ON THE UNITARY EQUIVALENCE AMONG THE COMPONENTS OF DECOMPOSITIONS OF REPRESENTATIONS OF INVOLUTIVE BANACH ALGEBRAS AND THE ASSOCIATED DIAGONAL ALGEBRAS

MASAMICHI TAKESAKI

(Received September 2, 1963)

Introduction. Let \( \varphi = \int_\Gamma \varphi(\gamma) d\mu(\gamma) \) be an irreducible decomposition of a representation \( \varphi \) of an involutive Banach algebra \( B \) over a measure space \( (\Gamma, \mu) \). As shown by several authors in [4], [8], [9], [13] etc., this decomposition cannot be regarded as a decomposition of the unitary equivalence class of \( \varphi \) into the unitary equivalence classes of \( \varphi(\gamma) \) except for some fairly nice cases, whereas this decompositon is determined only up to unitary equivalence. For instance, some representations can be decomposed in two ways that have no common components as in [8] and some two representations of quite different types can be decomposed into the direct integrals of the same components as in [13]. Therefore it comes into considerations what determines the unitary equivalence relation among the components \( \{ \varphi(\gamma) : \gamma \in \Gamma \} \) of the decomposition \( \varphi = \int_\Gamma \varphi(\gamma) d\mu(\gamma) \). For this question we shall answer in §1 that the algebraic relation between the commutant \( \varphi(\mathcal{R})' = M \) of \( \varphi(B) \) and the associated diagonal algebra \( A \) determines completely the unitary equivalence relation \( \mathcal{R} \) among the components \( \{ \varphi(\gamma) : \gamma \in \Gamma \} \). So we can regard \( \mathcal{R} \) as an algebraic invariant of the couple \((M, A)\). A. Guichardet used \( \mathcal{R} \) for characterization of discrete von Neumann algebras in [5]. We study the behavior of \( \mathcal{R} \) in more general situation.

In §2 we shall give the definitions of simplicity, smoothness and complete roughness of \( A \) in \( M \) using \( \mathcal{R} \). In §3 we shall reduce the study of smooth maximal abelian subalgebras to that of simple ones. §4 is devoted to show some relations between simple or completely rough maximal abelian subalgebras and regular, semi-regular or singular ones defined in [3]. Finally in §5 we shall give some examples of factors of type II and type III with simple maximal abelian subalgebras and completely rough ones simultaneously respectively.

1. Unitary equivalence relation. Let \( \Gamma \) be a standard Borel space\(^1\) and \( \mu \) a Borel measure on \( \Gamma \). Let \( A = L^\infty(\Gamma, \mu) \) be the commutative von Neumann

---

\(^1\) If a Borel space \((\Gamma, \mathcal{F})\) is Borel isomorphic to some separable complete metric space equipped with the Borel structure generated by closed sets, then we call it standard according to Mackey[9]. Calling the member of \( \mathcal{F} \) Borel set, we shall omit the letter \( \mathcal{F} \).
algebra consisting of all essentially bounded measurable functions over the measure space \((\Gamma, \mu)\). Suppose that \(A\) is imbedded in a von Neumann algebra \(M\) as a von Neumann subalgebra and that \(M\) has a faithful representation on a separable Hilbert space. Let \(\pi\) be a normal faithful representation of \(M\) onto a separable Hilbert space \(\mathcal{H}\) and let \(M'\) be the commutant algebra of \(\pi(M)\). Then we get a decomposition

\[ \mathcal{H} = \int_{\Gamma} \mathcal{H}_x(\gamma) \, d\mu(\gamma) \]

of \(\mathcal{H}\) over the measure space \((\Gamma, \mu)\) relative to \(\pi(A)\). \(\pi(A)\) becomes the algebra of all diagonalizable operators which is called the (associated) diagonal algebra and each operator in \(\pi(A)'\) is decomposable. Let \(\mathfrak{A}\) be a uniformly separable C*-algebra which is weakly dense in \(M'\). Then for \(\mathfrak{A}\) we can associate a family \(\{\varphi_x\}\) of representations of \(\mathfrak{A}\) in \(\mathcal{H}(\gamma)\) such that

\[ x = \int_{\Gamma} \varphi_x(x) \, d\mu(\gamma) \quad \text{for every } x \in \mathfrak{A}. \]

Besides, we can choose the family \(\{\varphi_x\}\) as follows; the function \(\gamma \rightarrow \langle \varphi_x(x), \xi(\gamma), \eta(\gamma) \rangle\) is Borel measurable over \(\Gamma\) for every \(x \in \mathfrak{A}\) and for every pair of \(\xi = \int_{\Gamma} \xi(\gamma) \, d\mu(\gamma), \eta = \int_{\Gamma} \eta(\gamma) \, d\mu(\gamma) \in \mathcal{H} \). We denote such family \(\{\varphi_x\}\) by \(\Phi\).

