2 in [2]. Concerning the latter difference, see e. g. the paper [4, (1. 9) and (1.10)].

The proofs of Lemmas 3 and 4 are similar.

LEMMA 5. *If* $\alpha > -1$, *then a necessary and sufficient condition for* $\theta_\alpha(t) \to \bar{s}$ *is that* $\psi_{\alpha+1}(t) \to 0$ *and* $\theta(t) \to \bar{s}(C)$.

See Bosanquet [1, Lemma 3].

LEMMA 6. *A necessary and sufficient condition that the allied series of* $\psi(t)$ *should be summable* (C) *for* $t = 0$ *to the sum* $\bar{s}$ *is that* $\theta(t) \to \bar{s}(C)$.

See Bosanquet [1, Lemma 5].

LEMMA 7. *If* $\alpha \geqq 0$ *and* $\psi_\alpha(t) = O(1)$, *then the allied series is either summable* $(C, \alpha + \delta)$, $\delta > 0$, *or not summable* (A). *A necessary and sufficient condition for summability to* $\bar{s}$ *is* $\theta_\beta(t) \to \bar{s}$ *for* $\beta > \alpha - 1$.

See Bosanquet [1, Theorem 6].

LEMMA 8. *Let $q > 0$ and $\Psi_q(t) = o(t^r)$ as $t \to 0$. If $q < r$, then there exists the limit*

$$\bar{s} = \lim_{t \to 0} \theta_{q-1}(t),$$

and

$$\theta_{q-1}(t) - \bar{s} = o(t^{r-q}).$$

In the case $q = r$ it is ambiguous.

PROOF. If $q > 0$, then we have

$$\psi_q(t) = (\pi/2) q (\theta_q(t) - \theta_{q-1}(t)),$$

by Bosanquet [1, (2.14)], and

$$\frac{d}{dt} (t^q \theta_q(t)) = q t^{q-1} \theta_{q-1}(t),$$

by the definition for $\theta_q(t)$ in §1, provided that $\theta_{q-1}(t)$ is defined. Hence, observing that $\theta_q(t)$ is defined for every $t > 0$ if $q > 0$, the assumption $\Psi_q(t) = o(t^r)$, i. e. $\psi_q(t) = o(t^{r-q})$ implies the existence of $\theta_{q-1}(t), t > 0$, and

(3. 7) $\theta_{q-1}(t) - \theta_q(t) = o(t^{r-q})$ as $t \to 0$.

Multiplying both sides of (3. 7) by $q t^{q-1}$ we have

$$\frac{d}{dt} (t^q \theta_q(t)) - q t^{q-1} \theta_q(t) = o(t^{r-1}),$$

and then

$$t^q \frac{d}{dt} \theta_q(t) = o(t^{r-1}),$$

which is

$$\frac{d}{dt} \theta_q(t) = o(t^{r-q-1}) \text{ as } t \to 0.$$

From the last relation, if $q < r$ then we have

$$[\theta_q(t)]_{t=t_1}^{t_2} = o(t_2^{r-q}) - o(t_1^{r-q}) \to 0 \ (t_1 \to 0, \ t_2 \to 0),$$

which implies the existence of $\lim_{t \to 0} \theta_q(t)$ ($= \bar{s}$, say), and $\theta_q(t) - \bar{s} = o(t^{r-q})$. Consequently, (3. 7) yields $\bar{s} = \lim \theta_{q-1}(t)$, and $\theta_{q-1}(t) - \bar{s} = o(t^{r-q})$.

In the case $q = r$, $\bar{s} = \lim \theta_{q-1}(t)$ exists if and only if $\theta(t) \to \bar{s}(C)$, i. e. $\bar{s}_n \to \bar{s}(C)$, by Lemmas 5 and 6. Thus, we get the lemma.

LEMMA 9. *Let $0 \le \beta$, $-1 \le c$, $0 < \beta - b \le \gamma - c$ and $-1 < c - b < 2$ (except the case $\beta - \gamma = b - c = 0$). If*

(3. 1) $$\int_0^t |\Psi_\beta(u)|\,du = o(t^{\gamma+1}),$$

and if

(3. 8) $$\bar{s}_n^c - A_n^c \bar{s} = O_L(n^b)$$

holds for some $\bar{s}$, *then we have* $\bar{t}_n^{c+1} = O_L(n^{b+1})$.

This lemma holds, a fortiori, when L in O_L is omitted.

