Let $f(x)$ be a continuous function with period $2\pi$ and $E_n$ be the set of equidistant nodal points situated in the interval $0 \leq x < 2\pi$, that is

$$\xi_0 + 2\pi j/(2n + 1) \quad (j = 0, 1, \ldots, 2n), \quad \text{(mod. } 2\pi)$$

where $\xi_0$ is any real number. Then the trigonometric polynomial of order $n$ coinciding with $f(x)$ on $E_n$ is

$$(1) \quad I_n(x,f) = \frac{1}{\pi} \int_0^{2\pi} f(t) D_n(x - t) d\omega_{2n+1}(t),$$

where $D_n(x)$ is the Dirichlet kernel and $\omega_{2n+1}(t)$ is a step function which is associated with $E_n$. (We shall refer to A. Zygmund [4, Chap. X] these notations and fundamental properties of trigonometric interpolation.) We denote the Fourier expansions of (1) by

$$(2) \quad I_n(x,f) = \sum_{k=-n}^{n} c_k^{(n)} e^{ikx},$$

where

$$c_k^{(n)} = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} d\omega_{2n+1}(t).$$

The $\{c_k^{(n)}\}$ are called the $k$-th Fourier-Lagrange coefficients and for a fixed $k$, $c_k^{(n)}$ is an approximate Riemann sum for the integral defining Fourier coefficient $c_k$ of $f(x)$, that is

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} dt.$$

Let us denote the partial sums of (1) by

$$I_{n,m}(x,f) = \sum_{k=-m}^{m} c_k^{(n)} e^{ikx} \quad (m \leq n),$$

in particular

$$I_n(x,f) = I_{n,n}(x,f).$$
Let $B_{n,v}(x,f)$ denote the arithmetic means of $I_{n,m}(x,f)$; thus

$$B_{n,v}(x,f) = \frac{1}{v+1} \sum_{m=0}^{v} I_{n,m}(x,f) \quad (v \leq n)$$

$$= \sum_{k=-v}^{v} \left( 1 - \frac{|k|}{v+1} \right) c_k^{(v)} e^{ikx}$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) K_v(t-x) d\omega_{2n+1}(t),$$

where $K_v(t)$ is the Fejér kernel, and set

$$B_{n}(x,f) = B_{n,n}(x,f).$$

In the present note, the author will investigate approximating properties of $B_{n,v}(x,f)$. These are analogous to the Fejér means of Fourier series. But the proofs are somewhat delicate.

**Theorem 1.** We have

(1°) $B_n(x,f) - f(x) = o(n^{-1})$

uniformly as $n \to \infty$, if and only if $f(x)$ is a constant.

(2°) $B_{n,v}(x,f) - f(x) = O(v^{-1})$

uniformly as $v \to \infty$ (for all $v \leq n$), if and only if $f(x)$ satisfies the Lipschitz condition of order 1.

**Proof.** (1°) From the formula (3),

$$\frac{1}{2\pi} \int_{0}^{2\pi} [f(x) - B_n(x,f)] e^{-ikx} dx = c_k - \left( 1 - \frac{|k|}{n+1} \right) c_k^{(n)}$$

$$= c_k - c_k^{(n)} + \frac{|k|}{n+1} c_k^{(n)}.$$

If $B_n(x,f) - f(x) = o(n^{-1})$ uniformly, then

$$c_k - c_k^{(n)} + \frac{|k|}{n+1} c_k^{(n)} = o(n^{-1}).$$

When a trigonometric polynomial of order $n$ has approximating degree $\varepsilon_n$ for $f(x)$, then the integral of $f(x)$ has the same degree of approximation by its Riemann sums, (Walsh-Sewell [3, Theorem 4]). Since $B_n(x,f)$ is a trigonometric polynomial of order $n$, and

$$f(x) - B_n(x,f) = o(n^{-1}), \quad f(x)e^{-ikx} - B_n(x,f)e^{-ikx} = o(n^{-1}) \quad (k \leq n),$$
we have
\[ \int_0^{2\pi} f(x)e^{-ikx}dx - c_k^{(n)} = o(n^{-1}), \quad \text{for fixed } k; \]
that is
\[ n(c_k - c_k^{(n)}) = o(1) \quad \text{as } n \to \infty. \]
Hence from (4)
\[ |k|c_k^{(n)} \to 0 \quad \text{as } n \to \infty, \]
and this means that \( c_k = 0, \ (k = \pm 1, \pm 2, \ldots) \). Thus we have \( f(x) = c_0 \). The converse is trivial.