The family \(\{\mathcal{H}_x(\gamma) : \gamma \in \Gamma\}\) of Hilbert spaces and the family \(\Phi\) are determined almost everywhere by the \(\mathcal{H}_x\) and the diagonal algebra \(\pi(A)\). Indeed, if \(\mathcal{H}_x\) is represented by a decomposition \(\mathcal{H}_x = \int_{\Gamma} \mathcal{H}_x(\gamma) \, d\mu(\gamma)\) with respect to \(\pi(A)\) and if \(\Phi\) is another associated family of representations of \(\mathfrak{A}\), then there exists a null set \(N \subset \Gamma\) and a family \(\{u_\gamma : \gamma \in \mathcal{C}(N)^2\}\) of unitary operators of \(\mathcal{H}_x(\gamma)\) onto \(\mathcal{H}_x(\gamma)\) such that \(u_\gamma \varphi_x(u_\gamma^{-1}) = \varphi_x\) for every \(\gamma \in \mathcal{C}(N)\).

Suppose that \(A_i = L^\infty(\Gamma_i, \mu_i)\) \((i = 1, 2)\) is a von Neumann subalgebra of \(M\), where \((\Gamma_i, \mu_i)\) \((i = 1, 2)\) are measure spaces as well as \((\Gamma, \mu)\). Then we get two decompositions

\[ \mathcal{H}_x = \int_{\Gamma_1} \mathcal{H}_1(\gamma_1) \, d\mu_1(\gamma_1) \quad \text{and} \quad \mathcal{H}_x = \int_{\Gamma_2} \mathcal{H}_2(\gamma_2) \, d\mu_2(\gamma_2) \]

of \(\mathcal{H}_x\) over \((\Gamma_1, \mu_1)\) and \((\Gamma_2, \mu_2)\) relative to \(\pi(A_1)\) and \(\pi(A_2)\) respectively and we fix these decompositions of \(\mathcal{H}_x\). Let \(\Phi^1 = \{\varphi_{x_1}\}\) and \(\Phi^2 = \{\varphi_{x_2}\}\) be families of representations of the C*-algebra \(\mathfrak{A}\) which are given by the decompositions of \(\mathcal{H}_x\) as well as \(\Phi\). We define a relation \(R^{\mathfrak{A}, \Phi^1, \Phi^2}\) between the points of \(\Gamma_1\) and \(\Gamma_2\) as follows; \(R^{\mathfrak{A}, \Phi^1, \Phi^2}(\gamma_1, \gamma_2)\) holds if and only if the representations \(\varphi_{x_1}\) and \(\varphi_{x_2}\) of \(\mathfrak{A}\) are unitarily equivalent.

Now let \(\mathcal{R}\) and \(\mathcal{R}'\) be two relations between the points of \(\Gamma_1\) and \(\Gamma_2\). We
define a relation $\equiv$ by the fact that there exist subsets $E_i$ of $\Gamma_i (i = 1, 2)$ with null complements such that $\otimes(\gamma_1, \gamma_2) \equiv \otimes(\gamma_1, \gamma_2)$ for every $(\gamma_1, \gamma_2) \in E_1 \times E_2$. Clearly this relation $\equiv$ is an equivalence relation. We denote the equivalence class of $\otimes$ under the relation $\equiv$ by $\otimes$.

**LEMMA 1.1.** $\otimes_{\Phi} \equiv \otimes_{\Phi}$ depends on neither $\Phi^1$ and $\Phi^2$ nor weakly dense uniformly separable $C^*$-subalgebra $\otimes$ of $M_\infty$. That is, for two weakly dense uniformly separable $C^*$-subalgebras $\otimes$ and $\otimes$ of $M_\infty$ and for associated families $\Phi^1$ and $\Phi^2$ ($i = 1, 2$) of representations of $\otimes$ and $\otimes$ ($i = 1, 2$), there exist Borel subsets $E_i$ of $\Gamma_i (i = 1, 2)$ with null complements such that $\otimes_\Phi(\gamma_1, \gamma_2)$ and $\otimes_\Phi(\gamma_1, \gamma_2)$ are equivalent for every $(\gamma_1, \gamma_2) \in E_1 \times E_2$.

**PROOF.** Let $\otimes$ be countable uniformly dense subalgebra of $\otimes$ over the rational complex number field $C_0$. Then $\otimes_{\Phi} \equiv \otimes_{\Phi}$ is equivalent to the fact that there exists a unitary $u$ of $\otimes(\gamma_1)$ onto $\otimes(\gamma_2)$ such that $u \phi_\gamma(x)u^{-1} = \phi_\gamma(x)$ for every $x \in \otimes$. Let $\{x_n\}$ be an enumeration of $\otimes$. For each $n$ there exists a sequence $\{y_{n,m}\}$ in $\otimes$ that converges strongly to $x_n$. By [2: Chap. II, §2, no 3 Prop. 4], there exist a subsequence $\{y_{n,m}\}$ and null sets $N_i \subseteq \Gamma_i (i = 1, 2)$ such that

$$\phi_\gamma(x_n) = \lim_{k \to \infty} \psi_{\gamma_i}(y_{n,m})$$

for every $\gamma_i \in \otimes_{\phi} (i = 1, 2)$.