PROOF. (I) The case $-1 < \gamma < \beta + 2$. If $\beta \neq \gamma$, then $\Psi_{\beta+1}(t) = o(t^{\gamma+1})$ which follows from (3. 1) yields, by a modified theorem of M. N. Obrechkoff, i. e. by Theorem 2^0,

(3. 9) $$\bar{s}_n^{\gamma+1+\eta} - A_n^{\gamma+1+\eta}\bar{s} = o(n^{\beta+1+\eta}), \quad \eta > 0,$$

where $\bar{s} = 0$ when $\beta < \gamma$, and $\bar{s} = \lim \theta_\beta(t)$ when $\beta < \gamma$. (3. 8) and (3. 9) with $\eta = 1$ imply

$$\bar{s}_n^{c+1} - A_n^{c+1}\bar{s} = o(n^{b+(\beta+2-b)/(\gamma+2-c)}),$$

by a L.S.Bosanquet's convexity theorem of Tauberian type. See, e.g. the paper [2, Lemma 2]. Hence, we have

(3.10) $$\bar{s}_n^{c+1} - A_n^{c+1}\bar{s} = o(n^{b+1}),$$

since $0 < \beta - b \leqq \gamma - c$.

On the other hand, from the definitions for $\bar{t}_n^{c+1}$ and $\bar{s}_n^c$ in §1 we have the identity

$$\bar{t}_n^{c+1} = n\bar{s}_n^c - (c+1)\bar{s}_{n-1}^{c+1},$$

which is written as

$$\bar{t}_n^{c+1} = n(\bar{s}_n^c - A_n^c\bar{s}) - (c+1)(\bar{s}_{n-1}^{c+1} - A_{n-1}^{c+1}\bar{s}),$$

independent of $\bar{s}$. Substituting (3. 8) and (3.10) into the last relation we get $\bar{t}_n^{c+1} = O_L(n^{b+1})$, which is the required.

Next, if $\beta = \gamma$ then we have $c < b$ and $0 < \beta - b < \gamma - c$ since the case $\beta - \gamma = b - c = 0$ is excepted. Taking the number γ_1 such as $\beta - b = \gamma_1 - c$ we see that

$$-1 < \gamma_1 < \beta + 2, \quad \beta \neq \gamma_1 \text{ and } \gamma_1 < \gamma,$$

by the assumptions. Thus, $\Psi_{\beta+1}(t) = o(t^{\gamma_1+1})$ gives the same conclusion as above.

(II) The case $\gamma \geqq \beta + 2$. This may be reduced to the case (I). Indeed, taking a number γ_2 such as $\gamma_2 = \beta + c - b + d$ for e.g.$d = -(c-b+1)/2$ we can make

$$-1 < \gamma_2 < \beta + 2, \quad \text{and} \quad \gamma_2 < \gamma,$$

under the assumptions.

Hence, we get the lemma.

4. Proofs of Theorem 1 and Corollaries 1, 2.

PROOF OF THEOREM 1. By Lemma 9, $\bar{s}_n^c - A_n^c \bar{s} = O_L(n^b)$ implies

$$\bar{t}_n^{c+1} = O_L(n^{b+1}),$$

under the rest conditions in the theorem. And, this holds of course without L in O_L. Hence, after Lemma 1, the assumptions in (I) of Theorem 1 conclude that

$$(4.\ 1) \qquad t_n^{r+1} = o(n^{q+1}), \qquad q = b + (r - c)\,(\beta - b)/(\gamma - c),$$

for $c < r < \gamma'$, γ' being defined by (2. 4)

On the other hand, we have the well known identities

$$\bar{t}_n^{r+1} = nA_n^r(\bar{\sigma}_n^r - \bar{\sigma}_{n-1}^{r+1})$$
$$= nA_n^{r+1}(\bar{\sigma}_n^{r+1} - \bar{\sigma}_{n-1}^{r+1}).$$

Thus, (3. 2) is written as, in two different ways,

$$(4.\ 2) \qquad\qquad \bar{\sigma}_n^r - \bar{\sigma}_{n-1}^{r+1} = o(n^{q-r}),$$

$$(4.\ 3) \qquad\qquad \bar{\sigma}_n^{r+1} - \bar{\sigma}_{n-1}^{r+1} = o(n^{q-r-1}),$$

since $A_n^r \simeq n^r/\Gamma(r + 1)$, $r > -1$, as $n \to \infty$.

If $q - r > 0$, then adding both sides of (4. 3) from $n = 1$ to n we get $\bar{\sigma}_n^{r+1} = o(n^{q-r})$, and then from (4. 2) $\bar{\sigma}_n^r = o(n^{q-r})$, which is the same thing as

$$(4.\ 4) \qquad\qquad \bar{s}_n^r - A_n^r \bar{s} = o(n^q),$$

where $\bar{s} = 0$.

