ON BOREL SERIES

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Introduction. In his book "Leçons sur la théorie des fonctions", É.Borel [1] considers certain infinite series, which in the theory of real variables, are sometimes referred to as Borel series, and which are defined by

$$(1) \qquad \sum_{n=1}^{\infty} (A_n/r_n^{m_n}),$$

$r_n^2 = (x_1 - a_n^{(1)})^2 + (x_2 - a_n^{(2)})^2 + \cdots + (x_h - a_n^{(h)})^2$, $m_n < m$. A_n, $n = 1, 2, \cdots$, is a sequence of real numbers and $\sum A_n$ is assumed to be convergent. $a_n^{(h)}$, $n = 1, 2, \cdots$ are $h \geqq 1$ sequences of real numbers; $x_1, \cdots, x_h$ are real variables and the exponents m_n are real positive numbers. According to a theorem of Borel, (1) converges almost everywhere.

If we put $A_n \equiv |A|^{p+q}/q$, where $|A_0| < |A| < 1$, A not necessarily real, p and q positive integers, and $m_n = 1$, $r_n = |x_1 - a_n^{(1)}| \equiv |x - (p/q)|$, i. e. if we identify $\{a_n^{(1)}\}$ with the somehow simply ordered double sequence $\{\{(p/q)\}\}$, then we obtain the special case of (1)

$$(2) \qquad f(A\,; x) \equiv \sum_{p,\,q=1}^{\infty} (|A|^{p+q}/|qx - p|).$$

In (2), let x be a fixed, real irrational number. Historically (2) was first discussed by H.Bruns [2], while dealing with the convergence of a trigonometric series to a bounded function for certain values of the parameters which occur. Since then, (2) has received considerable attention, e.g. [3], [4].

The principal purpose of this paper is to discuss convergence and divergence of (2) from an arithmetic point of view and in particular show that (2) may still converge for a subset of Liouville numbers x. The main tool will be K.Mahler's [5] classification of numbers as it is exposed in detail by Th.Schneider [6]. It would of course be nice to obtain an "if and only if" theorem to the effect that (2) diverges for all real irrational x that satisfy certain properties. Unfortunately, this appears unattainable at this time, owing in part to the method applied.

In the first paragraph, in which we shall point out a generalization of a theorem of E. Maillet, we shall find some aspects which are relevant to some

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parts of the later paragraphs.

1. On a theorem of Maillet. In his book "Irrationalzahlen", O. Perron [7] proves the

THEOREM (a). *If between two irrational numbers x and R the relation*

$$R = r_0 + r_1 x + \cdots + r_m x^m, \tag{3}$$

where $r_0,\cdots,r_m$ are rational numbers, holds, and if x is a Liouville number, then R is also a Liouville number.

He also proves the

THEOREM (b). *If between two irrational numbers x and S the relation*

$$S = \frac{p_0 + p_1 x + \cdots + p_m x^m}{q_0 + q_1 x + \cdots + q_m x^m}, \tag{4}$$

holds, where $p_0,p_1,\cdots,q_m$ are integers and not both p_m, q_m vanish, and if x is a Liouville number, then S is also a Liouville number.

If in (4), $m = 0, 1$ and if $p_0,\cdots,q_1$ are non-zero integers with $p_0 q_1 - p_1 q_0 \neq 0$, then we have the special case of this theorem proved by E. Maillet [8], [9]; in this case, also the converse of theorem (b) holds.

Both theorems above may be generalized if we replace "Liouville number" by "number satisfying the following criterion for transcendency" [6], page 34:

THEOREM (c). *If* 1.) p_n/q_n, $n = 1, 2,\cdots$, *with* $(p_n,q_n) = 1$, $q_{n+1} > q_n > 0$ *is an infinite sequence of quotients of integers, and if*

2.) *there exists a sequence of real numbers s_n, $n = 1, 2,\cdots$, such that* $\overline{\lim\limits_{n\to\infty}}\, s_n > 1$, *and if*

3.) *for an irrational number x the inequalities*

$$|q_n x - p_n| \leqq q_n^{-s_n}, \quad n = 1, 2,\cdots, \tag{5}$$

are satisfied, then x is a transcendental number.