Put $N_i = \bigcup_{n=1}^\infty N_i^n (i = 1, 2)$. Suppose that $\otimes_{\Phi} \equiv \otimes_{\Phi}$ holds for $(\gamma_1, \gamma_2)$. Then there exists a unitary $u$ of $\otimes(\gamma_1)$ onto $\otimes(\gamma_2)$ such that $u \phi(\gamma)(x)u^{-1} = \phi(\gamma)(x)$ for every $x \in \otimes$. Hence $\otimes_{\Phi} \equiv \otimes_{\Phi}$ holds. By symmetry $\otimes_{\Phi} \equiv \otimes_{\Phi}$ and $\otimes_{\Phi} \equiv \otimes_{\Phi}$ are equivalent on $\otimes \times \otimes$. Thus $(\gamma_i, \gamma_2)$ holds. By symmetry $\otimes_{\Phi} \equiv \otimes_{\Phi}$ and $\otimes_{\Phi} \equiv \otimes_{\Phi}$ are equivalent on $\otimes \times \otimes$. Therefore $\otimes_{\Phi} \equiv \otimes_{\Phi}$ holds.

According to Lemma 1.1, we can denote $\otimes_{\Phi} \equiv \otimes_{\Phi}$ by $\otimes_{\Phi}$, and $\phi(\gamma)(x)$ by $\phi(\gamma)(x)$ without the indication of the family $\Phi$.

Let $\gamma$ be the trivial representation of the scalar field $C$ onto countably infinite dimensional Hilbert space $\otimes$. For a representation $\pi$ of $M$ we define a representation $\pi \otimes \gamma$ onto $\otimes$ by $(\pi \otimes \gamma)(x)(\xi \otimes \eta) = (\pi(x) \xi) \otimes \eta$ for $x \in M$, $\xi \in \otimes$, and $\eta \in \otimes$.

**LEMMA 1.2.** $\otimes_{\Phi} \equiv \otimes_{\Phi}$.

**PROOF.** Let $\otimes$ and $\otimes$ be uniformly separable weakly dense $C^*$-subalgebras of $M_\infty$ and $B(\otimes)$ with units respectively. Then the uniform closure $\otimes$ of the set consiting of all $\sum_{i=1}^n x_i \otimes \gamma_i$, $x_i \in \otimes$, $\gamma_i \in \otimes$, which is $\otimes \otimes \otimes$ in the sense of Turumaru [14], is also uniformly separable weakly dense $C^*$-subalgebra of $M_\otimes$. From $\otimes \otimes \otimes = \otimes \otimes \otimes$, we have $\otimes \otimes \otimes = \otimes \otimes \otimes$. Hence $\otimes \otimes \otimes = \otimes \otimes \otimes$ for almost every $\gamma_i \in \Gamma_1$. 


(i = 1, 2) and for every $x \in \mathcal{A}$ and $y \in \mathcal{B}$. Suppose that $\mathfrak{M}_{\pi, \mathcal{D}_1, \mathcal{D}_2, \mathcal{A}_1, \mathcal{A}_2}(\gamma_1, \gamma_2)$ holds, that is, there exists a unitary $u$ of $\mathfrak{D}_1(\gamma_1)$ onto $\mathfrak{D}_2(\gamma_2)$ such that $ux^1(\gamma_1)u^{-1} = x^2(\gamma_2)$ for all $x \in \mathcal{A}$ and $y \in \mathcal{B}$, which implies $\mathfrak{M}_{\pi, \mathcal{D}_1, \mathcal{D}_2, \mathcal{A}_1, \mathcal{A}_2}(\gamma_1, \gamma_2)$, where $\pi$ is the family of representations $\psi_\gamma$ of $\mathbb{C}$ defined by $\psi_\gamma(x \otimes y) = \varphi_\gamma(x) \otimes y$ for $x \in \mathcal{A}$ and $y \in \mathcal{B}$.

Conversely suppose that $\mathfrak{M}_{\pi, \mathcal{D}_1, \mathcal{D}_2, \mathcal{A}_1, \mathcal{A}_2}(\gamma_1, \gamma_2)$ holds. There exists a unitary $u$ of $\mathfrak{D}_1(\gamma_1)$ onto $\mathfrak{D}_2(\gamma_2)$ such that $u(x \otimes y)^1(\gamma_1)u^{-1} = (x \otimes y)^2(\gamma_2)$ for all $x \in \mathcal{A}$ and $y \in \mathcal{B}$. Taking $x = I$, $u(I \otimes y)u^{-1} = I \otimes y$ for every $y \in \mathcal{B}$. Hence there exists a unitary $u$ of $\mathfrak{D}_1(\gamma_1)$ onto $\mathfrak{D}_2(\gamma_2)$ such that $u = u \otimes I$ by [7: p. 114, Lemma]. Since $(u \otimes I)(x^1(\gamma_1) \otimes y)(u \otimes I)^{-1} = x^2(\gamma_2) \otimes y$, we have $ux^1(\gamma_1)u^{-1} = x^2(\gamma_2)$. Hence $\mathfrak{M}_{\pi, \mathcal{D}_1, \mathcal{D}_2, \mathcal{A}_1, \mathcal{A}_2}(\gamma_1, \gamma_2)$ holds.