If $q - r = 0$, then (4. 2), i.e. $\sigma_n^r - \bar{\sigma}_{n-1}^{r+1} = o(1)$, and its analogues in which r and n are replaced by $r + 1$, $n - 1$; $r + 2$, $n - 2$; $\cdots$, conclude that Σb_n is summable (C, r) if and only if it is summable (C).

If $q - r < 0$, then (4. 3) implies

$$\bar{\sigma}_{n+p}^{r+1} - \bar{\sigma}_n^{r+1} = o(n^{q-r}) \qquad\qquad for\ p = 1, 2, \cdots,$$

and then the existence of $\lim \bar{\sigma}_n^{r+1}$ ($= \bar{s}$, say), and $\epsilon_n^{r+1} - \bar{s} = o(n^{q-r})$. Hence, from (4. 2) we have

$$(4.\ 5) \qquad\qquad \bar{s} = \lim_{n \to \infty} \bar{\sigma}_n^r,$$

and

$$\bar{\sigma}_n^r - \bar{s} = o(n^{q-r}),$$

which is the same thing as (4. 4).

In order to complete the proof it is sufficient to show that there exists $\lim \theta_r(t)$ whenever $q - r < 0$, and that this limit should be equal to the constant $\bar{s}$

in (4. 5). Now, since

$$q = b + (r - c)\rho, \ \rho = (\beta - b)/(\gamma - c),$$

by (4. 1), we have $0 < \rho \leqq 1$, and

(4. 6) $$q - r = (\beta - \gamma) + (\gamma - r)(1 - \rho),$$

which conclude that in the interval $c < r < \gamma'$,

    1^0 if $\beta \geqq \gamma$ (except the case 2^0 below), then steadily $q - r > 0$,

    2^0 if (and only if) $\beta - \gamma = b - c = 0$, then identically $q - r = 0$,

    3^0 if $\beta < \gamma$, then there exists an r such that $q - r < 0$.

    In the case 1^0, clearly $\bar{s}$ in (4. 4) may be arbitrary. In the case 2^0, the theorem is true if and only if $\theta(t) \to \bar{s}(C)$ by Lemma 6, as it is noticed in (N.B.2).

    In the case 3^0, taking an r such that $q - r < 0$ we have (4. 5), which implies

(4. 7) $$\theta(t) \to \bar{s}(C) \text{ as } t \to 0,$$

by Lemma 6. Since (4. 5) holds when r is, as we may, replaced by a slightly smaller one, we get

(4. 8) $$\psi_{r+1}(t) \to 0 \text{ as } t \to 0,$$

by a well known theorem due to L.S.Bosanquet. Cf. e. g. Theorem 4^0 with $\beta = \gamma$ in the last article. (4. 7) and (4. 8) imply $\bar{s} = \lim \theta_r(t)$ by Lemma 5.

    On the other hand, by Lemma 8 the condition

(4. 9) $$\Psi_{\beta+1}(t) = o(t^{\gamma+1}), \qquad \beta < \gamma,$$

which follows from (2. 1) implies the existence of $\lim \theta_\beta(t)$. So, we have

$$\bar{s} = \lim_{t \to 0} \theta_r(t) = \lim_{t \to 0} \theta_\beta(t).$$

Thus, we get (I) of Theorem 1.

The proof of (II) of Theorem 1 is quite similar.

Similarly, Theorems $2, 6$ and 1^0 follow from Lemmas $2, 3, 4$ and 9.

PROOF OF COROLLARY 1.   It is sufficient to show the existence of

(4.10) $$\bar{s} = \lim_{t \to 0} \theta_\alpha(t).$$

Applying Corollary 5 which will appear in the next article, and is independent of Theorem 1, to (4. 9) and $b_n = O_L(n^{-(1-\delta)})$, one obtains

$$\Psi_{\alpha'}(t) = o(t^{\alpha'}), \quad \alpha' = (\gamma + 1)\delta/(\gamma - \beta + \delta).$$

And, (4. 9) implies the existence of $\lim \theta_\beta(t)$. So, there exists $\lim \theta_{\alpha'-1}(t) \ (= \bar{s}$, say), by Lemma 5. Observing that $\alpha' - 1 = \alpha$, we get (4.10).

