ON THE DECOMPOSITION THEOREMS OF FOURIER TRANSFORMS WITH WEIGHTED NORMS

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1. Introduction. Littlewood and Paley [6] proved the following result;

For $f(x) \in L^p(-\pi, \pi)\,(1 < p < \infty)$, *let*

$$\hat{f}(\nu) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-i\nu x} dx, \tag{1.1}$$

$$f(x) \sim \sum_{\nu=-\infty}^{\infty} \hat{f}(\nu) e^{i\nu x}. \tag{1.2}$$

If

$$\Delta_n(x) = \begin{cases} \sum_{\nu=2^{n-1}}^{2^n-1} \hat{f}(\nu) e^{i\nu x} & n = 1, 2, \cdots \\ \hat{f}(0) & n = 0 \\ \sum_{\nu=-2^{-n}+1}^{-2^{-n-1}} \hat{f}(\nu) e^{i\nu x} & n = -1, 2, \cdots, \end{cases} \tag{1.3}$$

then

$$0 < A_p \leqq \int_{-\pi}^{\pi} \left\{ \sum_{n=-\infty}^{\infty} |\Delta_n(x)|^2 \right\}^{p/2} dx \Big/ \int_{-\pi}^{\pi} |f(x)|^p dx \leqq A'_p < \infty.$$

Concerning this theorem, discrete and integral analogues were proved by G.Sunouchi [12], [11] and recently J.Schwartz [8] gave a new proof. On the other hand, the theorem just cited was extended by I.I.Hirschman Jr. to the weighted L^p-class (Theorems 6 and 7) and the Fourier integral case with the weighted norms was investigated by D.L.Guy [2] (Theorems 1 and 2). However their proofs are complicated.

In the present note we shall prove the integral, discrete and ordinary cases with weighted norms with the idea of J.Schwartz. Our main methods depend upon the extended Marcinkiewicz interpolation theorem and the test for an operator to be weak type $(1, q)$ due to L. Hörmander [5] which are applied to vector-valued functions by J.Schwartz [8], and another tool is of a substitute of Parseval's relation.

§§2-6 and §§7-8 are devoted to the proof of Fourier integral and discrete cases respectively.

In §§9-11, we shall prove the theorems for Fourier series.

In §12 we shall consider some well known inequalities concerning the functions of Littlewood-Paley $g(\theta)$, $g^*(\theta)$, the function of Lusin $s(\theta)$ and that of Marcinkiewicz $\mu(\theta)$ in view of decomposition theorem. These considerations give unified real treatment of these functions.

2. The Integral case. The integral analogue of Littlewood-Paley's decomposition theorem with weighted norms is stated as follows.

THEOREM 1. *Let* $1 < p < \infty$, $-1 < \alpha < p-1$ *and* $f(x) \in L^p_\alpha$, *that is,* $|f(x)|^p|x|^\alpha \in L(-\infty, \infty)$. *Let* $\Delta_n(x) \equiv \Delta_n(x, f)$ *be the function whose Fourier transforms* $\hat{\Delta}_n(x)$ *is identical with that of* $f(x)$ *in the domain* $2^n \leqq |x| < 2^{n+1}$ *and vanishes outside this domain. Then,*

$$0 < A_{\alpha,p} \leqq \int_{-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} |\Delta_n(x)|^2 \right\}^{p/2} |x|^\alpha dx \Big/ \int_{-\infty}^{\infty} |f(x)|^p |x|^\alpha dx \leqq A'_{p,\alpha} < \infty. \tag{2. 1}$$

The inequalities in this note are to be interpreted as meanings, "if the majorant is finite, then the inequality is satisfied", and A, A_p, A'_p etc. are positive constants depending only on the indices submitted and may be different in each case.

For the proof of Theorem 1, we introduce the auxiliary function used by J. Schwartz. Let $\phi(x)$ be even C^∞ function equal to 1 for $1 \leqq |x| \leqq 2$ and equal to zero for $|x| \leqq 1/2$ or $|x| \geqq 3$, chosen so that its first few moments are zero. Let $\widehat{K}_1$ be the vector valued function with values in the two-sided Hilbert sequence-space l^2 as

$$\begin{aligned} \widehat{K}_1(x) &= (\cdots, \phi(2^n x), \phi(2^{n+1}x), \cdots) \\ &= (\cdots, \hat{k}_n(x), \hat{k}_{n+1}(x), \cdots) \end{aligned} \tag{2. 2}$$

and define $K_1(x)$ by

$$K_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixy} \widehat{K}_1(y) dy = (\cdots, k_n(x), k_{n+1}(x), \cdots), \tag{2. 3}$$

where

$$k_0(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixy} \phi(y) dy$$

and

$$k_n(x) = 2^{-n} k_0(2^{-n}x). \tag{2. 4}$$

For a scalar valued function $f(x)$, put

$$(\mathfrak{K}_1 f)(x) = \int_{-\infty}^{\infty} K_1(x-y) f(y) dy, \tag{2. 5}$$

and for a function $G(x)$ with values in Hilbert space l^2, put

$$(\mathfrak{L}_1 G)(x) = \int_{-\infty}^{\infty} K_1(x-y) G(y) dy. \tag{2. 6}$$

Then $\mathfrak{K}_1$ maps scalar valued functions into functions with values in l^2 and $\mathfrak{L}_1$ maps functions with values in l^2 into scalar valued functions. If $f(x)$ and $G(x)$ are suitably restricted, we have

$$\int_{-\infty}^{\infty} (\mathfrak{K}_1 f)(x) \overline{G(x)} dx = \int_{-\infty}^{\infty} f(x) \overline{(\mathfrak{L}_1 G)(x)} dx. \tag{2. 7}$$

From this equality we can easily deduce the following Lemma.

LEMMA 1. *Two inequalities*

$$\| \mathfrak{K}_1 f \|_{p,\alpha} \leqq A_{p,\alpha} \| f \|_{p,\alpha}$$

and

$$\| \mathfrak{L}_1 G \|_{q,\beta} < A_{p,\alpha} \| G \|_{q,\beta}$$

are equivalent, where $1 < p < \infty$, $1/p + 1/q = 1$, $\beta = (1-q)\alpha$ *and*

$$\| f \|_{p,\alpha} = \left\{ \int_{-\infty}^{\infty} |f(x)|^p |x|^\alpha dx \right\}^{1/p}.$$

First of all we show the L_α^2-case.

LEMMA 2. *If* $-1 < \alpha < 1$, *then we have*

$$\int_{-\infty}^{\infty} |(\mathfrak{K}_1 f)(x)|^2 |x|^\alpha dx \leqq A_\alpha \int_{-\infty}^{\infty} |f(x)|^2 |x|^\alpha dx \tag{2. 8}$$

and

$$\int_{-\infty}^{\infty} |(\mathfrak{L}_1 G)(x)|^2 |x|^\alpha dx \leqq A_\alpha \int_{-\infty}^{\infty} |G(x)|^2 |x|^\alpha dx. \tag{2. 9}$$

To prove this lemma we use the two theorems.

THEOREM A. (I. I. Hirschman, Jr., [4]). *If* $0 < \alpha < 2$ *and* $f \in L_\alpha^2 \cap L^2{}_\alpha$ *then*

$$\int_{E_n} |f(x)|^2 |x|^\alpha dx = A_{n,\alpha} \int_{E_n} \int_{E_n} |\hat{f}(x) - \hat{f}(y)|^2 |x-y|^{-n-\alpha} dx dy,$$

where E_n denotes the n-dimensional Euclidean space and $\hat{f}(x)$ is the Fourier transform of $f(x)$.

THEOREM B (I. I. Hirschman, Jr. [4]). *Let α, $0<\alpha<n$ be fixed. If $\Gamma(x)$ is a non-negative measurable function on E_n such that*

$$|\{x:\ \Gamma(x)\geqq a\}|\leqq Aa^{-n/\alpha}\qquad (0<a<\infty),$$

then

$$\int_{E_n}|\hat{f}(x)|^2\Gamma(x)dx\leqq A(\Gamma)\int_{E_n}|f(x)|^2|x|^\alpha dx.$$

PROOF OF LEMMA 2. By Lemma 1, it is sufficient to prove (2. 8) for $0<\alpha<1$ and (2. 9) for $0<\alpha<1$.

For a suitable $f(x)$, we have

$$(\widehat{\mathfrak{K}_1 f})(x)=\widehat{K}_1(x)\ \hat{f}(x),$$

therefore noting that support of $\hat{k}_n(x)$ is contained in $(2^{-n-1},\ 3\cdot 2^{-n})$ and that $\hat{k}_n(x)$ is uniformly bounded, (2. 8) with $\alpha=0$ is shown easily by Parseval's relation.

Now we assume $0<\alpha<1$. By definition

$$\int_{-\infty}^{\infty}|(\mathfrak{K}_1 f)(x)|^2|x|^\alpha dx=\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left|\int_{-\infty}^{\infty}k_n(x-y)f(y)dy\right|^2|x|^\alpha dx,$$

and using Theorem A,

$$\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}k_n(x-y)f(y)dy\right|^2|x|^\alpha dx$$
$$=A_\alpha\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\hat{f}(y)\hat{k}_n(y)-\hat{f}(x)\hat{k}_n(x)|^2|x-y|^{-1-\alpha}dxdy\leqq 2A^\alpha(I_n^1+I_n^2),$$

where

$$I_n^1=\int_{-\infty}^{\infty}|\hat{f}(x)|^2dx\int_{-\infty}^{\infty}|\hat{k}_n(x)-\hat{k}_n(y)|^2|x-y|^{-1-\alpha}dy,\tag{2.10}$$

and

$$I_n^2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\hat{f}(x)-\hat{f}(y)|^2|\hat{k}_n(y)|^2|x-y|^{-1-\alpha}dxdy.\tag{2.11}$$

By the remark just mentioned for $\hat{k}_n(x)$, we get

$$\sum_{n=-\infty}^{\infty}I^2_n=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\sum_{n=-\infty}^{\infty}|\hat{k}_n(y)|^2\right\}|\hat{f}(x)-\hat{f}(y)|^2|x-y|^{-1-\alpha}dxdy$$

$$\leqq A \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\hat{f}(x) - \hat{f}(y)|^2 |x-y|^{-1-\alpha} dy dx$$

$$\leqq A \int_{-\infty}^{\infty} |f(x)|^2 |x|^\alpha dx.$$

To estimate I_n^1, we consider for $x > 0$

$$J_n(x) = \int_{-\infty}^{\infty} |\hat{k}_n(x) - \hat{k}_n(y)|^2 |x-y|^{-1-\alpha} dy. \tag{2.12}$$

Let $2^{-m-1} \leqq x < 2^{-m}$, m being some integer. If $m-1 \leqq n \leqq m+3$, then we put

$$J_n(x) = \int_0^{2x} + \int_{2x}^{\infty}$$
$$= J_n^1(x) + J_n^2(x), \text{ say.}$$