**THEOREM 1.1.** Equivalence class of $\mathfrak{M}_{\pi, \mathcal{D}_1, \mathcal{D}_2, \mathcal{A}_1, \mathcal{A}_2}(\gamma_1, \gamma_2)$ under the relation $\equiv$ depends on neither $\mathcal{A}$ nor $\mathcal{B}$. That is, for normal faithfull representations $\pi$ and $\rho$ of $M$ onto $\mathfrak{D}_x$ and $\mathfrak{D}_y$, for uniformly separable weakly dense $C^*$-subalgebras $\mathcal{A}$ and $\mathcal{B}$ of $M_x$ and $M_y$ respectively and for families $\Phi^\pi$ and $\Psi^\rho$ of representations of $\mathcal{A}$ and $\mathcal{B}$ associated with the decomposition of $\mathfrak{D}_x$ and $\mathfrak{D}_y$ respectively ($i = 1, 2$), there exist subsets $E_1 \subset \Gamma_1$ and $E_2 \subset \Gamma_2$ with null complements such that $\mathfrak{M}_{\pi, \mathcal{D}_1, \mathcal{D}_2, \mathcal{A}_1, \mathcal{A}_2}(\gamma_1, \gamma_2)$ and $\mathfrak{M}_{\rho, \mathcal{D}_1, \mathcal{D}_2, \mathcal{A}_1, \mathcal{A}_2}(\gamma_1, \gamma_2)$ are equivalent each other for every $(\gamma_1, \gamma_2) \in E_1 \times E_2$.

**PROOF.** If $\pi$ and $\rho$ are unitarily equivalent, then Lemma 1.1 assures our mentions. By [8: p. 22, Lemma], we have $\pi \otimes L = \rho \otimes L$. Hence Lemma 1.2 assures our theorem.

According to Theorem 1.1, in the notation $\mathfrak{M}_{\pi, \mathcal{D}_1, \mathcal{D}_2, \mathcal{A}_1, \mathcal{A}_2}$ the letter $\pi$ does not have essential meaning. So we assume the von Neumann algebra $M$ to act on a fixed Hilbert space $\mathcal{H}$ from the beginning and we can denote $\mathfrak{M}_{\pi, \mathcal{D}_1, \mathcal{D}_2, \mathcal{A}_1, \mathcal{A}_2}$ by $\mathfrak{M}_{\mathcal{D}_1, \mathcal{D}_2, \mathcal{A}_1, \mathcal{A}_2}$. In the following, we denote $\mathfrak{M}_{\pi, \mathcal{D}_1, \mathcal{D}_2, \mathcal{A}_1, \mathcal{A}_2}$ by $\mathfrak{M}_{\pi, \mathcal{D}_1, \mathcal{D}_2}$, where $\pi$ means the identical representation of $M$. When we consider only one subalgebra $\mathcal{A} = L^\infty(\Gamma, \mu)$ of $M$, $\mathfrak{M}_{\pi, \mathcal{D}_1, \mathcal{D}_2}$ becomes an equivalence relation defined in the measure space $(\Gamma, \mu)$ which is simply denoted by $\mathfrak{M}_{\pi, \mathcal{D}_1, \mathcal{D}_2}$.

Now we shall give the interpretations of Theorem 1.1 to the decomposition theory of representations of involutive Banach algebras. In [13], in order to describe the structure of decompositions of some representations of certain $C^*$-algebras, at first we have studied the behavior of some special representation $\varphi_0$ of some special $C^*$-algebra $\mathcal{A}_0$ and next we have investigated the representation $\varphi$ of $C^*$-algebra $\mathcal{A}$ such that $\varphi_0(\mathcal{A}_0) = \varphi(\mathcal{A})$ by comparing the decompositions of $\varphi_0$ and $\varphi$ with respect to the diagonal algebras which are isomorphic under the isomorphism between $\varphi_0(\mathcal{A}_0)$ and $\varphi(\mathcal{A})$. According to Theorem 1.1, we can see the theoretical background of these arguments in [13]. Let $\mathcal{A}$ and $\mathcal{B}$ be two separable involutive Banach algebras and let $\varphi$ and $\psi$ be representations of $\mathcal{A}$ and $\mathcal{B}$ onto separable Hilbert spaces $\mathfrak{D}_\varphi$ and $\mathfrak{D}_\psi$ respectively. Suppose
COMPOSITIONS OF REPRESENTATIONS

That is, there exist a von Neumann algebra $M$ and two normal faithful representations $\pi$ and $\rho$ such that $\pi(M) = \phi(\mathfrak{A})$, $\rho(M) = \psi(\mathfrak{B})$ and $\sigma = \theta_0 \circ \pi$. Let $(\Gamma_1, \mu_1)$, $(\Gamma_2, \mu_2)$, $A_1$ and $A_2$ be as in Theorem 1.1.

Then $\phi$ (resp. $\psi$) is decomposed with respect to $\pi(A_1)$ and $\pi(A_2)$ (resp. $\sigma(A_1)$ and $\sigma(A_2)$) as follows:

$$
\theta = \int_{\Gamma_1} \phi^1(\gamma_1) d\mu_1(\gamma_1) \quad \text{and} \quad \phi = \int_{\Gamma_1} \phi^2(\gamma_1) d\mu_2(\gamma_1)
$$

(resp. $\psi = \int_{\Gamma_1} \psi^1(\gamma_1) d\mu_1(\gamma_1)$ and $\psi = \int_{\Gamma_1} \psi^2(\gamma_1) d\mu_2(\gamma_1)$).