Evidently, theorem (c) gives sufficient conditions for x to be a Liouville number, i.e. in case $\overline{\lim\limits_{n\to\infty}}\, s_n = +\infty$; cf. [6], page 3.

The generalizations for (3) and for (4), just mentioned, can be carried out —mutatis mutandis—by following the proofs of O. Perron, and we therefore omit the details of these proofs.

2. K. Mahler's classification of numbers. K. Mahler defines the arithmetic function

$$(6) \qquad w_n(H,x) = \text{Min}\left(\left| \sum_{\nu=0}^{n} a_\nu x^\nu \right| ; \; 0 \leqq |a_\nu| \leqq H, \right.$$

$$\left. a_\nu \textit{ integral rational}, \; \sum_{\nu=0}^{n} a_\nu x^\nu \neq 0 \right).$$

For $n \geqq 1$, $H \geqq 1$ is this function equal to at most 1, i.e. for $a_0 = 1$, $a_1 = a_2 = \cdots = a_n = 0$, and evidently does not increase with increasing n and H.

Further, Mahler forms the expressions

$$(7) \qquad w_n(x) = w_n = \overline{\lim_{H\to\infty}}\, [-\log w_n(H, x)/\log H], \; n = 1,2,\cdots \text{ and}$$

$$(8) \qquad w(x) = w = \overline{\lim_{n\to\infty}}\, [w_n(x)/n].$$

Evidently, for $n \geqq 1, 0 \leqq w_n \leqq \infty$, $0 \leqq w \leqq \infty$; further $w_{n+1}(H, x) \leqq w_n(H,x)$, $-\log w_{n+1}(H, x) \geqq -\log w_n(H, x)$; hence $w_{n+1}(x) \geqq w_n(x)$. Thus, w is either a non-negative finite quantity, or positive infinite.

Let μ be the smallest index for which $w_\mu = \infty$, in case such an index exists. Otherwise, whenever w_n remains bounded for all n, let $\mu = \infty$. Hence μ is determined uniquely. Consequently, for a finite μ, $w = \infty$; and μ and w cannot both have finite values for the same x. Thus, there remain four possibilities for the values of μ and w according to which the number x may be classified. We say that x is an

A-number, if $w = 0$, $\quad \mu = \infty$
S-number, if $0 < w < \infty$, $\quad \mu = \infty$
T-number, if $w = \infty$, $\quad \mu = \infty$
U-number, if $w = \infty$, $\quad \mu < \infty$.

We do not know whether the class of T-numbers is empty or not. The A-numbers are identical with the class of algebraic numbers, however, in our study we consider the subset of irrational A-numbers only. The S-, T-, and U-numbers are transcendental numbers. In our study of (2) we need certain estimates of (6) which follow from the above definitions of the A-, S-, T-, and U-numbers.

3. Convergence of f for irrational A-, S-, and T-numbers, as well as *U*-numbers of index $\mu \geqq 2$. In the case of the A-numbers we have [6], page 68, the

THEOREM (d). *Every algebraic number is an A-number, and every A-number is an algebraic number.*

The proof of theorem (d) which we omit, provides us with the following inequality

(9) $$w_n(H, x) > cH^{1-s}, \quad c = c(x, n) > 0,$$

and $s > 0$ is the degree of the algebraic number x.

From the definition of S-, and T-numbers one obtains [6], pages 67-68 estimates for (6). In case of the S-numbers: For a given x there exists a $\theta_0 = \theta_0(x) > 0$, such that for every $\varepsilon > 0$

(10) $$w_n(H, x) > c_n H^{-(\theta_0+\varepsilon)n}, \ n = 1,2,\cdots, \ \textit{and} \ c_n = c(x, n, \varepsilon) > 0.$$

In case of the T-numbers we have

(11) $$w_n(H, x) > c_n H^{-\theta_n n}, \ n = 1,2,\cdots, \text{ where } c_n = c(x, n, \theta_n) > 0,$$
$$\theta_n = \theta(x, n) > 0, \text{ and } \overline{\lim_{n\to\infty}} \ \theta_n = \infty; \ c_n \text{ and } \theta_n \text{ are independent of } H.$$