PROOF OF COROLLARY 2.   (4. 9) and $\psi(t) = O_L(t^-)$ imply

$$\Psi_q(t) = o(t^r), \ r = -\delta + (\gamma + 1 + \delta)q/(\beta + 1)$$

for $1 \leqq q < \beta + 1$, by a M.Riesz's theorem. See, e. g. the paper [2, Lemma 5]. Here, we may put

$$q = \alpha + 1 \qquad (\alpha = \beta\delta/(\gamma - \beta + \delta)),$$

since $1 < \alpha + 1 < \beta + 1$ is easily verified. We then have

$$r - q = -\delta + (\gamma + 1 + \delta)(\alpha + 1)/(\beta + 1) - (\alpha + 1) = (\gamma - \beta)/(\beta + 1) > 0.$$

Hence, one obtains $\Psi_{\alpha+1}(t) = o(t^{\alpha+1})$, which together with (4. 9) assures the existence of the limit (4.10).

Quite analogously we can prove the rest of the corollaries.

5. Theorems 3, 4 and 5. The proofs of the following Theorems 3, 4 and 5 do not differ in principle from those of Theorems 3, 4 and 5 in the papers [2, 3] respectively.

THEOREM 3. *Let* $-1 \leqq \beta, \ 0 \leqq c$ *and* $0 < \gamma + 1 - c \leqq \beta + 1 - b$, $[(\beta + 1)(c - 1) < b\gamma]$. (I) *If* $\bar{s}_n^\beta - A_n^\beta \bar{s} = o(n^\gamma)$, *or more generally*

$$(5.\ 1) \qquad \sum_{\nu=1}^n |\bar{s}_\nu^\beta - A_\nu^\beta \bar{s}| = o(n^{\gamma+1}), \ as \ n \to \infty,$$

holds for some constant $\bar{s}$, *and if*

$$(5.\ 2) \qquad \Psi_c(t) = O(t^b) \ as \ t \to 0,$$

then we have

$$(5.\ 3) \qquad \Psi_r(t) = o(t^q), \qquad q = b + (r - c)(\beta + 1 - b)/(\gamma + 1 - c),$$

as $t \to 0$, *for* $c < r < \gamma + 1$.

(II) *If* (5. 1) *holds for some constant* $\bar{s}$, *and*

$$(5.\ 2)' \qquad \Psi_c(t) = O_L(t^b) \ as \ t \to 0,$$

then we have (5. 3) *for* $c + 1 \leqq r < \gamma + 1$, *provided that* $c < \gamma$.

REMARK 2. In Theorem 3 (and also in Theorem 4 below), we have the following three cases concerning the constant $\bar{s}$.

1^0 If $\beta < \gamma$, then $\bar{s}$ may be arbitrary in view of (5. 1), and so we may put $\bar{s} = 0$.

2^0 If $\beta > \gamma$, then $q > r$ occurs in the interval $c < r < \gamma + 1$, and by Lemma 8, for such r (5. 3) implies the existence of $\lim \theta_{r-1}(t)$, and then that of $\lim \theta_\gamma(t)$ since $\gamma > r - 1$. Hence, observing that $\bar{\sigma}_n^{\beta+1} \to \bar{s}$ by (5. 1), we conclude that $\bar{s}$ should be equal to $\lim \theta_\gamma(t)$ by Lemma 6.

3^0 If $\beta = \gamma$, then by a modified S.Izumi's theorem (cf. Theorem 4^0 below), (5. 1) implies $\psi_{\gamma+1+\eta}(t) \to 0$ $(\eta > 0)$, and then $\theta_{\gamma+\eta}(t) \to \bar{s}$ by Lemmas 6 and 5.

In particular, $\bar{s} = \lim \theta_{\gamma+1}(t)$.

Concerning Theorem 5 below it is slightly different. If $\beta = \gamma$, then we may put $\bar{s} = 0$.

THEOREM 4. *Let* $-1 \leqq \beta, 0 \leqq c$ *and* $0 < \gamma + 1 - c \leqq \beta + 1 - b$, $[(\beta + 1)(c - 1) < b\gamma]$. *If* (5. 1) *holds for some constant* $\bar{s}$, *and if* $\bar{s}_n^{b-1} - A_n^{b-1}\bar{s} = O(n^{c-1})$, *or more generally*

$$(5.\ 4) \qquad \sum_{\nu=n}^{2n} (|\bar{s}_\nu^{b-1} - A_\nu^{b-1}\bar{s}| - (\bar{s}_\nu^{b-1} - A_\nu^{b-1}\bar{s})) = O(n^c) \ as \ n \to \infty,$$

then we have the conclusion in (I) *of Theorem* 3.