Then we get the following

**Corollary 1.** There exist null sets $N_1 \subset \Gamma_1$ and $N_2 \subset \Gamma_2$ such that $\phi(\gamma_1)$ is equivalent to $\psi(\gamma_1)$ for every $(\gamma_1, \gamma_2) \in \mathbb{C} N_1 \times \mathbb{C} N_2$.

**Proof.** Putting $\mathfrak{A} = \phi(\mathfrak{A})$ and $\mathfrak{B} = \psi(\mathfrak{B})$, the decomposition $\phi = \int_{\Gamma_1} \phi^1(\gamma_1) d\mu_1(\gamma_1)$ and $\psi = \int_{\Gamma_1} \psi^1(\gamma_1) d\mu_1(\gamma_1)$ (i = 1, 2) give the associated families $\Phi^1$ and $\Phi^2$ (i = 1, 2) of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Then the relations $\phi(\gamma_1) \cong \psi(\gamma_2)$ and $\psi(\gamma_1) \cong \psi(\gamma_2)$ are equivalent to $\mathbb{M}_{\mathfrak{A}, \mathfrak{B}}^{\phi, \psi} (\gamma_1, \gamma_2)$ and $\mathbb{M}_{\mathfrak{A}, \mathfrak{B}}^{\phi, \psi} (\gamma_1, \gamma_2)$ respectively. Hence Theorem 1.1 implies our mention.

Corollary 1 states that the unitary equivalence among the components of representations is completely determined by the algebraic relation between the commutant algebra and the associated diagonal algebra.

**Corollary 2.** Let $A_i = L^\infty (\Gamma_i, \mu_i)$ be imbedded in a von Neumann algebra $M_i$ acting on a Hilbert space $\mathfrak{H}_i$ (i = 1, 2). Let $\mathfrak{A}$ and $\mathfrak{B}$ be two separable involutive Banach algebras and let $\phi_1$, $\phi_2$, $\psi_1$, and $\psi_2$ be two representations of $\mathfrak{A}$ (resp. $\mathfrak{B}$) onto $\mathfrak{H}$ and $\mathfrak{B}$ respectively such that $\phi_1(\mathfrak{A}) = M_1$ and $\phi_2(\mathfrak{A}) = M_2$ (resp. $\psi_1(\mathfrak{B}) = M_3$ and $\psi_2(\mathfrak{B}) = M_4$). Then $\phi_1$ and $\psi_1$ are decomposed with respect to $\phi_i$ as follows (i = 1, 2);

$$
\phi_i = \int_{\Gamma_i} \phi_i(\gamma_1) d\mu_i(\gamma_1) \quad \text{and} \quad \psi_i = \int_{\Gamma_i} \psi_i(\gamma_1) d\mu_i(\gamma_1) \quad (i = 1, 2).
$$

If $(\phi_1 \oplus \phi_2)(\mathfrak{A}) = (\psi_1 \oplus \psi_2)(\mathfrak{B})$, then the relation $\phi_1(\gamma_1) \equiv \phi_2(\gamma_2)$ of $\gamma_1$ and $\gamma_2$ is equivalent to $\psi_1(\gamma_1) \equiv \psi_2(\gamma_2)$ except for some negligible part.

**Proof.** Putting $\phi = \phi_1 \oplus \phi_2$, $\psi = \psi_1 \oplus \psi_2$, $\phi(\mathfrak{A}) = \phi(\mathfrak{B}) = M_i = (\Gamma_i, \mu_i)$ and $A = A_i \oplus A_2$, we have $M_i \supset M_i \oplus M_2 \supset A_i \oplus A_2 = A = L^\infty (\Gamma, \mu)$. Application of Corollary 1 to $\phi$, $\psi$, $M$ and $A$ assures our mention.
REMARK. If \( \varphi_1 \) and \( \varphi_2 \) (resp. \( \psi_1 \) and \( \psi_2 \)) are disjoint representations, then our assumption \( (\varphi_1 \oplus \varphi_2) (\mathfrak{A}) = (\varphi_1 \oplus \varphi_2) (\mathfrak{B}) \) is automatically satisfied. Indeed, if \( \varphi_1 \) and \( \varphi_2 \) are representations of quite different types, it may happen that there exists a Borel isomorphism \( \Theta \) of \( \Gamma_1 \) onto \( \Gamma_2 \) such that \( \varphi_2(\gamma_1) \sim \varphi_2(\Theta(\gamma_1)) \) for all \( \gamma_1 \in \Gamma_1 \) (cf. [13]), though, of course, \( \mu_2 \) and \( \Theta(\mu_1) \) are disjoint.

Suppose that there is an isomorphism \( \theta \) of \( A_1 = L^\infty(\Gamma_1, \mu_1) \) onto \( A_2 = L^\infty(\Gamma_2, \mu_2) \). By [5:§1, Prop. 1], there exist null sets \( N_1 \subseteq \Gamma_1 \) and \( N_2 \subseteq \Gamma_2 \) respectively and a one-to-one measurable mapping \( \Theta \) of \( \mathcal{C} N_2 \) onto \( \mathcal{C} N_1 \) such that \( \Theta(a) = a(\Theta(\gamma_2)) \) for every \( a \in A_1 \) and for every \( \gamma_2 \in \mathcal{C} N_2 \) and \( \Theta(\mu_2) \) is equivalent to \( \mu_1 \).