Using the fact that $|\hat{k}_n(x) - \hat{k}_n(y)| < 2^n A |x-y|$ or $J_n^1(x)$ and $|\hat{k}_n(x) - \hat{k}_n(y)| \leqq A$ for $J_n^2(x)$, we get

$$J_n^1(x) = A \cdot 2^{2n} \int_0^{2x} |x-y|^{1-\alpha} dy \leqq A_\alpha x^{-\alpha},$$

and

$$J_n^2(x) = A \int_{2x}^{\infty} |x-y|^{-1-\alpha} dy \leqq A_\alpha x^{-\alpha},$$

hence

$$J_n(x) \leqq A_\alpha x^{-\alpha}, \qquad \text{for } m-1 \leqq n \leqq m+3$$

If $n > m+3$ or $n > m-1$, then observing $\hat{k}_n(x) = 0$,

$$J_n(x) \leqq A \int_{2^{-n-1}}^{3.2^{-n}} |x-y|^{-1-\alpha}\, dy$$

and

$$\sum J_n(x) \leqq A \left(\int_0^{3\cdot 2^{-m-4}} + \int_{2^{-m+1}}^{\infty} \right) |x-y|^{-1-\alpha}\, dy$$
$$\leqq A \left(\int_0^{3x/8} + \int_{2x}^{\infty} \right) |x-y|^{-1-\alpha}\, dy$$
$$\leqq A_\alpha x^{-\alpha},$$

where the summation is taken over $\{n : n \leqq m-2 \text{ or } m+4 \leqq n\}$. Therefore

$$\sum_{n=-\infty}^{\infty} J_n(x) \leqq A_\alpha x^{-\alpha}, \tag{2.13}$$

and

$$\sum_{n=-\infty}^{\infty} I_n^1 \leqq A_\alpha \int_{-\infty}^{\infty} |\hat{f}(x)|^2 |x|^{-\alpha}\, dx$$
$$\leqq A_\alpha \int_{-\infty}^{\infty} |f(x)|^2 |x|^{\alpha}\, dx,$$

applying Theorem B. Collecting these inequalities, we get (2. 8) for $0 \leqq \alpha < 1$.

Now we prove the inequality (2. 9) for $0 < \alpha < 1$.

Let $G(x) = (\cdots, g_n(x), g_{n+1}(x), \cdots)$ belong to $L_\alpha^2(l^2)$ and $L^2(l^2)$, i. e. $\int_{-\infty}^{\infty} |G(x)|^2 |x|^\alpha dx < \infty$ and $\int_{-\infty}^{\infty} |G(x)|^2 dx < \infty$, then

$$(\widehat{\mathfrak{L}_1 G})(x) = \sum_{n=-\infty}^{\infty} \hat{g}_n(x)\hat{k}_n(x). \tag{2.14}$$

Again by Theorem A,

$$\int_{-\infty}^{\infty} |(\mathfrak{L}_1 G)(x)|^2 |x|^\alpha\, dx$$
$$= A_\alpha \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| \sum_{n=-\infty}^{\infty} \{ \hat{k}_n(x)\hat{g}_n(x) - \hat{k}_n(y)\hat{g}_n(y) \} \right|^2 |x-y|^{-1-\alpha} dxdy$$
$$\leqq 2A_\alpha(I^1 + I^2), \tag{2.15}$$

where

$$I^1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| \sum_{n=-\infty}^{\infty} \hat{g}_n(x) \{\hat{k}_n(x) - \hat{k}_n(y)\} \right|^2 |x-y|^{-1-\alpha} dxdy \tag{2.16}$$

and

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| \sum_{n=-\infty}^{\infty} \hat{k}_n(x) \{\hat{g}_n(x) - \hat{g}_n(y)\} \right|^2 |x-y|^{-1-\alpha} dxdy. \tag{2.17}$$

Applying Schwarz inequality, we get,

$$I^1 \leqq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} |\hat{g}_n(x)|^2 \right\} \left\{ \sum_{n=-\infty}^{\infty} |\hat{k}_n(x) - \hat{k}_n(y)|^2 \right\} |x-y|^{-1-\alpha} dxdy, \tag{2.18}$$

and

$$I^2 \leqq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} |\hat{k}_n(x)|^2 \right\} \left\{ \sum_{n=-\infty}^{\infty} |\hat{g}_n(x) - \hat{g}_n(y)|^2 \right\} |x-y|^{-1-\alpha} dxdy. \tag{2.19}$$

Hence we have the required inequality by using the estimate (2.13) for (2.18) and Theorem A for (2.19).

3. In order to generalize Lemma 2, we shall state here some preliminary theorems.

The first is the Marcinkiewicz interpolation theorem extended by Stein-Weiss [10] and we can state in the following form without essential change of their proof.

Let $(M, \mathfrak{M}, \mu)$ and $(N, \mathfrak{N}, \nu)$ be two measurable spaces and $\alpha_0, \alpha_1, \beta_0$ and β_1 be positive measurable functions with respect to μ and ν respectively. Let define the measure μ_s on $\mathfrak{M}$ and ν_r on $\mathfrak{N}$ by

$$d\mu_s = \alpha_0^{1-s}\,\alpha_1{}^s d\mu \qquad \text{and} \qquad d\nu_r = \beta_0^{1-r}\beta_1{}^r d\nu \tag{3. 1}$$

for $0 \leqq r,\ s \leqq 1$. Let $1 \leqq p_j \leqq q_j \leqq \infty$ $(j = 0, 1)$, $p_0 \neq p_1$, $q_0 \neq q_1$,

$$1/p_t = (1-t)/p_0 + t/p_1, \qquad (0 \leqq t \leqq 1) \tag{3. 2}$$

$$1/q_t = (1-t)/q_0 + t/q_1, \qquad (0 \leqq t \leqq 1) \tag{3. 3}$$

and $s(t) = (tp_t)/p_1$, $r(t) = (tq_t)/q_1$ and set

$$d\xi = \left(\frac{\beta_0^{q_1}}{\beta_1^{q_0}} \right)^{1/(q_1-q_0)} d\nu. \tag{3. 4}$$

THEOREM C. *Let X and Y be two Banach spaces and T be sublinear operator, mapping functions defined on M and having values in X into functions defined on N and having values in Y. Suppose that T has the following two properties;*

(i) *The domain of T includes $L_{\mu_0}^{p_0}(X) \cup L_{\mu_1}^{p_1}(X)$, where $L_\mu^p(X)$*[1] *denotes the L^p-space with values in X.*

(ii) *If f is in $L_{\mu_j}^{p_j}(X)$ $(j = 0, 1)$, let*

$$F_y = \{x \in N\colon \|k(x)(Tf)(x)\|_Y > y\}$$

where $k = (\beta_0/\beta_1)^{1/(q_0-q_1)}$ and $y > 0$. Then, we have

$$\xi(F_y) \leqq \left\{ \frac{A_j}{y} \|f\|_{p_j,\mu_j} \right\}^{q_j}. \tag{3. 5}$$

Then, T is defined on $L_{\mu_{s(t)}}^{p_t}(X)$ for $0 < t < 1$ and if f is in this space

$$\|Tf\|_{q_t,\,\mu_{r(t)}} \leqq A_t \|f\|_{p_t,\,\mu_{s(t)}}. \tag{3. 6}$$

Next two lemmas are the variations of L.Hörmander's theorem [5] (see also, J.Schwartz [8]).

LEMMA 3. *With above X, Y, let $K(x)$, $x \in (-\infty, \infty)$ be the bounded linear operators of X into Y. Suppose that $K(x)$ is locally integrable; suppose that there exist constant A, A', A'' such that*

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1) In special case $d\mu(x) = |x|^\alpha dx$, L_μ^p and $||p||_{p,\mu}$ will be denoted by L_α^p and $||f||_{p,\alpha}$ respectively. If $d\mu = dx$, we write simply L^p and $||f||_p$.

$$\int_{|x|\geqq A} |||K(t(x-y))-K(tx)|||\,dx \leqq A't^{-1} \tag{3.7}$$

for all y, $|y| \leqq A''^{-1}$ *and*

$$|||K(x)||| \leqq A'|x|^{-1} \tag{3.8}$$

for all x. *Put for function* $f(x)$ *having values in* X

$$(\mathfrak{K}f)(x) = \int_{-\infty}^{\infty} K(x-y)f(y)dy \tag{3.9}$$

Suppose also that for some $0 \leqq \alpha < 1$,

$$\|\mathfrak{K}f\|_{2,-\alpha} \leqq A\|f\|_{2,-\alpha}. \tag{3.10}$$

Then

$$\mu_{-\alpha}(\{x : \|(\mathfrak{K}f)(x)\|_Y > a\}) \leqq A/a\|f\|_{1,-\alpha} \tag{3.11}$$

where $d\mu_{-\alpha} = |x|^{-\alpha}dx$.

REMARK 1. Though this lemma is stated for $L^p_{-\alpha}$-space on $(-\infty, \infty)$, it holds for $L^p_{-\alpha}$-space on finite interval, e.g. $(-\pi, \pi)$, and for the space $l^p_{-\alpha}$ consisting of sequences $\{f(n)\}$ such that $\|f\|_{p,-\alpha} = \left\{\sum_{n=-\infty}^{\infty} |f(n)|^p(|n|+1)^{-\alpha}\right\}^{1/p} < \infty$. Since the proofs of other cases are similar, we omit it.

To prove Lemma 3, we need the following,

LEMMA 4. *Let* $u \in L^1_{-\alpha}(X)$, $0 \leqq \alpha < 1$, *where* X *is some Banach space and let* $s > 0$. *Then we can write*

$$u = v + \sum_{k=1}^{\infty} w_k \tag{3.12}$$

where $v \in L^1_{-\alpha}(X)$ *and* $w_k \in L^1_{-\alpha}(X) \cap L^1(X)$,

$$\|v\|_{1,-\alpha} + \sum_{k=1}^{\infty} \|w_k\|_{1,-\alpha} \leqq A_\alpha \|u\|_{1,-\alpha}, \tag{3.13}$$

$$\|v(x)\|_X \leqq A_\alpha \cdot s, \ a.e.\ x \in (-\infty, \infty), \tag{3.14}$$

and for certain disjoint cubes I_k,

$$\text{support of } w_k \subset I_k, \qquad k = 1, 2, \cdots, \tag{3.15}$$

$$\sum_{k=1}^{\infty} \mu_{-\alpha}(I_k) \leqq 1/s\|u\|_{1,-\alpha} \tag{3.16}$$

and

$$\int_{I_k} w_k(x)dx = 0, \qquad k = 1, 2, \cdots. \tag{3.17}$$

PROOF OF LEMMA 4. Our proof is similar to that of L. Hörmander's.