THEOREM 5. *Let* $0 \leqq c$ *and* $0 < \gamma - c \leqq \beta - b$, $[(c - 1)\beta < b(\gamma - 1)]$. *If*

$$\Psi_\gamma(t) = o(t^\beta) \ as \ t \to 0,$$

and if $s_u^{b-1} - A_n^{b-1}\,\bar{s} = O(n^{c-1})$, *or more generally* (5. 4) *holds for some constant* $\bar{s}$, *then we have*

$$\Psi_r(t) = o(t^q), \qquad c = b + (r - c)(\beta - b)/(\gamma - c),$$

as $t \to 0$, *for* $c < r < \gamma$.

When $b = 0$ and $c = 1$, Theorem 3 can be slightly modified as follows.

THEOREM 3⁰. *If* $0 < \gamma < \beta + 1$, *and* $\bar{s}_n^\beta - A_n^\beta\,\bar{s} = O(n^\gamma)$, *or more generally*

$$(5.\ 1)' \qquad \sum_{\nu=1}^{n} |\bar{s}_\nu^\beta - A_\nu^\beta\bar{s}| = O(n^{\gamma+1})$$

holds for some $\bar{s}$, *then we have*

$$\Psi_r(t) = o(t^{(r-1)(\beta+\nu)/\gamma}), \qquad 1 \leqq r < \gamma + 1.$$

From Theorem 4 we have the following

THEOREM 4⁰. *If* $-1 \leqq \beta$, $-1 < \gamma < \beta + 1$ *and* $\bar{s}_n^\beta - A_n^\beta\bar{s} = o(n^\gamma)$, *or more generally* (5. 1) *holds for some* $\bar{s}$, *then we have*

$$\Psi_{\gamma+1+\eta}(t) = o(t^{\beta+1+\eta}), \quad \eta > 0.$$

In this theorem, the case $\gamma \geqq \beta + 1$ is trivial.

Theorems 3, 4 and 5 give the following corollaries respectively.

COROLLARY 3. *Let* $0 < \delta$ *and* $-1 < \gamma < \beta$, $[\delta\gamma < \beta + 1]$. *If*

$$(5.\ 1) \qquad \sum_{\nu=1}^{n} |\bar{s}_\nu^\beta - A_\nu^\beta\bar{s}| = o(n^{\gamma+1})$$

holds for some $\bar{s}$, and if either of the two conditions

$$\begin{cases} \psi(t) = O(t^{-\delta}), \\ \psi(t) = O_L(t^{-\delta}), \ \beta - \gamma \leqq \delta\gamma, \end{cases}$$

is satisfied, then we have $\bar{s} = \lim \theta_{\alpha-1}(t)$, *and*

$$\Psi_\alpha(t) = o(t^\alpha), \ \alpha = \delta(\gamma + 1)/(\beta - \gamma + \delta).$$

Here, we prove the existence of the limit $\bar{s}$. Indeed, (5. 1) together with $-1 < \gamma < \beta$ implies $\bar{\sigma}_n^{\beta+1} \to \bar{s}$, and then by Lemma 6 $\theta(t) \to \bar{s}\,(C)$. Hence, $\Psi_\alpha(t) = o(t^\alpha)$, $\alpha > 0$, assures that $\bar{s} = \lim \theta_{\alpha-1}(t)$ by Lemma 5.

COROLLARY 4. *Let* $0 < \gamma < 1$ *and* $-(1 - \delta) < \gamma < \beta$. *If* (5. 1) *holds for some* $\bar{s}$, *and* $b_n = O_L(n^{-(1-\delta)})$, *then we have* $\bar{s} = \lim \theta_{\alpha-1}(t)$, *and*

$$\Psi_\alpha(t) = o(t^\alpha), \ \alpha = \delta(\beta + 1)/(\beta - \gamma + \delta).$$

COROLLARY 5. *Let* $0 < \delta < 1$ *and* $\delta < \gamma < \beta$. *Then*

$$\Psi_\gamma(t) = o(t^\beta) \text{ and } b_n = O_L(n^{-(1-\delta)})$$

imply

$$\Psi_\alpha(t) = o(t^\alpha), \quad \alpha = \beta\delta/(\beta - \gamma + \delta),$$

and the existence of $\bar{s} = \lim \theta_{\alpha-1}(t)$.

COROLLARY 3⁰. *If* $0 < \gamma < \beta$, *and* (5. 1)′ *holds for some* $\bar{s}$, *then we have* $\bar{s} = \lim \theta_{\alpha-1}(t)$, *and*

$$\Psi_\alpha(t) = o(t^\alpha), \ \alpha = (\beta + 1)/(\beta - \gamma + 1).$$

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For more precise references see the paper [2].

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