**Theorem 1.2.** Suppose that there exists a unitary \( u \) of \( M \) such that \( uA_1u^{-1} = A_2 \). Let \( \Theta \) be the measurable mapping of \( \mathcal{C} N_2 \) onto \( \mathcal{C} N_1 \) associated with the isomorphism \( \theta \) of \( A_1 \) onto \( A_2 \) induced by \( u \), where \( N_1 \) and \( N_2 \) are the null subsets of \( \Gamma_1 \) and \( \Gamma_2 \) defined above respectively. Then \( \mathbb{R}^\infty_{\mu_1, \mu_2} \Theta^{-1}(\gamma_1) \) holds for almost every \( \gamma_1 \in \mathcal{C} N_1 \).

**Proof.** Let \( M \) act on a Hilbert space \( \mathfrak{H} \). Let \( \varphi_1 = \int_{\Gamma_1} \delta^1(\gamma_1) d\mu_1(\gamma_1) \) and \( \varphi_2 = \int_{\Gamma_2} \delta^2(\gamma_2) d\mu_2(\gamma_2) \) be the decompositions of \( \varphi \) with respect to \( A_1 \) and \( A_2 \) respectively. Applying [2: Chap. II, §7 Theorem 4] to \( u \) and \( \Theta \), there exist a null set \( N_1 \subseteq \Gamma_1 \) and a unitary \( u(\gamma_1) \) of \( \mathfrak{H} \) onto \( \mathfrak{H} \) such that \( u(a) = a(\Theta(\gamma_2)) \) for every \( a \in A_1 \) and for every \( \gamma_2 \in \mathcal{C} N_2 \). Continuity of \( u^* \)
and $uxu^{-1}$ implies $x' = uxu^{-1}$. On the other hand, we have $uxu^{-1} = x$ for every $x \in \mathfrak{A}$. Hence we have $x = x'$ for every $x \in \mathfrak{A}$. Then for $x \in \mathfrak{A}$ there exists a null set $N_x \subset \Gamma_x$ such that $x'(\gamma) = \phi^*_n(x)$ for every $\gamma \in \mathcal{C}N_x$. Putting $N = \bigcup_{x \in \mathfrak{A}} N_x$, $N_2$ is a null set and we have

$$u(\gamma_1)\phi^*_n(x)u(\gamma_1)^{-1} = x'(\Theta^{-1}(\gamma_1)) = \phi^*_n(\theta^{-1}(\gamma_1))$$

for every $x \in \mathfrak{A}$ and for every $\gamma_1 \in \mathcal{C}(\Theta(N_2) \cap N)$. By the continuity of $u(\gamma_1)$ and $\phi^*_n(\gamma_1)^{-1}$ we have $\phi^*_n \equiv \phi^*_n(\Theta^{-1}(\gamma_1))$ for almost every $\gamma_1 \in \Gamma_1$, that is, $\mathfrak{R}_{n_1,n_2}^{\mathfrak{A},\mathfrak{A},\Gamma_1}$ holds for almost every $\gamma_1 \in \Gamma_1$.

**Remark.** When an abelian subalgebra $A$ of $M$ is represented in two ways as $A \cong L^\infty(\Gamma_1, \mu_1)$ and $A \cong L^\infty(\Gamma_2, \mu_2)$, taking $A = A_1 = A_2 = A$ and $u = I$ in Theorem 1.2, there exists null sets $N_1 \subset \Gamma_1$, $N_2 \subset \Gamma_2$ and a one-to-one measurable mapping $\Theta$ from $\mathcal{C}N_1$ onto $\mathcal{C}N_2$ such that $\Theta(\mu_1) \approx \mu_1$ and $\mathfrak{R}_{n_1,n_2}^{\mathfrak{A},\mathfrak{A},\Gamma_1}$ holds for every $\gamma_1 \in \mathcal{C}N_1$. Hence the behaviors of the equivalence relations $\mathfrak{R}_{n_1,n_2}^{\mathfrak{A},\mathfrak{A},\Gamma_1}$ and $\mathfrak{R}_{n_1,n_2}^{\mathfrak{A},\mathfrak{A},\Gamma_1}$ in the measure spaces $(\Gamma_1, \mu_1)$ and $(\Gamma_2, \mu_2)$ are almost isomorphic. That is, we can say that the equivalence relation $\mathfrak{R}_{n_1,n_2}^{\mathfrak{A},\mathfrak{A},\Gamma_1}$ depends only on the algebraic relation of $M$ and $A$.

In order to study the behavior of $\mathfrak{R}_{n_1,n_2}^{\mathfrak{A},\mathfrak{A},\Gamma_1}$, we set the following.