Divide the whole real axis into the intervals I_j having same measure such that one of them has origin at its end point and

$$\mu_{-\alpha}(I_j) \geqq \frac{1}{s} \|u\|_{1,-\alpha}, \qquad j = 1, 2, \cdots. \tag{3.18}$$

Divide each interval into two intervals of same measure and let I_{1j} be those intervals on which the mean of u is not less than s, then we have

$$s\mu_{-\alpha}(I_{1j}) \leqq \int_{I_{1j}} \|u\|_X d\mu_{-\alpha} \leqq 2s\mu_{-\alpha}(I_{1j}). \tag{3.19}$$

We define $v(x)$ and $w_{1j}(x)$ by

$$v(x) = \frac{1}{|I_{1j}|} \int_{I_{1j}} u(y)dy \quad \text{if } x \in I_{1j},\ j = 1, 2, \cdots, \tag{3.20}$$

$$w_{1j}(x) = \begin{cases} u(x) - v(x) & \text{if } \in I_{1j} \\ 0 & \text{if } \notin I_{1j} \end{cases}, \ j = 1, 2, \cdots. \tag{3.21}$$

Next we divide I_{1j} into two intervals with same measure and repeat the above process getting new sequence of intervals I_{2j}. Then we extend the definition (3.20) and (3.21). Continuiting in this way, we get the sequences of functions w's and intervals I's; for simplicity we write them by $\{w_k\}$, $\{I_k\}$. If we define

$$v(x) = u(x) \qquad \text{for } x \notin O \equiv \bigcup_{k=1}^{\infty} I_k,$$

then (3.12) (3.15), (3.16) and (3.17) hold clearly. Let $x \in I_k$ for some k, then we have

$$\frac{1}{|I_k|} \int_{I_k} \|u(x)\|_X dx \leqq \frac{A_\alpha}{\mu_{-\alpha}(I_k)} \int_{I_k} \|u(x)\|_X d\mu_{-\alpha}(x), \tag{3.22}$$

A_α not depending on u or I_k. In order to prove (3.22), we denote I_k by $(a - h, a + h)$ and we may assume $a > 0$. Noting $a \geqq h$ by our construction, we have

$$\int_{I_k} \|u(x)\|_X |x|^{-\alpha} dx \geqq \frac{1}{(a+h)^\alpha} \int_{I_k} \|(x)\|_X dx$$

$$\geqq \frac{1}{2^\alpha a^\alpha} \int_{I_k} \|u(x)\|_X dx.$$

In the case $a \geqq h \geqq a/2$, we get

$$\mu_{-\alpha}(I_k) = \int_{a-h}^{a+h} x^{-\alpha}\, dx \leqq \int_0^{2a} x^{-\alpha}\, dx \leqq \frac{2^{1-\alpha}}{1-\alpha} a^{1-\alpha}$$
$$\leqq A_\alpha a^{-\alpha} |I_k|,$$

and in the case $a/2 > h > 0$,

$$\mu_{-\alpha}(I_k) = \int_{a-h}^{a+h} x^{-\alpha} dx \leqq \frac{2h}{(a-h)^\alpha} \leqq \frac{2^\alpha |I_k|}{a^\alpha}.$$

Therefore we have (3.22).

By (3.20), (3.22) and (3.19), we get

$$\|v(x)\|_X \leqq \frac{1}{|I_k|} \int_{I_k} \|u(x)\|_X d\mu_{-\alpha}$$
$$\leqq \frac{A_\alpha}{\mu_{-\alpha}(I_k)} \int_{I_k} \|u(x)\|_X d\mu_{-\alpha} \leqq 2A_\alpha \cdot s. \tag{3.23}$$

If $x \notin O$, then for arbitrary small interval I,

$$\frac{1}{\mu_{-\alpha}(I)} \int_I \|u(x)\|_X d\mu_{-\alpha} \leqq 2 \cdot s,$$

hence we get $\|u(x)\|_X \leqq 2s$ for a. e. $x \notin O$. Thus (3.14) is proved.

To prove (3.13) we first note

$$\|v\|_{1,-\alpha} = \left(\int_{CO} + \int_O \right) \|v(x)\|_X d\mu_{-\alpha}$$
$$\leqq \|u\|_{1,-\alpha} + \sum_{k=1}^{\infty} \frac{\mu_{-\alpha}(I_k)}{|I_k|} \int_{I_k} \|u(y)\|_X dy$$

and using the inequality (3.22), we have

$$\|v\|_{1,-\alpha} \leqq (A_\alpha + 1) \|u\|_{1,-\alpha}.$$

Hence

$$\|v\|_{1,-\alpha} + \sum_{k=1}^{\infty} \|w_k\|_{1,-\alpha} \leqq (2A_\alpha + 3) \|u\|_{1,-\alpha}.$$

REMARK 2. In the case of $L^1_{-\alpha}$-space on $(-\pi, \pi)$, it is sufficient for our purpose to prove this lemma for $s > A_\alpha \|u\|_{1,-\alpha}$, A_α being some constant. It is convenient to divide $(-\pi, \pi)$ into four intervals I_j and take $A_\alpha = 2/\pi^{1-\alpha}$, then (3.18) holds.

REMARK 3. In the discrete case $l^1_{-\alpha}$, intervals I_j are successions of integers and may not be divisible into two intervals of same measure, therefore, we must make subdivision in the way that measure of each interval is not greater than two times of others.

PROOF OF LEMMA 3. Decompose $u \in L^1_{-\alpha}(x)$ in the way of Lemma 4. Let k be fixed and denote w_k by w and I_k by $I=(a-h,\ a+h)$, where we may assume $a>0$ and $a-h \geqq 0$. By virture of (3. 3) we may assume that (9. 7) holds with $A=1$, $A''=1/2$.

$$\begin{aligned}\int_{y\notin(a-2h,a+2h)} &\|(\mathfrak{K}w)(y)\|_Y d\mu_{-\alpha}(y)\\ &=\int_{y\notin(a-2h,a+2h)}\Big\|\int_I\{K(y-x)-K(y-a)\}w(x)dx\Big\|_Y|y|^{-\alpha}dy\\ &\leqq\int_I\|w(x)\|_X dx\int_{y\notin(a-2h,a+2h)}|||K(y-x)-K(y-a)|||\,|y|^{-\alpha}dy\\ &=\int_{-h}^{h}\|w(x+a)\|_X dx\int_{|y|\geqq 2h}|||K(y-x)-K(y)|||\,|y+a|^{-\alpha}dy. \end{aligned}\tag{3.24}$$

Now we show

$$|x+a|^{\alpha}\int_{|y|\geqq 2h}|||K(y-x)-K(y)|||\,|y+a|^{-\alpha}dy\leqq A_\alpha, \tag{3.25}$$

for $|x|\leqq h$. It is sufficient to consider the integral over $(-3a/2,-a/2)\cap(-\infty,-2h)$ by hypothesis (3. 7). By (3. 8) we get that $|||K(y)|||$, $|||K(x-y)|||\leqq Aa^{-1}$ for $y\in(-3a/2,\ -a/2)\cap(-\infty,\ -2h)$ and $x\in(-h,h)$, therefore noting $|x+a|^{\alpha}\leqq 2^{\alpha}a^{\alpha}$, we have (3.25).

By (3.24) and (3.25),

$$\int_{y\notin(a-2h.a+2h)}\|(\mathfrak{K}w)\|(y)\|_Y d\mu_{-\alpha}(y)\leqq A_\alpha\int_{x\in I}\|w(x)\|_X d\mu_{-\alpha}(x). \tag{3.26}$$

If we put $w'=\sum_{k=1}^{\infty}w_k$, then it follows from (3.16), (3.26) that there exists a set E of measure at must $A_\alpha s^{-1}\|u\|_{1,-\alpha}$ such that

$$\int_{y\notin E}\|(\mathfrak{K}w')(y)\|_Y d\mu_{-\alpha}(y)\leqq A_\alpha\|w'\|_{1,-\alpha}\leqq A_\alpha\|u\|_{1,-\alpha} \tag{3.27}$$

By well known argument we have from (3.27) that

$$\mu_{-\alpha}(\{y:\|(\mathfrak{K}w')(y)\|_Y>t\})\leqq A_\alpha(t^{-1}+s^{-1})\|u\|_{1,-\alpha}. \tag{3.28}$$

On the other hand it follows from (3.14) and hypothesis (3.10) that

$$\|(\mathfrak{K}v)\|_{2,-\alpha}\leqq A_\alpha s^{1/2}\|v\|_{1,-\alpha}^{1/2}\leqq A_\alpha s^{1/2}\|u\|_{1,-\alpha}^{1/2}. \tag{3.29}$$

Since $u=v+w$, we get by (3.28) and (3.29)

$$\mu_{-\alpha}(\{y : \|(\mathfrak{R}u)(y)\|_{r} > t\}) \leqq A_{\alpha}(t^{-1} + s^{-1}+t^{-2}s)\|u\|_{1.-\alpha}. \tag{3.30}$$

If we chose $s = t$, we get

$$\mu_{-\alpha}(\{y : \|(\mathfrak{R}u)(y)\|_{r} > t\}) \leqq A_{\alpha}t^{-1}\|u\|_{1.-\alpha}.$$

This completes our proof.

4. LEMMA 5. *Let $K_1(x)$, $x \in (-\infty, \infty)$ be the function defined by* (2. 3), *then*

$$\int_{|x|\geqq 1} |K_1(t(x-y)) - K_1(tx)|\,dx \leqq A/t, \tag{4. 1}$$

for all $t > 0$ and y, $|y| \leqq 1/2$.

This Lemma was proved by J. Schwartz [8], but for the sake of completeness we show directely.

PROOF. Since $\phi \in C^\infty$ and its first few moments are zero, we have

$$k_0(x) = O(x^{-2}),\ k_0'(x) = O(x^{-3}) \text{ as } x \to \infty, \tag{4. 2}$$

and

$$k_0(x) = O(1),\ k_0'(x) = O(x) \text{ as } x \to 0. \tag{4. 3}$$

Suppose $2^m \leqq x < 2^{m+1}$, m being some integer, then remembering $k_n(x) = 2^{-n} k_0(2^{-n}x)$,

$$|k_n'(x)| \leqq \begin{cases} A2^{-2n}\cdot 2^{-n}x & \text{for } n \geqq m \\ A2^{-2n}(2^{-n}x)^{-3} & \text{for } n \leqq m+1. \end{cases}$$

Therefore, we get

$$|K_1'(x)|^2 = \Big(\sum_{n=-\infty}^{m+1} + \sum_{n=m+2}^{\infty}\Big)|k'_n(x)|^2 \leqq A/x^4.$$

Hence

$$\int_{|x|>t} |K_1(x-yt) - K(x)|\,dx = \sum_{k=1}^{\infty}\int_{t\cdot 2^k>|x|\geqq 2^k\_t} |K_1(x-yt) - K_1(x)|\,dx$$
$$\leqq \sum_{k=1}^{\infty}\frac{A\cdot t|y|}{(2^{k-1}t-2^{-1}t)^2}2^k t \leqq A \qquad \text{if } |y| \leqq 1/2.$$

Similarly we can get the following by using the first inequalities in (4. 2) and (4. 3),

LEMMA 6. *We have*

$$|K_1(x)| \leqq A|x|^{-1}. \tag{4. 4}$$

LEMMA 7. *Let* $1 < p \leqq 2$, *and* $-1 < \alpha < p-1$, *then with the notation of* §2,

$$\int_{-\infty}^{\infty} |(\mathfrak{K}_1 f)(x)|^p |x|^\alpha dx \leqq A_{p,\alpha} \int_{-\infty}^{\infty} |f(x)|^p |x|^\alpha dx, \tag{4. 5}$$

and

$$\int_{-\infty}^{\infty} |(\mathfrak{K}_1 G)(x)|^p |x|^\alpha dx \leqq A_{p,\alpha} \int_{-\infty}^{\infty} |G(x)|^p |x|^\alpha dx. \tag{4. 6}$$

PROOF. We show (4. 5) only, a proof of (4. 6) is similar.