(2') We suppose that
\[ f(x) - B_{n,v}(x,f) = O(v^{-1}) \]
uniformly as \( n \geq v \to \infty \). Since the unit ball of \( L^\infty \) space is weak* compact, there exist a bounded function \( g(x) \) and a subsequence \( \{v_p\} \) of \( v \) such as
\[
\lim_{n \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \nu_p(f(x) - B_{n,v}(x,f))e^{-ikx}dx = \frac{1}{2\pi} \int_0^{2\pi} g(x)e^{-ikx}dx
\]
for all integral \( k \). The first term is
\[
\lim_{v \to \infty} \left\{ \nu_p(c_k - c_k^{(n)}) + \frac{\nu_p |k|}{\nu_p + 1} c_k^{(n)} \right\} \quad (v_p \leq n_p).
\]
On the other hand if we take \( v = n, \) then the above Walsh-Sewell result yields
\[
n(c_k - c_k^{(n)}) = O(1) \quad (k \leq n).
\]
When we set \( v = [n^{1-\delta}] \) \((0 < \delta < 1) \) and select a subsequence \( \{n_p\} \) and we set
\[ v_p = [n_p^{1-\delta}], \]
then
\[
\nu_p(c_k - c_k^{(n_p)}) = n_p^{1-\delta}(c_k - c_k^{(n_p)}) \]
\[ = (n_p)^{-\delta}n_p(c_k - c_k^{(n)}) = o(1) \]
from (7). Since \( c_k^{(n_p)} \to c_k \), from (5) and (6) we conclude
\[ |k|c_k = \frac{1}{2\pi} \int_0^{2\pi} g(x)e^{-ikx}dx, \quad g(x) \in L^\infty(0,2\pi), \]
for all integral \( k \). This means that \( |k|c_k \) are the Fourier coefficients of \( g(x) \) which belongs to the class \( L^n(0, 2\pi) \). Since \( \{c_k\} \) are Fourier coefficients of \( f(x) \), it is easy to see that \( f'(x) \in L^n(0, 2\pi) \). This is equivalent to that \( \widehat{f}(x) \) satisfies the Lipschitz condition of order 1.

Conversely if \( \widehat{f}'(x) \in L^n(0, 2\pi) \), then the Fejér means \( \sigma_n(x, f) \) of \( f(x) \) are the best approximation (A. Zygmund [4, I, p. 123]), that is

\[
\sigma_n(x, f) = O(\nu^{-1}).
\]

\( B_n, \nu(x, f) \) is a linear method of approximation, and

\[
B_n, \nu(x, f) - f(x) = B_n, \nu(x, f) - \sigma_n(x, f) - \sigma_n(x, f) + \{B_n, \nu(x, \sigma_n) - \sigma_n\}
= P_n, \nu(x) + Q_n, \nu(x),
\]

say. \( B_n, \nu(x) \) transforms any bounded function to some bounded function, so from (8)

\[
P_n, \nu(x) = O(\nu^{-1}) \quad (\nu \leq n).
\]

On the other hand \( \sigma_n(x) \) is a \( \nu \)-th order polynomial and \( \nu \leq n, \)

\[
I_n(x, \sigma_n(x)) = \sigma_n(x) = \sum_{k=-\nu}^{\nu} \left( 1 - \frac{|k|}{\nu + 1} \right) c_k e^{ikx},
\]

and

\[
B_n, \nu(x, \sigma_n(x)) = \sum_{k=-\nu}^{\nu} \left( 1 - \frac{|k|}{\nu + 1} \right)^2 c_k e^{ikx}.
\]

Hence

\[
Q_n, \nu(x) = B_n, \nu(x, \sigma_n(x)) - \sigma_n(x)
= \sum_{k=-\nu}^{\nu} \left\{ \left( 1 - \frac{|k|}{\nu + 1} \right)^2 \right\} c_k e^{ikx} - \sum_{k=-\nu}^{\nu} \left( 1 - \frac{|k|}{\nu + 1} \right) c_k e^{ikx}
= -\frac{1}{\nu + 1} \sum_{k=-\nu}^{\nu} \left( 1 - \frac{|k|}{\nu + 1} \right) |k| c_k e^{ikx}.
\]

From the assumption \( \widehat{f}'(x) \in L^n(0, 2\pi) \), the arithmetic means of Fourier series of \( \widehat{f}(x) \) is bounded. Consequently

\[
Q_n, \nu(x) = O(\nu^{-1}).
\]

Collecting the estimates of \( P_n, \nu(x) \) and \( Q_n, \nu(x) \), we have the desired result.

**Theorem 2.** If \( f(x) \) belongs to the Lipschitz class of order \( \alpha \) (0 < \( \alpha < 1 \))
then
\[ B_{n,v}(x,f) - f(x) = O(\nu^{-\alpha}) \quad (\nu \leq n) \]
and if \( f(x) \) belongs to the Lipschitz class of order 1, then
\[ B_{n,v}(x,f) - f(x) = O(\nu^{-1}\log \nu) \quad (\nu \leq n). \]
More generally if \( f(x) \) belongs to the class \( \Delta^1 \), then
\[ B_{n,v}(x,f) - f(x) = O(\nu^{-1} \log \nu). \]

**Proof.** \( B_{n,v}(x,f) \) maps any bounded function to some bounded function. Hence applying author's another result (G. Sunouchi [2, Theorem 1]) to Theorem 1, we get Theorem 2.

Theorem 2 has been proved by Ruban and Krasilinikoff [1] with another method.

**Literature**


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