**Theorem 1.3.** Let $A_1 = L^\infty(\Gamma_1, \mu_1)$ and $A_2 = L^\infty(\Gamma_2, \mu_2)$ be two abelian von Neumann subalgebras of a von Neumann algebra $M$ acting on a Hilbert space $\mathfrak{H}$. Let $\mathfrak{H} = \bigoplus_{i=1}^{\infty} \mathfrak{H}^i(\gamma_i)$ be the decompositions of $\mathfrak{H}$ with respect to $A_1$ and $A_2$ respectively. Let $\mathfrak{H}$ be a uniformly separable weakly dense C*-subalgebra of $M$ and let $\Phi^i = \{\phi^i_n : \gamma_i \in \Gamma_i\}$ and $\Phi^2 = \{\phi^2_n : \gamma_2 \in \Gamma_2\}$ be families of representations of $\mathfrak{H}$ associated with the decompositions of $\mathfrak{H}$. Then the graph of $\mathfrak{R}_{n_1,n_2}^{\mathfrak{A},\mathfrak{A},\Gamma_1}$ in $\Gamma_1 \times \Gamma_2$ is an analytic subset of $\Gamma_1 \times \Gamma_2$. Besides, if $A_1$ and $A_2$ are maximal abelian in $M$ then there exist null sets $N_1 \subset \Gamma_1$ and $N_2 \subset \Gamma_2$ such that the graph of $\mathfrak{R}_{n_1,n_2}^{\mathfrak{A},\mathfrak{A},\Gamma_1}$ in $(\Gamma_1 - N_1) \times (\Gamma_2 - N_2)$ is a Borel subset of $(\Gamma_1 - N_1) \times (\Gamma_2 - N_2)$.

**Proof.** Let $R$ be the graph of $\mathfrak{R}_{n_1,n_2}^{\mathfrak{A},\mathfrak{A},\Gamma_1}$. Putting $\Gamma_1^n = \{\gamma_i \in \Gamma_i ; \dim. \mathfrak{H}^i(\gamma_i) = n\}$, $i = 1, 2$, $\Gamma_i^n$ becomes a Borel subset of $\Gamma_i$ for each $n$, $i = 1, 2$, and we have $\bigcup_{n=1}^{\infty} \Gamma_1^n = \Gamma_1$, $i = 1, 2$, and $R \subset \bigcup_{n=1}^{\infty} (\Gamma_1^n \times \Gamma_2^n) \cup (\Gamma_1^n \times \Gamma_2^n)$. So we may assume that there exists a fixed Hilbert space $\mathfrak{H}_0$ such that $\mathfrak{H}^i(\gamma_i) = \mathfrak{H}_0$ for each $\gamma_i \in \Gamma_i$, $i = 1, 2$. Let $B = B(\mathfrak{H}_0)$ be the algebra of all bounded operators on $\mathfrak{H}_0$ equipped with the Borel structure induced by the weak topology. Then $B$ is a standard Borel space, since $B$ is covered by countably many metrizable compact subsets. For each $x \in \mathfrak{A}$ the function $(\gamma_1, \gamma_2, u) \in \Gamma_1 \times \Gamma_2 \times U \rightarrow (x^1(\gamma_1), x^2(\gamma_2), u) \in B \times B \times U$ becomes a Borel function, where $U$ means the unitary
group of $B$. Besides the function $(x, y, u) \in B \times B \times U \to ux - yu \in B$ is a Borel function. Indeed, let $\{\xi_n\}$ be a complete normalized orthogonal system of $\mathcal{S}_0$, then we have 
\[
(ux - yu)\xi_n, \xi_m) = (ux \xi_n, \xi_m) - (yu \xi_n, \xi_m)
\]
Since each member of summands is a Borel function of $(x, y, u) \in B \times B \times U$, $(ux - yu)\xi_n, \xi_m)$ is a Borel function of $(x, y, u)$. For each $\xi, \eta \in \mathcal{S}_0$
\[
(ux - yu)\xi_n, \eta) = \sum_{n,m} (\xi_n, \xi_m)(ux - yu)\xi_n, \xi_m)
\]
is a Borel function of $(x, y, u)$. Hence the function $(x, y, u)\to ux - yu$ is a Borel function. After all, the set
\[
A = \{(x, y) \in \Gamma_1 \times \Gamma_2 \times U : ux^2 = y^2 u \forall x \in \mathfrak{H}\}
\]
is a Borel subset of a standard Borel space $\Gamma_1 \times \Gamma_2 \times U$. $R$ is the projection of $A$ to $\Gamma_1 \times \Gamma_2$, so that $R$ is analytic.

If $A_1$ and $A_2$ are maximal abelian in $M$, then there exist null sets $N_1 \subset \Gamma_1$ and $N_2 \subset \Gamma_2$ such that $\varphi_{i1}$ and $\varphi_{i2}$ are irreducible representations for every $\gamma_i \in \Gamma_1 - N_1$ and $\gamma_i \in \Gamma_2 - N_2$. Hence $(\gamma_1, \gamma_2) \in R \cap (\Gamma_1 - N_1) \times (\Gamma_2 - N_2)$ is equivalent to $\mathcal{J}(\varphi_{i1}, \varphi_{i2}) > 0$, where $\mathcal{J}(\varphi_{i1}, \varphi_{i2})$ means the linear dimension of the space of all bounded operators $u$ such that $\varphi_{i1}(x) = \varphi_{i2}(x)u$ for all $x \in \mathfrak{H}$. But $\mathcal{J}(\varphi_{i1}, \varphi_{i2})$ is a Borel function of $(\gamma_1, \gamma_2)$ by [9; Theorem 8.2]. Thus, $R \cap (\Gamma_1 - N_1) \times (\Gamma_2 - N_2)$ is a Borel subset of $(\Gamma_1 - N_1) \times (\Gamma_2 - N_2)$.