Operator $\mathfrak{K}_1$ mapping L^1 into $L^1(l^2)$ is weak type (1, 1) by Lemmas 2 and 3 with $\alpha = 0$. On the other hand this mapping is strong type (2. 2) by Lemma 2 with $\alpha = 0$ and applying Theorem C, we get the inequality (4. 5) in case $\alpha = 0$. Thus we get (4. 5) for $p = p_0$, $1 < p_0 < 2$, $\alpha = \alpha_0 = 0$ and for $p = p_1 = 2$, $\alpha = \alpha_1$, $0 \leqq \alpha_1 < 1$ by Lemma 2, therefore using Theorem C again, we have (4. 5) for $0 \leqq \alpha < p-1$ and $1 < p \leqq 2$.

In the case $-1 < \alpha < 0$, operator $\mathfrak{K}_1$ is weak type (1. 1) with respect to measure $|x|^\alpha dx$ by Lemma 2 and 3 and strong type (2. 2), therefore by Theorem C, inequality (4. 5) is proved for $-1 < \alpha < 0$ and $1 < p \leqq 2$. Thus Lemma is proved.

LEMMA 8. *Lemma 7 holds for* $1 < p < \infty$ *and* $-1 < \alpha < p-1$.

A proof is obvious by Lemmas 1 and 7.

5. The last lemma is the following which is due to J. Schwartz [8].

LEMMA 9. *Let* $1 < p < \infty$, $-1 < \alpha < p-1$ *and for each N, let* $\mathfrak{K}_N$ *be the transformation in* $L_\alpha^p(l^2)$ *which maps the vector whose n-th component has the Fourier transform* $h_n(x)\hat{f}_n(x)$ *for* $n \leqq N$, *and* $f_n(x)$ *for* $n > N$. *Then there exists a finite constant A independent on N such that the norm of* $\mathfrak{K}_N$, *regarded as mapping of* $L_\alpha^p(l^2)$ *into itself, is at most A.*

Lemma 9 was proved by J. Schwartz for $\alpha = 0$ and since a proof of the case for $-1 < \alpha < p-1$ does not differ from it except to use the following theorem, we omit it. We need only Corollary 1 below to prove Theorem 1, which follows from Theorem 2 only.

THEOREM 2. *Let* $1 < p < \infty$, $-1 < \alpha < p-1$ *and X be* L^q *space*, $1 < q < \infty$, *on any measure space* $(S, \mathfrak{F}, m)$. *If we defined the conjugate function* $\widetilde{f}$ *by*

$$\widetilde{f}(x) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy,$$

for $f(x) \in L_\alpha^p(X)$, $x \in (\infty, \infty)$,

$$\tilde{f}(x) = \frac{1}{\pi} P.V. \int_{-\pi}^{\pi} \frac{f(y)}{2\tan(x-y)/2} dy,$$

for periodic $f(X) \in L_\alpha^p(X)$ *with period* 2π *and*

$$\tilde{f}(n) = \frac{1}{\pi} \sum_{m \neq n} \frac{f(m)}{n-m},$$

for $\{f(n)\} \in l_\alpha^p(X)$, *then*

$$\|\tilde{f}\|_{p,\alpha} \leqq A_{p,q,\alpha} \|f\|_{p,\alpha}, \tag{5. 1}$$

where norms denote $L_\alpha^p(X)$*-norms on* $(-\infty, \infty)$ *or on* $[-\pi, \pi]$ *or* $l_\alpha^p(X)$*-norms respectively.*

PROOF. We show the case $L_\alpha^p(X)$ on $(-\infty, \infty)$, other cases will be proved similarly. Since the kernel of mapping $Tf = \tilde{f}$ satisfies the condition (3. 1) in Lemma 3, T is weak type (1. 1) with respect to the measure $d\mu^\alpha = |x|^\alpha dx$, $-1 < \alpha \leqq 0$. On the other hand we have

$$\int_{-\infty}^{\infty} |\tilde{f}(x, s)|^q d\mu_\alpha \leqq A_{q,\alpha} \int_{-\infty}^{\infty} |f(x, s)|^q q\mu_\alpha,$$

for $1 < q < \infty$, $-1 < \alpha < q - 1$ and $s \in S$ (e.g. see Hirschman [3], where we find the proof of compact case, but other cases may be proved by the same way.) Integrating above inequality we get (5. 1) for $p = q$. Therefore by interpolating argument we have (5. 1) for $1 < p \leqq q$. The validity of our theorem for $1 < q < p$ will follow from adjoint argument.

COROLLARY 1. *Let* $(\cdots, f_n(x), f_{n+1}(x), \cdots) \in L_\alpha^p(l^q)$, $1 < p < \infty$, $1 < q < \infty$, $-1 < \alpha < p-1$, *and define*

$$S_n(x\,; u_n, v_n) = \int_{u_n}^{v_n} e^{ixy} \hat{f}_n(y) dy,$$

then

$$\int_{-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} |s_n(x\,; u_n, v_n)|^q \right\}^{p/q} |x|^\alpha dx \leqq A_{p,\alpha} \int_{-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} |f_n(x)|^q \right\}^{p/q} |x|^\alpha dx.$$

PROOF OF THEOREM 1. Let $\Phi(x) \in L_\alpha^p(l^2)$ and its n-th component $\varphi_n(x)$ have Fourier transform $\hat{\varphi}_n(x)$. Consider an operator which maps $\Phi(x)$ to the vector with n-th component $\psi_n(x)$ defined by

$$\hat{\psi}_n(x) = \begin{cases} \hat{\varphi}_n(x) & \text{if } 2^n \leqq |x| < 2^{n+1} \\ 0 & \text{elsewhere,} \end{cases} \tag{5. 2}$$

then the norm of this operator mapping $L^p_\alpha(l^2)$ into itself is bounded by Corollary 1. Therefore if we put $\Phi(x) = (\mathfrak{K}_1 f)(x)$, then the right hand of (2. 1) is obvious by Lemma 8. The left hand follow at once from the equality

$$(\widehat{\mathfrak{L}_1 G})(x) = \sum_{n=-\infty}^{\infty} \hat{g}_n(x)\hat{k}_n(x).$$

Thus our proof is completed.

REMARK 4. By Corollary 1, we may easily modify the left hand of (2. 1) in the following way;

If $(\cdots, g_n(x), g_{n+1}(x), \cdots) \in L^p_\alpha(l^2)$, $1 < p < \infty$, $-1 < \alpha < p-1$ *is suitably restricted then there exists* $f \in L^p_\alpha$, *such that*

$$\hat{f}(x) = \hat{g}_n(x) \qquad \text{for } 2^n \leqq |x| < 2^{n+1}$$

and

$$\int_{-\infty}^{\infty} |f(x)|^p |x|^\alpha dx \leqq A_{p,\alpha} \int_{-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} |g_n(x)|^2 \right\}^{p/2} |x|^\alpha dx.$$

6. Now we state the Marcinkiewicz type theorem.

THEOREM 3. *Let* $1 < p < \infty$, $-1 < \alpha < p-1$. *For each* x, *let* $\lambda(x)$ *be a bounded operator in* l^2, *suppose that* $\lambda(x)$ *is bounded and that its variation satisfies*

$$\operatorname*{vari}_{2^n \leqq |x| < 2^{n+1}} \lambda(x) \leqq A, \quad n = 0, \pm 1, \pm 2, \cdots. \tag{6. 1}$$

Let $\mathfrak{M}$ *be the mapping defined by*

$$(\widehat{\mathfrak{M} f})(x) = \lambda(x)\hat{f}(x) \quad \text{in } 2^n \leqq |x| < 2^{n+1},$$

for $f \in L^p_\alpha$ *then* $\mathfrak{M}$ *is a bounded mapping of the space* $L^p_\alpha(l^2)$ *into itself.*

A proof follows at once from Theorem 1.

COROLLARY 2. *In Theorem* 3, *the hypothesis* (6. 1) *may be replaced by assumption*

$$|||\lambda(x)||| \leqq A|x|^{-1}.$$

A proof is obvious.

7. The Discrete Case. In this section we consider the weighted form of a theorem in G. Sunouchi [12], that is,

THEOREM 4. *Let* $1 < p < \infty$, $-1 < \alpha < p-1$ *and* $\{f(k)\} \in l^p_\alpha$, *that is,*

$\|f\|_{p,\alpha}=\left\{\sum_{k=-\infty}^{\infty}|f(k)|^p(|k|+1)^\alpha\right\}^{1/p}<\infty$. *Suppose $\{f(k)\}$ be a Fourier coefficient of some integrable function $\hat{f}(\theta)$,*

$$\hat{f}(\theta)\sim\sum_{k=-\infty}^{\infty}f(k)e^{2\pi ik\theta}.$$

and define

$$\delta_n(k)=\int_{2^{-n}}^{2^{-n+1}}f(\theta)e^{-2\pi ik\theta}d\theta \quad n=1,2,\cdots, \tag{7. 1}$$

then

$$0<A_{p.\alpha}\leqq\sum_{k=-\infty}^{\infty}\left\{\sum_{n=1}^{\infty}|\delta_n(k)|^2\right\}^{p/2}(|k|+1)^\alpha\Big/\sum_{k=-\infty}^{\infty}|f(k)|^p(|k|+1)^\alpha$$

$$\leqq A'_{p,\alpha}<\infty. \tag{7. 2}$$

We can proceed with a proof in the way of integral case, so we sketch it only.

Now let $\hat{K}_2(x)$ be vector $(\hat{k}_1(x),\ \hat{k}_2(x),\cdots)$, where $\hat{k}_n(x)$ $(n\geqq 2)$ are periodic with period 1 and identical with the functions of (2. 2) for $0\leqq x<1$ and $\hat{k}_1(x)$ equal to the function of (2. 2) for $1/2\leqq x<3/2$ and is periodic with period 1.