For the U-numbers of index $\mu \geqq 2$ we have

(11′) (= (17) below) $$w_1(H, x) \geqq H^{-\lambda_0+1}.$$

Combining (9), (10), (11), as well as (14) and (17) in connection with theorem (f) of paragraph 4 below, we can now prove

THEOREM 1. *If x is an irrational A-, S-, or T-number, or if x is a U-number of index $\mu \geqq 2$, then the expression (2) for f converges.*

PROOF. Let $0 \neq B(x) = \sum_{\nu=0}^{1} a_\nu x^\nu$ be a binomial with integral rational coefficients of height $H = \mathrm{Max}(|a_0|, |a_1|) > 0$. We observe that it suffices to consider binomials since the decisive estimates for the A-, S-, and T-numbers hold for all polynomials. In the latter cases then, let $c = c(x) > 0$, $\gamma > 0$ be constants. In case of S-, and T-numbers c may still depend upon γ. In case of T-numbers γ may still depend on x. But then it follows from (9), (10), and (11) that

(12) $$0 \neq \left| \sum_{\nu=0}^{1} a_\nu x^\nu \right| > w_1(H, x) > cH^{-\gamma},$$
$$H = \mathop{\mathrm{Max}}_{\nu=0}^{1} (|a_\nu|),$$

which holds for all binomials of height H. Now, by (2) we have for any term the estimate

(13) $$\frac{|A|^{p+q}}{|qx - p|} < |A|^{p+q} c^{-1} H^\gamma \leqq c^{-1} |A|^p |A|^q (pq), \ H = \mathrm{Max}\,(p, q) \leqq pq,$$

and it is evident that the double series extended over the right hand side of (13) converges since $|A| < 1$. In case of the U-numbers of index $\mu \geqq 2$, we have available inequality (11′), which at once leads to an estimate of the form (13) and hence the expression (2) for f converges, q.e.d.

Examples for A-numbers, such that f converges, are easily exhibited. An example for S-numbers is the base of natural logarithms $x = e$, in this case then, f converges. Another example is Ludolph's number π. Let $x = \pi$. In this case we have available K. Mahler's [10] approximation $|q\pi - p| > q^{-41}$ for all positive integers $p, q \geqq 2$. It follows that f converges in this case. π is an S-, or T-number. As an example to the U-numbers of index $\mu \geqq 2$ we mention the following [7], page 185 : If $y_0 = x_0^2 = [b_0, b_1, b_2, \cdots]$ is a Liouville number represented by its simple continued fraction expansion with $b_{n+1} = q_n^{2n-1}$, where q_n is the n-th approximation denominator to y_0, then $y_0^{1/2} = x_0$ is not a Liouville number, and hence, the expression (2) for f converges for x_0. Evidently, x_0 is a U-number of index $\mu = 2$. J.W. LeVeque [11] has studied U-numbers of index $\mu \geqq 1$.

4. K. Mahler's *U*-numbers of index $\mu = 1$, and divergence of *f*. The definition of the U-numbers implies, there is a finite $\mu = \mu(x)$, so that for every $n \geqq \mu$, the quantity $w_n(x) = \infty$, or $\overline{\lim}_{H\to\infty} [- \log w_n(H, x)/\log H] = \infty$. Thus, to every $\theta > 0$ and every fixed $n \geqq \mu$ an infinite subsequence H_λ may be extracted for which $[- \log w_n(H_\lambda, x)/\log H_\lambda] > \theta n$ is astisfied. Hence there are polynomials with integral rational coefficients of height $H \leqq H_\lambda$, for which to an arbitrarily large H_λ and an arbitrary given θ the inequality

$$(14) \qquad 0 \neq \left| \sum_{\nu=0}^{n} a_\nu x^\nu \right| < H_\lambda^{-\theta n}, \quad |a_\nu| \leqq H_\lambda, \quad \lim_{\lambda\to\infty} H_\lambda = \infty$$

holds. Since H_λ may be arbitrary large, there are infinitely many polynomials to every fixed θ and fixed n.

Further, [6], page 1, we require Liouville's

THEOREM (e). *If x is an algebraic number of degree $n > 1$, then there exists a constant $c = c(x) > 0$, such that for all integral rational p, q, $q > 0$, the inequality*

$$(15) \qquad |x - (p/q)| > cq^{-n}$$

holds.