Let

$$K_2(m)=\int_0^1\hat{K}_2(x)e^{-2\pi imx}dx=(k_1(m),\ k_2(m),\cdots), \tag{7. 3}$$

then we can use the same estimate as (4. 2) and (4. 3), therefore replacing m for x, we get

$$\sum_{|m|\geqq l}|K_2(m-l)-K_2(m)|\leqq A \qquad \text{for } |l|\leqq[m/2], \tag{7. 4}$$

and

$$|K_2(m)|\leqq A/(|m|+1) \qquad \text{for all } m=0,\ \pm1,\cdots, \tag{7. 5}$$

For a scalar valued sequence $\{f(m)\}$, put

$$(\mathfrak{K}_2f)(m)=\sum_{l=-\infty}^{\infty}K_2(m-l)f(l) \tag{7. 6}$$

and for a sequence $\{G(m)\}$ with values in one sided sequence space l^2, put

$$(\mathfrak{L}_2G)(m)=\sum_{l=-\infty}^{\infty}K_2(m-l)G(l). \tag{7. 7}$$

If $\{f(m)\}$ and $\{G(m)\}$ are suitably restricted, we have

$$\sum_{m=-\infty}^{\infty} (\mathfrak{K}_2 f)(m)\overline{G(m)} = \sum_{m=-\infty}^{\infty} f(m)\,\overline{(\mathfrak{L}_2 G)(m)}.$$

By this relation we get Lemma 1 replacing $\mathfrak{K}_1, \mathfrak{L}_1$, by $\mathfrak{K}_2, \mathfrak{L}_2$ and L^p_α-norms by l^p_α-norms.

LEMMA 10. *Let* $-1<\alpha<1$, $f \in l^2_\alpha$, *and* $G \in l^2_\alpha(l^2)$, *then*

$$\sum_{m=-\infty}^{\infty} |(\mathfrak{K}_2 f)(m)|^2(|m|+1)^\alpha \leqq A_\alpha \sum_{m=-\infty}^{\infty} |f(m)|^2(|m|+1)^\alpha, \tag{7. 8}$$

and

$$\sum_{m=-\infty}^{\infty} |(\mathfrak{L}_2 G)(m)|^2(|m|+1)^\alpha \leqq A_\alpha \sum_{m=-\infty}^{\infty} |G(m)|^2(|m|+1)^\alpha. \tag{7. 9}$$

Since a proof is similar to that of Lemma 2, we omit it, but we must use the followings in place of Theorems A and B.

THEOREM E (A.Devinatz and I.I.Hirschman, Jr. [1]). *If* $f \in l^2_\alpha$, $0<\alpha<1$ *and* $\hat{f}(\theta) \sim \Sigma f(m)e^{-2\pi i m\theta}$, *then*

$$A_\alpha \sum_{m=-\infty, m\neq 0}^{\infty} |f(m)|^2(|m|+1)^\alpha \leqq \int_0^1\int_0^1 |\hat{f}(\theta)-\hat{f}(\varphi)|^2\{\sin \pi|\theta-\varphi|\}^{-1-\alpha}d\varphi d\theta$$
$$\leqq A'_\alpha \sum_{m=-\infty, m\neq 0}^{\infty} |f(m)|^2(|m|+1)^\alpha.$$

THEOREM F (special case of Pitt's theorem). *With above notations*

$$\int_0^1 |\hat{f}(x)|^2(\sin \pi x)^{-\alpha}\,dx \leqq A_\alpha \sum_{m=-\infty}^{\infty} |f(m)|^2(|m|+1)^\alpha.$$

By (7. 4), (7. 5), Remark 1 and Lemma 10, operators $\mathfrak{K}_2$ and $\mathfrak{L}_2$ are weak type (1. 1) with respect to the measure concentrated in integers only and having mass $(|m|+1)^\alpha$ at m $(-1<\alpha\leqq 0)$. Hence by interpolating arguments and discrete analogue of Lemma 1, we get,

LEMMA 11. *Let* $1<p<\infty$, $-1<\alpha<p-1$ *and* $f \in l^p_\alpha$, $G \in l^p_\alpha(l^2)$, *then*

$$\sum_{m=-\infty}^{\infty} |(\mathfrak{K}_2 f)(m)|^p(|m|+1)^\alpha \leqq A_{p,\alpha} \sum_{m=-\infty}^{\infty} |f(m)|^p(|m|+1)^\alpha, \tag{7.10}$$

$$\sum_{m=-\infty}^{\infty} |(\mathfrak{L}_2 G)(m)|^p(|m|+1)^\alpha \leqq A_{p,\alpha} \sum_{m=-\infty}^{\infty} |G(m)|^p(|m|+1)^\alpha. \tag{7.11}$$

8. LEMMA 12. *Suppose that* $\{f_n(m)\} \in l^p_\alpha(l^q)$, *where* $1 < p < \infty$, $-1 < \alpha < p-1$, $1 < q < \infty$ *and that there exist integrable* $\hat{f}_n(\theta)$ *such that* $\hat{f}_n(\theta) \sim \sum' f_n(m) e^{-2\pi i m}$.
Let

$$f_n(m\,;t_n) = \int_0^{t_n} \hat{f}_n(\theta) e^{-2\pi i m\theta} d\theta,\ 0 \leqq t_n \leqq 1,$$

then

$$\sum_{m=-\infty}^{\infty} \left\{ \sum_{n=1}^{\infty} |f_n(m\,;\,t_n)|^q \right\}^{p/q} (|m|+1)^\alpha \leqq A_{p,\alpha} \sum_{m=-\infty}^{\infty} \left\{ \sum_{n=1}^{\infty} |f_n(m)|^q \right\}^{p/q} (|m)+1)^\alpha. \quad (8.1)$$

PROOF. This lemma was proved by G.Sunouchi [12] for $\alpha = 0$ and $q = 2$. In this case we can follow his proof. Denote the characteristic function of $(0, t)$ by $\chi_t(\theta)$, then

$$\int_0^1 \chi_t(\theta) e^{-2im\theta} d\theta = \begin{cases} (1 - e^{-2\pi i m t})/2\pi i m & \text{for } m \neq 0 \\ t & \text{for } m = 0, \end{cases}$$

and

$$\begin{aligned} f_n(m\,;t_n) &= \int_0^1 \hat{f}_n(\theta) \chi_{t_n}(\theta) e^{-2\pi i m\theta} d\theta \\ &= \sum_{l=-\infty}^{\infty}{}' \frac{1 - e^{-2\pi i l t_n}}{2\pi i l} f_n(m-l) + t_n f_n(m) \\ &= \sum_{l=-\infty}^{\infty}{}' \frac{f_n(m-l)}{2\pi i l} + e^{2\pi i m t_n} \sum_{l=-\infty}^{\infty}{}' \frac{e^{-2\pi i (m-l) t_n} f_n(m-l)}{2\pi i l} \\ &\quad + t_n f_n(m), \end{aligned} \quad (8.\ 2)$$

where $\sum'$ denotes the summation for $l \neq 0$. Therefore by Theorem 2, we get (8. 1).

PROOF OF THEOREM 4. The left hand of (7. 2) follows from

$$(\hat{\mathfrak{L}}_2 G)(\theta) = \sum_{n=1}^{\infty} \hat{k}_n(\theta) \hat{g}_n(\theta),\ G = \{g_n\},$$

and (7.11). The right hand of (7. 2) follows from (7.10) and Lemma 12.

We can prove Theorem 5 using the analogue of Lemma 9, but without it we can prove the following (see G.Sunouchi [12]),

THEOREM 5. *If* $\lambda(\theta)$ *is a function such that*

$$|\lambda(\theta)| \leqq M,\ \int_{2^{-n}}^{2^{-(n-1)}} |d(\theta)| \leqq M, \qquad n = 1, 2, \cdots,$$

and $f \in l^p_\alpha$, $1 < p < \infty$, $-1 < \alpha < p-1$,

$$\hat{f}(\theta) \sim \sum f(m) e^{2\pi i m\theta}$$

then

$$\lambda(\theta)\hat{f}(\theta) \sim \sum g(m) e^{2\pi i m\theta},\ g \in l^p_\alpha$$

and

$$\|g\|_{p,\alpha} \leqq A_{p,\alpha} M \|f\|_{p,\alpha}.$$

9. Now we prove the decomposition theorem generalized by I.I.Hirschman, Jr. along the line of integral and discrete case.

In this and next sections we use the following notation; $f \in L^p_\alpha$ implies $\|f\|_{p\alpha} = \left\{ \int_{-\pi}^{\pi} |f(x)|^p |x|^\alpha dx \right\}^{1/p} < \infty$ and $\hat{f}(n)$ represent Fourier coefficients of $f(x)$.

THEOREM 6. *Let* $1 < p < \infty$, $-1 < \alpha < p-1$ *and* $f \in L^p_\alpha$, *then*

$$0 < A_{p,\alpha} \leqq \int_{-\pi}^{\pi} \left\{ \sum_{n=-\infty}^{\infty} |\Delta_n(x)|^2 \right\}^{p/2} |x|^\alpha dx \Big/ \int_{-\pi}^{\pi} |f(x)|^p x^\alpha dx \leqq A'_{p,\alpha} < \infty, \tag{9. 1}$$

where $\Delta_n(x)$ *are the functions defined by* (1. 3).

First we define the two sided vector $K_3(x) = (\cdots, k_n(x), k_{n+1}(x), \cdots)$. Let us denote

$$\Delta(t) = \begin{cases} 1 - |t| & \text{if } |t| \leqq 1 \\ 0 & \text{if } |t| > 1, \end{cases}$$

$$\Delta_\lambda(t) = \Delta(t/\lambda)$$

and

$$\tau_n(x) = 2\Delta_{2|n|}(x) - \Delta_{|n|}(x),\ n \neq 0.$$

Define Fourier coefficients $\hat{k}_n(x)$ by

$$\hat{k}_0(m) = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m = 0, \end{cases}$$

$$\hat{k}_n(m) = \tau_{2^{n-2}}(m - 3\cdot 2^{n-2}), \qquad m = 0, \pm 1, \cdots \ (n \geqq 1)$$

$$\hat{k}_n(m) = \tau_{2^{-n+2}}(m + 3\cdot 2^{-n+2}), \quad m = 0, \pm 1, \cdots \ (n \leqq -1)$$

so that

$$k_n(x) = \sum_{m=-\infty}^{\infty} \hat{k}_n(m) e^{imx}, \tag{9. 2}$$

and

$$k_0(x) = 1,$$

$$k_n(x) = \begin{cases} 2e^{2nxt}\cos x, & \text{if } |n| = 1 \\ \dfrac{e^{3\cdot 2^{(n-2)}ix}(\cos 2^{(n-2)}x - \cos 2^{(n-1)}x)}{2^{(n-3)}(\sin x/2)^2} & \text{if } n > 1 \\ \dfrac{e^{-3\cdot 2^{-(n+2)}xt}(\cos 2^{-(n+2)x} - \cos^{-(n+1)}x)}{2^{-(n+3)}(\sin x/2)^2} & \text{if } n < -1. \end{cases} \tag{9. 3}$$

LEMMA 13.

$$\int_{\pi\geqq|x|\geqq t} |k_3(x-ty) - k_3(x)|dx \leqq A, \tag{9. 4}$$

for all $|y| \leqq 1/2$, $\pi \geqq t > 0$.

PROOF. Let $n \geqq 2$. By definition of $k_n(x)$, we have

$$\begin{aligned} |k_n(x-ty) - k_n(x)| &\leqq |k_n(x-ty) + |k_n(x)| \\ &< \frac{A}{2^{n-3}(|x| - t|y|)^2} + \frac{A}{2^{n-3}x^2} \\ &< A2^{-n}x^{-2}, \end{aligned} \tag{9. 5}$$

for $\pi \geqq |x| \geqq t$, $|y| \leqq 1/2$ and $n = 2, 3, 4,\cdots$ On the other hand, since

$$\begin{aligned} \frac{d}{dx} k_n(x) = \frac{2\cdot e^{3\cdot 2^{n-2}xt}}{(2\sin x/2)^2}\Big[& 3i(\cos 2^{n-2}x - \cos 2^{n-1}x) \\ & + (2\sin 2^{n-1}x - \sin 2^{n-2}x) \\ & - \frac{\cos x/2, (\cos 2^{n-2}\alpha - \cos 2^{n-1}x)}{2^{n-2} \sin x/2}\Big], \end{aligned}$$

we get

$$|k_n(x-ty) - k_n(x)| < A2^n t|y|^{-1}, \tag{9. 6}$$

for all $\pi \geqq |x| \geqq t$, $|y| \leqq 1/2$ and $n = 2, 3,\cdots$.

Hence for arbitrary positve integer N, we have

$$\begin{aligned} J^2(x) &\equiv \sum_{n=0}^{\infty} |k_n(x-ty) - k_n(x)|^2 \\ &= \Big(\sum_{n=0}^{N} + \sum_{n=N+1}^{\infty}\Big) |k_n(x-ty) - k_n(x)|^2 \\ &\leqq A(2^{2N}t^2|x|^{-2N} + 2^{-2N}x^{-4}), \end{aligned} \tag{9. 7}$$

for all $\pi \geqq |x| \geqq t, t > 0, |y| \leqq 1/2$, applying (9. 5) for the second term and (9. 6) for the first term. For the sum $\sum_{n=-\infty}^{-1} |k_n(x-ty) - k_n(x)|^2$ we get the similar

estimation.
Therefore

$$\int_{\pi \geqq |x| \geqq t} |K_3(y - ty) - K_3(x)|\,dx$$
$$\leqq A \int_{\pi \geqq |x| \geqq t} \left(\frac{2^N t}{|x|} + \frac{1}{2^N x^2} \right) dx$$
$$\leqq A \int_t^1 \left(\frac{2^N t}{x} + \frac{1}{2^N x^2} \right) dx + A' \tag{9. 8}$$

Let $2^{-m+1} \geqq t > 2^{-m}$, m being positive integer, then the last integral is less than

$$I = \int_{2^{-m}}^1 \left(\frac{2^N t}{x} + \frac{1}{2^N x^2} \right) dx$$
$$= \sum_{\nu=1}^{m} \int_{2^{-\nu}}^{2^{-\nu+1}} = \sum_{\nu=1}^{m} I_\nu, \text{ say.} \tag{9. 9}$$

If we choose $N = [(\nu + m - 1)/2]$, then

$$I \leqq 2^N t 2^\nu \cdot 2^{-\nu} + 2^{-N} 2^{2\nu} \cdot 2^{-\nu} \leqq 2^{N-m+1} + 2^{\nu-N}$$
$$\leqq 2^{(\nu-m+1)/2} + 2^{(\nu-m+3)/2} \leqq 3\sqrt{2}(2^{\nu/2}\sqrt{t}).$$

Hence

$$I = \sum_{\nu=1}^{m} I_\nu \leqq \sum_{\nu=1}^{m} 3\sqrt{2}(2^{\nu/2}\sqrt{t}) \leqq A \frac{1}{\sqrt{t}} \sqrt{t} = A. \tag{9.10}$$

By (9. 8), (9. 9) and (9.10) we have (9. 4).

LEMMA 14.

$$|K_3(x)| \leqq A|x|^{-1}, \textit{ for } \pi \geqq |x| > 0. \tag{9.11}$$

PROOF. By definition of $k_n(x)$

$$|k_n(x)| \leqq A 2^{-|n|} x^{-2} \text{ and } A 2^{|n|}$$

for $|x| \leqq \pi$ and $n = 0, \pm 1, + 2, \cdots$. Hence

$$\sum_{n=-\infty}^{\infty} |k_n(x)|^2 = \left(\sum_{|n| \leqq N} + \sum_{|n| > N} \right) |k_n(x)|^2$$
$$\leqq A(2^{2N} + 2^{-2N} x^{-4}).$$

Choosing N so that $2^{-N+2} > x \geqq 2^{-N+1}$, we have

$$\sum_{n=-\infty}^{\infty} |k_n(x)|^2 \leqq A x^2.$$

10. For a scalar valued measurable function $f(x)$, $x \in (-\pi, \pi)$, we put

$$(\mathfrak{K}_3 f)(x) = \int_{-\pi}^{\pi} K_3(x-y)f(y)dy \tag{10.1}$$

and for function $G(x)$ of two sided sequence space l^2,

$$(\mathfrak{L}_3 G)(x) = \int_{-\pi}^{\pi} K_3(x-y)G_3(y)dy. \tag{10.2}$$

Since

$$\int_{-\pi}^{\pi} (\mathfrak{K}_3 f)(x)\,\overline{G(x)}\,dx = \int_{-\pi}^{\pi} f(x)\,\overline{(\mathfrak{L}_3 G)(x)}\,dx, \tag{10.3}$$

we have the analogue of Lemma 1 replacing $\mathfrak{K}_1$ and $\mathfrak{L}_1$ by $\mathfrak{K}_3$ and $\mathfrak{L}_3$ respectviely.

LEMMA 15. *Let* $-1 < \alpha < 1$, *then*

$$\int_{-\pi}^{\pi} |(\mathfrak{K}_3 f)(x)|^2 |x|^\alpha dx \leqq A\alpha \int_{-\pi}^{\pi} |f(x)|^2 |x|^\alpha dx, \tag{10.4}$$

$$\int_{-\pi}^{\pi} |\mathfrak{L}_3(G)(x)|^2 |x|^\alpha\, dx \leqq A^\alpha \int_{-\pi}^{\pi} |G(x)|^2 |x|^\alpha dx. \tag{10.5}$$

To prove lemma 15, we use the following.

THEOREM G. (I.I. Hirschman, Jr [3]). *If* $f(x) \in L^2_\alpha$ *and* $0 < \alpha < 1$, *then*

$$A_\alpha \int_{-\pi}^{\pi} |f(x)|^2 |x|^\alpha dx \leqq \sum_{l=-\infty}^{\infty} \sum_{k=l+1}^{\infty} |\hat{f}(k) - \hat{f}(l)|^2 |k-l|^{-k-\alpha}$$
$$\leqq A'_\alpha \int_{-\pi}^{\pi} |f(x)|^2 |x|^\alpha dx, \tag{10.6}$$

where $\hat{f}(k)$ *mean Fourier coefficients of* $f(x)$ *now and later.*

THEOREM H. (Spacial case of Pitt's theorem). *Let* $0 \leqq \alpha < 1$, *then*

$$\sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 (|k|+1)^{-\alpha} \leqq A_\alpha \int_{-\pi}^{\pi} |f(x)|^2 |x|^\alpha dx \tag{10.7}$$

PROOF OF LEMMA 15. Our proof is almost same as before.
If $\alpha = 0$, then Lemma is easily shown by Parseval's relation and uniform boundedness of $\sum_{n=-\infty}^{\infty} |\hat{k}_n(m)|^2$. Suppose $0 < \alpha < 1$. Applying Theorem G, for n-th

component $k_n * f$ of $\mathfrak{R}_3 f$,

$$\int_{-\pi}^{\pi} |(k_n*f)(x)|^2 |x|^\alpha dx \leqq A_\alpha \sum_{m=-\infty}^{\infty} \sum_{l=m+1}^{\infty} |\hat{f}(l)\hat{k}_n(l) - \hat{f}(m)\hat{k}_n(m)|^2 |l-m|^{-1-\alpha}$$
$$\leqq 2A_\alpha(I_n^1 + I_n^2), \text{ say,} \tag{10.8}$$

where

$$I_n^1 = \sum_{m=-\infty}^{\infty} \sum_{l=m+1}^{\infty} |\hat{f}(l) - \hat{f}(m)|^2 |\hat{k}_n(l)|^2 |l-m|^{-1-\alpha}, \tag{10.9}$$

$$I_n^2 = \sum_{m=-\infty}^{\infty} \sum_{l=m+1}^{\infty} |\hat{k}_n(l) - \hat{k}_n(m)|^2 |\hat{f}(m)|^2 |l-m|^{-1-\alpha}. \tag{10.10}$$

Using Theorem G again,

$$\sum_{n=-\infty}^{\infty} I_n^1 = \sum_{m=-\infty}^{\infty} \sum_{l=m+1}^{\infty} |\hat{f}(l) - \hat{f}(m)|^2 \left\{ \sum_{n=-\infty}^{\infty} |\hat{k}_n(l)|^2 \right\} |l-m|^{-1-\alpha}$$
$$\leqq A \sum_{m=-\infty}^{\infty} \sum_{l=m+1}^{\infty} |\hat{f}(l) - \hat{f}(m)|^2 |l-m|^{-1-\alpha}$$
$$\leqq A_\alpha \int_{-\pi}^{\pi} |f(x)|^2 |x|^\alpha dx. \tag{10.11}$$

Concerning with the term $I^2{}_n$

$$I_n^2 = \sum_{m=-\infty}^{\infty} |\hat{f}(m)|^2 \sum_{l=m+1}^{\infty} |\hat{k}_n(l) - \hat{k}_n(m)|^2 |l-m|^{-1-\alpha}$$
$$= \sum_{m=-\infty}^{\infty} |\hat{f}(m)|^2 J_n(m), \tag{10.12}$$

where

$$J_n(m) = \sum_{l=m+1}^{\infty} |\hat{k}_n(l) - \hat{k}_n(m)|^2 |l-m|^{-1-\alpha}.$$

If we have

$$\sum_{n=-\infty}^{\infty} J_n(m) \leqq A(|m|+1)^{-\alpha} \qquad \text{for } m = 0, \pm 1, \pm 2, \cdots, \tag{10.13}$$

then applying Theorem H,

$$\sum_{n=-\infty}^{\infty} I^2{}_n = \sum_{m=-\infty}^{\infty} |\hat{f}(m)|^2 \left\{ \sum_{n=-\infty}^{\infty} J_n(m) \right\}$$
$$\leqq A \sum_{n=-\infty}^{\infty} |\hat{f}(m)|^2 (|m|+1)^{-\alpha}$$

$$\leqq A_\alpha \int_{-\pi}^{\pi} |f(x)|^2 |x|^\alpha dx. \tag{10.4}$$

Hence by (10.11) and (10.14), we have

$$\int_{-\pi}^{\pi} |(\mathfrak{R}_3 f)(x)|^2 |x|^\alpha \leqq A_\alpha \sum_{n=-\infty}^{\infty} (I_n^1 + I_n^2)$$
$$\leqq A_\alpha \int_{-\pi}^{\pi} |f(x)|^2 |x|^\alpha dx.$$

Therefore (10.14) is proved.