Theorem (e) then permits the

DEFINITION 1. Every irrational number x is called a *Liouville number* if one can find to every natural number λ a rational fraction p_λ/q_λ, so that the inequalities

$$(16) \qquad |x - (p_\lambda/q_\lambda)| < q_\lambda^{-\lambda}, \qquad q_\lambda > 1,$$

hold.

Evidently, by (15), it follows that every Liouville number is transcendental.

THEOREM (f). *A C-number x has index $\mu = 1$, if and only if it is a*

Liouville number.

PROOF. If x is a Liouville number, then there exist for every natural number λ, integers p and $q > 1$, so that $|qx - p| < q^{-\lambda+1}$. Hence it follows from (14) that every Liouville number is a U-number with index $\mu = 1$. Conversely, if x is not a Liouville number, then there exists a λ_0 so that for all integers p, q the inequalities $|qx - p| \geqq q^{-\lambda_0+1}$ hold. Hence we have by (6)

$$w_1(H, x) \geqq H^{-\lambda_0+1}, \tag{17}$$

and thus by (7)

$$w_1(x) \leqq \lambda_0 - 1,$$

which is against our hypothesis that x is a U-number with index $\mu = 1$. Thus the Liouville numbers are identical with the U-numbers of index $\mu = 1$, q. e. d.

In paragraph 1, we did already observe that theorem (c) gives sufficient conditions for x to be a Liouville number, namely in case $\overline{\lim_{n\to\infty}}\, s_n = +\infty$. We have the following

LEMMA 1. *If* $s_n = [m(p_n + q_n)/\log q_n]$, $n = 1,2,\cdots$, *is a sequence of real numbers with integers* $q_{n+1} > q_n > 1$, $p_n \geqq 0$, $n \geqq 1$, *and* $m = -\log |A| > 0$, *then* $\lim s_n = +\infty$.

PROOF. All q_n are positive integers and $q_n \to \infty$, as $n \to \infty$. Since also $p_n \geqq 0$, we observe that the assertion follows at once from $(q_n/\log q_n) \to \infty$, as $n \to \infty$, q. e. d.

THEOREM 2. *If x is a Liouville number satisfying the hypotheses of theorem* (c), *as well as those of lemma* 1, *then the expression* (2) *for f diverges.*

PROOF. By theorem (c), x permits the approximations

$$|q_n x - p_n| \leqq q_n^{-s_n}, \quad n = 1, 2, \cdots. \tag{18}$$

Now, from the double series (2) we extract a sequence of terms $f(n)$, so that by (18) for these terms

$$f(n) = \frac{|A|^{p_n+q_n}}{|q_n x - p_n|} \geqq |A|^{p_n+q_n} q_n^{s_n} = 1, \qquad \text{for } n \geqq 1. \tag{19}$$

Thus the single series extended over the terms $f(n)$, $n \geqq 1$, diverges and *a fortiori* the double series (2) diverges, q. e. d.

In the following we restrict ourselves essentially to a certain subset of

Liouville numbers that may be constructed by means of simple continued fractions.

DEFINITION 2. An infinite continued fraction is said to be simple, if all partial numerators equal + 1, and if the partial denominators are all positive integers, except for the first one b_0, which may be any integer.

From now on, the letter b will denote a partial denominator of a simple continued fraction and the letters p and q, with subscripts n in general, will denote numerators and denominators of convergents to certain real numbers represented by their simple continued fraction developments.

Again without proof, we state [7], page 179.

THEOREM (g). *A simple continued fraction* $[b_0, b_1, b_2, \cdots]$ with convergents p_n/q_n *is a Liouville number if and only if to every arbitrary large number* k *an index* n^1 *may be found, such that* $b_{n+1} > q_n^k$ *for all* $n \geqq n^1$.

This theorem implies that the Liouville number $x_0 = [b_0, b_1, b_2, \cdots]$ permits to every positive k, as soon as n is sufficiently large, the approximations

$$|x_0 - (p_n/q_n)| < q_n^{-k}, \ (x_0 > 0, \text{ w. l. g.}) \tag{20}$$

for by the approximation theorem of continued fractions [9], page 37, we have with $\{p_n/q_n\}$ the sequence of convergents to x_0 that $|x_0 - (p_n/q_n)| < b_{n+1}^{-1}q_n^{-2}$, and since $b_{n+1} \geqq q_n^{k-2}$ for all $n \geqq n^1$, and for every arbitrary large number k, (20) follows; in theorem (g) above, k was replaced by $k+2$.