Now we verify (10.13). If $2 \geqq |m|$ then $J_n(m) = 0$ for $n \leqq -1$ and $n \geqq 2$. For any case $J_n(m) \leqq \sum_{l=m+1}^{\infty} |l-m|^{-1-\alpha} \leqq A < \infty$, therefore (10.3) is proved in this case. If $m \geqq 3$ then $J_n(m) = 0$ for $n \leqq 2$.

Let us fix $m \geqq 3$. If $m \geqq 5 \cdot 2^{n-2}$ then $\hat{k}_n(m) = \hat{k}_n(l) = 0$ and $J_n(m) = 0$. If $m \leqq 2^{n-2}$, then $\hat{k}_n(m) = 0$ and

$$J_n(m) \leqq \sum_{l=2^{n-2}+1}^{5\cdot 2^{n-2}} |\hat{k}_n(l)|^2 (l-m)^{-1-\alpha}$$
$$\leqq \sum_{l=2^{n-2}+1}^{5\cdot 2^{n-2}} \left\{ \frac{4(l-2^{n-2})}{5\cdot 2^{n-2} - 2^{n-2}} \right\}^2 (l-2^{n-2})^{-1-\alpha}$$
$$= 8 \sum_{l=1}^{2^n} 2^{-2n} l^{1-\alpha} \leqq A_\alpha 2^{-\alpha n}.$$

If $2^{n-2} < m < 5\cdot 2^{n-2}$ then

$$J_n(m) \leqq \sum_{l=m+1}^{5\cdot 2^{n-2}-1} |\hat{k}_n(l) - \hat{k}_n(m)|^2 (l-m)^{-1-\alpha}$$
$$\leqq \sum_{l=m+1}^{5\cdot 2^{n-2}-1} \left\{ \frac{4(l-m)}{5\cdot 2^{n-2} - 2^{n-2}} \right\}^2 (l-m)^{-1-\alpha}$$
$$\leqq A_\alpha 2^{-\alpha n} \leqq A_\alpha m^{-\alpha}.$$

Hence we get

$$\sum_{n=-\infty}^{\infty} J_n(m) = \sum_{\{n; m \leqq 2^{n-2}\}} + \sum_{\{n; 2^{n-2} < m < 5\cdot 2^{n-2}\}}$$
$$\leqq A \sum_{\{n; m \leqq 2^{n-2}\}} 2^{-\alpha n} + 3A_\alpha m^{-\alpha}$$
$$\leqq A_\alpha m^{-\alpha}, \qquad \text{for } m \geqq 3. \tag{10.15}$$

For $m \leqq -3$, we can prove in the same way.

To prove (10.5), we denote n-th component of $G(x)$ by $g_n(x)$, then by Theorem G and

$$(\mathfrak{L}_3 G)(m) = \sum_{n=-\infty}^{\infty} \hat{k}_n(m)\hat{g}_n(m),$$

we have

$$\int_{-\pi}^{\pi} |(\mathfrak{L}_3(G)(x)|^2 |x|^\alpha dx$$

$$\leqq A_\alpha \sum_{l=-\infty}^{\infty} \sum_{m=l+1}^{\infty} \left| \sum_{n=-\infty}^{\infty} \{\hat{k}_n(m)\hat{g}_n(m) - \hat{k}_n(l)\hat{g}(l)\} \right|^2 \cdot |m-l|^{-1-\alpha} = 2A_\alpha(I' + I''),$$

where

$$I' = \sum_{l=-\infty}^{\infty} \sum_{m=l+1}^{\infty} \left| \sum_{n=-\infty}^{\infty} \hat{k}_n(m)\{\hat{g}_n(m) - \hat{g}_n(l)\} \right|^2 |m-l|^{-1-\alpha},$$

$$I'' = \sum_{l=-\infty}^{\infty} \sum_{m=l+1}^{\infty} \left| \sum_{n=-\infty}^{\infty} \hat{g}_n(l)\{\hat{k}_n(m) - \hat{k}_n(l)\} \right|^2 |m-l|^{-1-\alpha}.$$

By Shwartz inequality and Theorem G

$$I' \leqq \sum_{l=-\infty}^{\infty} \sum_{m=l+1}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} |\hat{k}_n(m)|^2 \right\} \left\{ \sum_{n=-\infty}^{\infty} |\hat{g}_n(m) - \hat{g}_n(l)|^2 |m-l|^{-1-\alpha} \right.$$

$$\leqq A \sum_{l=-\infty}^{\infty} \sum_{m=l+1}^{\infty} \sum_{n=-\infty}^{\infty} \left| \hat{g}_n(m) - \hat{g}_n(l) \right|^2 |m-l|^{-1-\alpha}$$

$$\leqq A_\alpha \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} |g_n(x)|^2 |x|^\alpha dx = A^\alpha \int_{-\pi}^{\pi} |G(x)|^2 |x|^\alpha dx.$$

On the other hand, using Schwartz inequality again

$$I'' \leqq \sum_{l=-\infty}^{\infty} \sum_{m=l+1}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} |\hat{g}_n(l)|^2 \right\} \left\{ \sum_{n=-\infty}^{\infty} |\hat{k}_n(m) - \hat{k}_n(l)|^2 |m-l|^{-1-\alpha} \right\}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} |\hat{g}_n(l)|^2 \left\{ \sum_{\nu=-\infty}^{\infty} J_\nu(l) \right\} \leqq A_\alpha \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} |\hat{g}_n(l)|^2 (|l|+1)^{-\alpha}$$

$$\leqq A_\alpha \int_{-\pi}^{\pi} |G(x)|^2 |x|^\alpha dx.$$

Therefore a proof is completed.

By Lemma 15, 13, 14 and Remark 1 in §3, operators $\mathfrak{K}_3$ and $\mathfrak{L}_3$ are weak types (1, 1) with respect to the measure $|x|^\alpha dx$ $(0 \geqq \alpha > -1)$. Hence applying interpolating arguments and analogue of Lemma 1, we have the following

LEMMA 16. *If $1 < p < \infty$ and $-1 < \alpha < p-1$, then for $f \in L^p_\alpha$ and*

$G \in L^p_\alpha(l^2)$

$$\|\mathfrak{K}_3 f\|_{p,\alpha} \leqq A_{p,\alpha}\|f\|_{p,\alpha}, \tag{10.16}$$

$$\|\mathfrak{L}_3 G\|_{p,\alpha} \leqq A_{p,\alpha}\|G\|_{p,\alpha}. \tag{10.17}$$

11. LEMMA 17. *Let $f_n \in L^p_\alpha$, $1 < p < \infty$, $-1 < \alpha < p-1$ and let $S_n(x; k_n)$ be the k_n-th partial sum of Fourier series of f_n, then*

$$\int_{-\pi}^{\pi} \left\{ \sum_{n=-\infty}^{\infty} |S_n(x;k_n)|^2 \right\}^{p/2} |x|^\alpha dx \leqq A_{p\,\alpha} \int_{-\pi}^{\pi} |f(x)|^p |x|^\alpha dx. \tag{11.1}$$

PROOF. This Lemma follows from Theorem 2 at once.

PROOF OF THEOREM 6. The left hand of (5. 1) is obvious by (10.17), formula

$$(\widehat{\mathfrak{L}_3 G})(m) = \sum_{n=-\infty}^{\infty} \hat{k}_n(m)\, \hat{g}_n(m),$$

where $G(x) = \{\Delta_n(x,f)\}$. The right hand of (5. 1) follows from (10.16) and Lemma 17.

Next Theorem follows from Theorem 5 (e.g. see [7]).

THEOREM 7. *If $\{\lambda_n\}$ is a sequence such that*

$$|\lambda_n| \leqq M, \sum_{2^n-1 \geqq |\nu| \geqq 2^{n-1}} |\lambda_\nu - \lambda_{\nu+1}| \leqq M$$

and $f \in L^p_\alpha$, $1 < p < \infty$, $-1 < \alpha < p-1$, then

$$\Sigma \hat{f}(n)\lambda_n e^{inx}$$

is the Fourier series of an $h(x) \in L^p_\alpha$ and

$$\|h\|_{p,\alpha} \leqq MA_{p,\alpha}\|f\|_{p,\alpha}.$$

12. From above results, we can give a real proof of the theorem on the functions of Littlewood-Paley, Lusin and Marcinkiewicz. Up to the present these theorems were all proved by the complex methods.

For any function $\varphi(z)$ regular in $|z| < 1$, the Littlewood-Paley functions $g(\theta)$, $g^*(\theta)$ are defined by

$$g(\theta) \equiv g(\theta, \varphi) = \left\{ \int_0^1 (1-\rho)|\varphi'(\rho e^{i\theta})|^2 d\rho \right\}^{1/2}$$

and

$$g^*(\theta) \equiv g^*(\theta, \rho) = \left\{ \int_0^1 (1-\rho)\chi^2(\rho, \theta) d\rho \right\}^{1/2}$$

where

$$\chi(\rho,\theta)=\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-\rho^2}{1-2\rho\cos t+\rho^2}|\varphi'(\rho e^{i(\theta+t)})|^2dt\right\}^{1/2}.$$

Lusin's function $s(\theta)$ is defined by

$$s(\theta)\equiv s_\delta(\theta,\varphi)=\left\{\iint_{\Omega_\delta(\theta)}|\varphi'(x+iy)|^2dxdy\right\}^{1/2}$$

where $\Omega_\delta(\theta)$ means the open domain bounded by the two tangents from $z=e^{i\theta}$ to circle $|z|=\delta<1$ and by the more distant are of $|z|=\delta$ between the points of contact.