In order to emphasize the new meaning of the p_n's and q_n's as numerators and denominators of convergents to a simple continued fraction, we re-name the sequence s_n, $n = 1, 2, \cdots$ of real numbers of theorem 2 by the symbol φ_n, $n = 1,2, \cdots$. Then to theorem 2, we have at once the

COROLLARY. *If* $x_0 = [b_0, b_1, b_2, \cdots]$ *is a Liouville number with* $b_{n+1} > q_n^{\varphi_n - 1}$ *and where* $\varphi_n = \lfloor m(p_n + q_n)/\log q_n \rfloor$, $n = 1,2, \cdots$, *then the expression* (2) *for* f *diverges.*

PROOF. The proof follows at once from theorem (g) and (20), as well as from the argument that led to (19) in theorem 2, q. e. d.

EXAMPLE. We construct the continued fraction $x_0 = [b_0, b_1, b_2, \cdots]$, whose partial denominators are generated by means of the recursion formula $q_n = b_n q_{n-1} + q_{n-2}$, $n = 1, 2, \cdots$. By definition $q_{-1} = 0$, $q_0 = 1$. Then we put $b_0 = 0$, so that with $b_1 = 1$, $q_1 = 1 \cdot 1 + 0 = 1$; we put $b_2 = q_1^{q_1} = 1$, so that $q_2 = 1 \cdot 1 + 1 = 2$; we put $b_3 = q_2^{q_2} = 4$, so that $q_3 = 4 \cdot 2 + 1 = 9$; we put $b_4 = q_3^{q_3}$, etc. In general, we put $b_{n+1} = q_n^{q_n}$. Now, since $\lim_{n \to \infty} q_n = \infty$, x_0 is a Liouville number by theorem (g).

By theorem 2 we must have that $b_{n+1} > q_n^{\varphi_n-1}$, where $\varphi_n = [m(p_n + q_n)/\log q_n]$, $n = 1,2,\cdots$. W.l.g. we put $m = 1$ in this last formula. Then sinec $b_0 = 0$, $p_n < q_n$ for all $n \geqq 2$. Now it is evident that for $n \geqq \bar{n}$, $q_n > (2q_n/\log q_n) > [(p_n + q_n)/\log q_n] - 1$, and the expression (2) for f diverges for this number x_0.

5. *U*-numbers of index $\mu = 1$, and convergence of *f*. In order to discuss convergence we require a theorem of Lagrange [9], page 44, which we state without proof,

THEOREM (h). *If p_n/q_n, $n \geqq 1$, is the n-th convergent to x_0, and if p/q is a fraction different from p_n/q_n with $0 < q \leqq q_n$, then* $|qx_0 - p| \geqq |q_{n-1}x_0 - p_{n-1}| > |q_nx_0 - p_n|$.

LEMMA 2. *If the strictly monotone increasing sequence of natural numbers b_n, $n = 1, 2, \cdots$, b_0 any integral rational number, is such that $b_n + 1 \leqq q_{n-1}^{\psi_{n-1}-1}$, where $\psi_{n-1} = [(mq_{n-2}/\log q_{n-1})] - c$, $c > 3$ a fixed real constant, $n = 1,2,\cdots$, then $\lim_{n\to\infty} \psi_n = +\infty$.*

PROOF. From the recursion formula $q_n = b_nq_{n-1} + q_{n-2}$, for $n \geqq 2$, we have $q_n < (b_n + 1)q_{n-1}$ and hence
$(\log q_n/q_{n-1}) < [\log (b_n + 1) + \log q_{n-1}/q_{n-1}] \leqq [(\psi_{n-1} - 1)\log q_{n-1} + \log q_{n-1}/q_{n-1}]$ $= (\psi_{n-1} \log q_{n-1}/q_{n-1}) = (mq_{n-2}/q_{n-1}) - (c\log q_{n-1}/q_{n-1}) < (mq_{n-2}/q_{n-1}) \leqq (m/b_{n-1})$ since $q_{n-1} \geqq b_{n-1}q_{n-2}$. Since m is a fixed constant, and since $b_{n-1} \to \infty$, as $n \to \infty$, we have the assertion, q. e. d.

LEMMA 3. *Under the hypotheses of lemma 2 and those of theorem (c), and $q_n^{\psi_n-1-r} \leqq b'_{n+1} = b_{n+1} + 1 \leqq q_n^{\psi_n-1}$, for $n > n_0(r)$, $r \geqq r_1 > 0$ a fixed real number, $x_0 = [b_0, b_1, b_2, \cdots]$ is a Liouville number.*

PROOF. We consider the sequence of all natural numbers $n = n_0 + 1$, $n_0 + 2, \cdots$, $n > n_0(r)$; then by [9], page 37 we have $|x_0 - [b_0, b_1, b_2, \cdots, b_{n_0}, b_n]| < (1/b'_{n+1}q_n^2)$, where b'_{n+1} is the partial denominator immediately following b_n. Now, we construct the partial denominators b'_{n+1} by means of the condition $b'_{n+1} = b_{n+1} + 1 \geqq q_n^{\psi_n-1-r}$ subject to the upper bound for b'_{n+1}, for $n = n_0 + 1$, $n_0 + 2, \cdots, n > n_0(r)$. Then

$$|x_0 - [b_0, b_1, b_2, \cdots, b_{n_0}, b_n]| = |x_0 - (p_n/q_n)| \leqq (1/q_n^{\psi_n+1-r}).$$

By lemma 2, $\psi_n + 1 - r \to \infty$, as $n \to \infty$. By theorem (c) the sequence $\{\psi_n + 1 - r\}$ plays the role of the sequence $\{s_n\}$. Hence x_0 is a Liouville number, q. e. d.

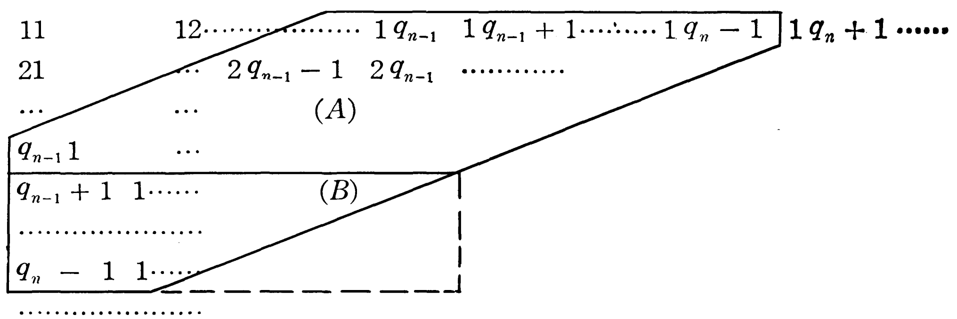
LEMMA 4. *If δ_n is defined by $q_n^{\psi_n+\delta_n} = q_n^{\psi_n} + q_{n-1}$, then under the hypo-*

theses of lemma 2, $\lim_{n\to\infty} \delta_n = 0$.

PROOF. Evidently $q_n^{\delta_n} = 1 + q_{n-1}q_n^{-\psi_n}$ and $0 \leqq \delta_n < [\log(1 + q_n^{1-\psi_n})/\log q_n]$. Upon taking the limit, we have the assertion, q.e.d.

THEOREM 3. *If* $x_0 = [b_0, b_1, b_2, \cdots]$ *is a Liouville number satisfying the hypotheses of lemmas* 3 *and* 4, *the expression* (2) *for* f *converges.*

PROOF. We shall add (2) by diagonals. Thus we consider the square array



where the q_n's are the denominators of the n-th convergents to the Liouville number x_0. The p and q of (2) run along columns and rows, respectively. Requiring that $q_{n-1} + 1 \leqq p + q < q_n + 1$ for all $n \geqq 1$, we can now order the double sequence $\{\{p/q\}\}$ into a single sequence and within this interval the pairs (p, q) may be ordered arbitrarily. The number of terms in the diagonal area (A) of (S) is readily seen to be $(q_n - q_{n-1})q_{n-1}$, and the number of terms in the remaining triangular area (B) of (S) is $(q_n - q_{n-1})(q_n - q_{n-1} - 1)./2$ Thus, denoting the total number of terms arising from (A) and from (B) for each positive integer n, subject to the condition $q_{n-1} + 1 \leqq p + q < q_n + 1$, by $g(n)$, we have $g(n) = (q_n - q_{n-1})(q_n + q_n - 1)/2$, $n = 1, 2, \cdots$. Formally, we rewrite (2)