Let f be integrable and periodic with period 2π, and F be integral of f, then Marcinkiewicz's function is defined by

$$\mu(\theta)\equiv\mu(\theta,f)=\left\{\int_0^{\pi}\frac{|F(\theta+t)+F(\theta-t)-2F(\theta)|^2}{t^3}dt\right\}^{1/2}.$$

THEOREM 8. *Let $1<p<\infty$ and $-1<\alpha<p-1$. Suppose that $\varphi(\theta)\sim\sum_{n=0}^{\infty}c_ne^{in}\in L^p_\alpha$ and $f\in L^p_\alpha$, then we have*

$$A_{p,\alpha}\|\varphi\|_{p,\alpha}\leqq\|g\|_{p,\alpha}\leqq A'_{p,\alpha}\|\varphi\|_{p,\alpha},$$
$$A_{p,\alpha}\|\varphi\|_{p,\alpha}\leqq\|g^*\|_{p,\alpha}\leqq A'_{p,\alpha}\|\varphi\|_{p,\alpha},$$
$$A_{p,\alpha,\delta}\|\varphi\|_{p,\alpha}\leqq\|s\|_{p,\alpha}\leqq A'_{p,\alpha,\delta}\|\varphi\|_{p,\alpha},$$

and

$$A_{p,\alpha}\|f\|_{p,\alpha}\leqq\|\mu\|_{p,\alpha}\leqq A'_{p,\alpha}\|f\|_{p,\alpha}.$$

PROOF. First we prove the right hands of inequalities. It is known that there are the following relations between these functions;

$$g(\theta)\leqq A_\delta s(\theta), \tag{12.1}$$
$$s_\delta(\theta)\leqq A_\delta g^*(\theta)\text{ and }\mu(\theta)\leqq Ag^*(\theta). \tag{12.2}$$

(for (12.1) see e.g. Zygmund [16, vol II; p.210] and for (12.2) see Zygmund [17]).

On the other hand it is well known that

$$Ag^*(\theta)\leqq\left\{\sum_{n=1}^{\infty}\frac{|s_n(\theta)-\sigma_n(\theta)|^2}{n}\right\}^{1/2}\leqq A'g^*(\theta), \tag{12.3}$$

where $s_n(\theta)$ and $\sigma_n(\theta)$ mean the partial sums and $(C,1)$ means of $\Sigma c_ne^{n\theta}$ respectively.

By Zygmund's method [16, vol II; p.230], we have

$$\int_{-\pi}^{\pi}\left\{\sum_{n=1}^{\infty}\frac{|s_n-\sigma_n|^2}{n}\right\}^{p/2}|x|^{\alpha}dx$$
$$=\int_{-\pi}^{\pi}\left\{\sum_{k=0}^{\infty}\sum_{n=2^k}^{2^{k+1}-1}\frac{|s'_n|^2}{n(n+1)^2}\right\}^{p/2}|x|^{\alpha}dx$$
$$\leqq A_{p,\alpha}\int_{-\pi}^{\pi}\left\{\sum_{k=0}^{\infty}\sum_{n=2^k}\frac{|s'_{2^k}|^2}{n(n+1)^2}\right\}^{p/2}|x|^{\alpha}dx$$
$$\leqq A_{p,\alpha}\int_{-\pi}^{\pi}\left\{\sum_{k=0}^{\infty}|s_{2^k}-\sigma_{2^k}|^2\right\}^{p/2}|x|^{\alpha}dx, \tag{12.4}$$

using Lemma 17. Following Zygmund [15], we get

$$\sum_{k=0}^{\infty}|s_{2^k}-\sigma_{2^k}|^2\leqq\sum_{k=0}^{\infty}\frac{1}{(2^k+1)^2}\left|\sum_{j=0}^{2^k}(s_{2^k}-s_j)\right|^2$$
$$\leqq\sum_{k=0}^{\infty}\frac{1}{2^k+1}\sum_{j=0}^{2^k}|s_{2^k}-s_j|^2$$
$$=\sum_{k=0}^{\infty}\frac{1}{2^k+1}\sum_{i=1}^{k}\sum_{j=2^{i}-1}^{2^i-1}|s_{2^k}-s_j|^2+\sum_{k=0}^{\infty}\frac{|s_{2^k}-s_0|^2}{2^k+1}. \tag{12.5}$$

Hence

$$\int_{-\pi}^{\pi}\left\{\sum_{k=0}^{\infty}|s_{2^k}-\sigma_{2^k}|^2\right\}^{p/2}|x|^{\alpha}dx$$
$$\leqq A_p\int_{-\pi}^{\pi}\left\{\sum_{k=0}^{\infty}\frac{1}{2^k+1}\sum_{i=0}^{k}\sum_{j=2^i-1}^{2^i-1}|s_{2^k}-s_j|^2\right\}^{p/2}|x|^{\alpha}dx$$
$$+A_p\int_{-\pi}^{\pi}\left\{\sum_{k=0}^{\infty}\frac{|s_{2^k}-s_0|^2}{2^k+1}\right\}^{p/2}|x|^{\alpha}dx$$
$$\leqq A_{p,\alpha}\int_{-\pi}^{\pi}\left\{\sum_{k=0}^{\infty}\frac{1}{2^k+1}\sum_{j=1}^{k}2^j|s_{2^{k+1}-1}-s_{2^j-1}|^2\right\}^{p/2}|x|^{\alpha}dx$$
$$+A_p\int_{-\pi}^{\pi}\left\{\sum_{k=0}^{\infty}\frac{|s_{2^k}-s_0|^2}{2^k+1}\right\}^{p/2}|x|^{\alpha}dx. \tag{12.6}$$

Since $|s_{2^{k+1}-1}-s_{2^j-1}|\leqq|\Delta_j|+|\Delta_{j+1}|+\cdots+|\Delta_{k+1}|$, we get

$$\sum_{k=0}^{\infty}\frac{1}{2^k+1}\sum_{j=1}^{k}2^j|s_{2^{k+1}-1}-s_{2^j-1}|^2$$
$$\leqq\sum_{k=0}^{\infty}\frac{1}{2^k+1}\sum_{j=1}^{k}2^j\left(\sum_{i=j}^{k+1}|\Delta_j|^2 2^{i/2}\right)\left(\sum_{i=j}^{\infty}2^{-i/2}\right)$$
$$\leqq A\sum_{k=0}^{\infty}\frac{1}{2^k+1}\sum_{j=1}^{k}\sum_{i=j}^{k+1}|\Delta_i|^2 2^{(i+j)/2}$$

$$\leqq A\sum_{i=0}^{\infty}|\Delta_i|^2. \tag{12.7}$$

The integrand of the remaining term may be estimated by the same way. Therefore we have

$$\int_{-\pi}^{\pi}\left\{\sum_{n=1}^{\infty}\frac{|s_n-\sigma_n|^2}{n}\right\}^{p/2}|x|^{\alpha}dx\leqq A_{p,\alpha}\int_{-\pi}^{\pi}\left\{\sum_{a=0}^{\infty}|\Delta_n|^2\right\}^{p/2}|x|^{\alpha}dx. \tag{12.8}$$

By Theorem 6, (12.1), (12.2), (12.3) and (12.8), we have the right hand of our inequalities.

In order to prove the left hand, set for $f\in L_{\alpha}^{p}$

$$h(\theta)\equiv h(\theta,f)=\left\{\int_0^1\rho^{-2}(1-\rho)|f_{\theta}(\rho,\theta)|^2d\rho\right\}^{1/2},$$

where $f_{\theta}(\rho,\theta)$ is the derivative of Poisson integral of f with respect to θ. By Zygmund [17],

$$h(\theta)\leqq A\mu(\theta) \tag{12.9}$$

and if f is the real part of φ, then $h(\theta,f)\leqq g(\theta,\varphi)$ clearly and by conjugacy method (cf. Zygmund [16, vol II; p.215]) we have

$$\|f\|_{p,\alpha}\leqq A_{p,\alpha}\|h\|_{p.\alpha}. \tag{12.10}$$

Therefore the opposite inequalities are obvious.

13. Above arguments hold for the integral analogue. For the function $\phi(z)$ regular in right half-plane $\Re_e z>0$, the analogues of above functions are defined by

$$g(\tau)=\left\{\int_0^{\infty}\sigma|\phi'(\sigma+i\tau)|^2d\sigma\right\}^{1/2},$$

$$g^*(\tau)=\left\{\int_0^{\infty}\sigma\chi^2(\sigma,\tau)d\sigma\right\}^{1/2},$$

where

$$\chi(\sigma,\tau)=\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma}{\sigma^2+(\tau-u)^2}|\phi'(\sigma+iu)|^2du\right\}^{1/2},$$

and

$$s(\tau)=\left\{\iint_{\Omega\delta(\tau)}|\phi'(\sigma+iu)|^2d\sigma du\right\}^{1/2},$$

where $\Omega_{\delta}(\tau)=\{(\sigma,u):|\tau-u|<\delta\sigma\}$. The last function is

$$\mu(\tau)=\left\{\int_0^{\infty}\frac{|F(\tau+u)+F(\tau-u)-2F(\tau)|^2}{u^3}du\right\}^{1/2}$$

where F is indefinite integral of f in $L^p(p>1)$.

THEOREM 9. *Let* $1<p<\infty$ *and* $-1<\alpha<p-1$. *Suppose that* $\phi(\tau)$ *is the boundary function of the function* $\phi(\sigma+i\tau)$ *regular in right half-plane and belongs to* L^p_α, *and* f *is in* L^p_α, *then we have the analogous inequalities in Theorem* 8.

PROOF. Waterman [13] and [14] proved that

$$g(\tau) \leqq A_\delta s(\tau) \leqq A'_\delta g^*(\tau) \tag{13.1}$$

and

$$\mu(\tau) \geqq A g^*(\tau). \tag{13.2}$$

On the other hand if we put for $\phi(\tau)$ having locally integrable Fourier transform $\hat{\phi}(x)$,

$$s(\omega, \tau) = \int_0^\omega \hat{\phi}(x) e^{ix\tau} dx$$

and

$$\sigma(\omega, \tau) = \int_0^\omega \frac{\omega - x}{\omega} \hat{\phi}(x) e^{ixt} dx,$$

then

$$g^*(\tau) = A \int_0^\infty \frac{|s(\omega, \tau) - \sigma(\omega, \tau)|^2}{\omega} d\omega$$

(see, G. Sunouchi [11]). Therefore we may follow the above inequalities (12.4), (12.5), (12.6) and (12.7) term by term, and we will get $\|g^*\|_{p\cdot\alpha} \leqq A_{p,\alpha}\|\phi\|_{p,\alpha}$. Hence the first part of Theorem is proved.

For the remaining part, we set for $f \in L^p_\alpha$

$$\omega(\tau) = \left\{ \int_0^\infty \sigma |f_\tau(\sigma, \tau)|^2 d\sigma \right\}^{1/2},$$

where $f_\tau(\sigma, \tau)$ is the derivative of Poisson integral of f with respect to τ, then it holds that

$$\omega(\tau) \leqq A\mu(\tau]$$

and

$$\|f\|_{p,\alpha} \leqq A_{p,\alpha}\|\omega\|_{p,\alpha},$$

by conjugacy (see Waterman [14]). It is clear that if f is real part of ϕ, then $\omega(\tau, f) \leqq g(\tau, \phi)$. Therefore we have the opposite inequalities.

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