$$(2') \qquad \sum_{p,q=1}^{\infty} (|A|^{p+q}/|qx_0 - p|) = f(p, q) + \sum_{n=1}^{\infty}\Big[\sum_{\substack{1\leq p,q \\ q_{n-1}\leq p+q<q_n+1}} (|A|^{p+q}/|qx_0 - p|)\Big],$$

where $f(p, q)$ equals a non-negative constant, possibly arising from the terms before the first diagonal containing the first approximation denominator of x_0. A rough estimate for the number of terms in the inner sum of (2′), in our addition by diagonals, shows that $g(n) < q_n^2$. Now, consider x_n, the n-th complete quotient of x_0 [9], page 34, defined by $x_n = [b_n, b_{n+1}, \cdots]$, $n = 1, 2, \cdots$; thus the $(n + 1)$-st complete quotient will be $x_{n+1} = [b_{n+1}, b_{n+2}, \cdots]$ or what is the same $x_{n+1} = b_{n+1} + (1/x_{n+2})$. Since every $b_n, n \geqq 1$, is a positive integer, it follows from definition 2 that $x_{n+1} < b_{n+1} + 1$ and therefore, because of our hypothesis $b_{n+1} + 1 \leqq q_n^{\psi_n - 1}$, $x_{n+1} < q_n^{\psi_n - 1}$. Then, from the theory of continued fractions [9], page 37,

$$q_n x_{n+1} + q_{n-1} < q_n^{\psi_n} + q_{n-1}.$$

Since further $q_n x_0 - p_n = (-1)^n/(q_n x_{n+1} + q_{n-1})$, and going over to absolute amounts

$$|q_n x_0 - p_n| = |q_n x_{n+1} + q_{n-1}|^{-1} > |q_n(b_{n+1} + 1) + q_{n-1}|^{-1} \geqq$$

$$|q_n^{\psi_n} + q_{n-1}|^{-1} = |q_n^{\psi_n + \delta_n}|^{-1},\ \delta_n \geqq 0,\ n \geqq n_1, \tag{21}$$

by lemma 4. Now, by theorem (h) $|qx_0 - p| \geqq |q_n x_0 - p_n|$, $0 < q \leqq q_n$, and by (21), we obtain as an estimate for the inner sum of (2′)

$$f(n\,; p, q) \equiv \sum_{\substack{1 \leq p, q \\ q_{n-1}+1 \leqq p+q < q_n+1}} (|A|^{p+q}/|qx_0 - p|) \leqq g(n)|A|^{q_{n-1}+1}|q_n x_0 - p_n|^{-1}$$

$$< |A|^{q_{n-1}+1} q_n^{\psi_n + \delta_n + 2} = |A| q_n^{\delta_n + 2 - c}.$$

Now, $|A| < 1$, and $q_n \geqq n$. Also $c > 3$, or $c = 3 + \tau$, for $\tau > 0$; let $\varepsilon = (\tau/2)$. Then, for $n \geqq n_2(\varepsilon)$, $\delta_n < \varepsilon = (\tau/2)$, and $\delta_n + 2 - c < (\tau/2) + 2 - 3 - \tau = -1 - (\tau/2)$ and

$$f(n\,; p, q) < n^{-1-(\tau/2)} \tag{22}$$

so that the sum extended over the terms of the inner sum of (2′) converges, hence the expression for (2) converges, q. e. d.

EXAMPLE. An example for such a Liouville number may easily be obtained, if we proceed similarly as for the example to the corollary of theorem 2. This time we use the notion of the largest integer not exceeding a given real number, i.e. we consider $[\psi_n]$, since all b_n for $n \geqq 1$ are to be positive integers. We omit the details.

K. Mahler [12] has shown, that all non-S-numbers have measure zero on the real line, hence our expression (2) for f diverges for at most a set of real numbers of measure zero.

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