2. Classification of abelian von Neumann subalgebras. Let $A = L^\infty(\Gamma, \mu)$ be an abelian von Neumann subalgebra of $M$. Then $\mathcal{R}_{\text{vN}} = \mathcal{R}$ is an equivalence relation associated with $M$ and $A$ defined in the measure space $(\Gamma, \mu)$. Let $\mathcal{G}$ be the Borel space of all $\mathcal{R}$-equivalence classes in $\Gamma$ equipped with the quotient Borel structure of the Borel structure of $\Gamma$ under $\mathcal{R}$. If $\mathcal{G}$ is countably separated Borel space, then for each Borel set $S \subset \Gamma$ the space $\mathcal{S}$ of all $\mathcal{R}$-equivalence classes in $S$ equipped with the quotient Borel structure of the Borel structure of $S$ is so. Hence we can set the following definition by Theorem 1.1 and Theorem 1.2.

**Definition 2.1.** If there exists a Borel null set $N \subset \Gamma$ for any $\mathcal{R}$ associated with $M$ and $A$ such that $(\Gamma - N)/\mathcal{R}$ is countably separated, then we call $A$ smooth in $M$. If $Ae$ is not smooth in $eMe$ for each nonzero projection $e$ of $A$, we call $A$ completely rough in $M$. If for any $\mathcal{R}$ there exists a Borel null set $N \subset \Gamma$ such that $\mathcal{R}(\gamma, \gamma')$ implies $\gamma = \gamma'$ for each $(\gamma, \gamma') \in (\Gamma - N) \times (\Gamma - N)$, then we call $A$ simple in $M$. Of course, simple subalgebra is also smooth.

**Lemma 2.1.** An abelian subalgebra $A = L^\infty(\Gamma, \mu)$ of $M$ is smooth if and only if for any $\mathcal{R}_{\text{vN}}$ there exists a Borel subset $N \subset \Gamma$ and an analytic
subset $E$ of $\Gamma$ such that $\mu(N) = 0$ and such that $E$ contains one and only one element in common with each $\mathcal{R}^{\kappa, \tau}$-equivalence class in $\Gamma - N$. Besides if $A$ is smooth, then we can choose $E$ to be a Borel subset of $\Gamma$.

**Proof.** Denote $\mathcal{R}^{\kappa, \tau} = \mathcal{R}$. Suppose that $A$ is smooth. Eliminating a Borel null set from $\Gamma, \Gamma = \Gamma/\mathcal{R}$ is countably separated, so that $\Gamma$ is an analytic Borel space by [9: Cor. of Theorem 5.1]. Hence there exists a Borel $\tilde{\mu}$-null set $N \subset \Gamma$ such that $\Gamma - N$ is standard by [Theorem 6.1], where $\tilde{\mu}$ is the quotient measure of $\mu$ in $\Gamma$. Let $r$ be the natural mapping of $\Gamma$ onto $\Gamma$. Then $r$ is a Borel mapping from the standard Borel space $\Gamma - r^{-1}(\hat{N})$ onto the standard Borel space $(\Gamma - r^{-1}(\hat{N})) \times (\Gamma - \hat{N})$ is its Borel subset. From [9: Theorem 6.3] we conclude the existence of a Borel null set $\hat{N}_{1} \subset \Gamma$ and a Borel mapping $\phi$ from $\Gamma - \hat{N}$ to $\Gamma$ such that $\phi(r(\gamma)) = \hat{\gamma}$ for every $\gamma \in \Gamma - \hat{N}$. Since $\phi$ is one-to-one, its image $E$ is a required subset of $\Gamma$ by [9: Theorem 3.2].

Conversely, suppose that there exist an analytic set $E \subset \Gamma$ and a Borel null set $N \subset \Gamma$ as in the statement of our Lemma. Then $r$ is a one-to-one Borel mapping of $E$ onto $(\Gamma - N)/\mathcal{R} = (\Gamma - N)'$. Hence if $r$ is a Borel isomorphism then $(\Gamma - N)'$ is analytic Borel space, so that $(\Gamma - N)/\mathcal{R}$ is countably separated. So it suffices to show that $r$ is a Borel isomorphism, that is, to show that $r(F)$ is a Borel subset of $(\Gamma - N)'$ for every relative Borel subset $F$ of $E$. Hence we shall show that $r^{-1}(F)$ is a Borel subset of $\Gamma - N$. Let $R$ be the graph of $\mathcal{R}$ in $(\Gamma - N)' \times (\Gamma - N)$. Then we have $r^{-1}(F) = \text{pr}_2(F \times (\Gamma - N) \cap R)$, where $\text{pr}_2$ is defined by $\text{pr}_2(\gamma, \gamma') = \gamma'$ for $(\gamma, \gamma') \in \Gamma \times \Gamma$. Since $F$ is a relative Borel subset of the analytic set $E, F$ is analytic. Hence $r^{-1}(F)$ is analytic by Theorem 1.3. Similarly $r^{-1}(E - F)$ is also analytic. Since $r^{-1}(F)$ and $r^{-1}(E - F)$ are complementary subsets of $\Gamma - N$, they are both Borel sets. This completes the proof.

**Lemma 2.2.** Let $A$ be an abelian subalgebra of a von Neumann algebra $M$. 1°. If there exists a partition of unit $\sum_{n=1}^{\infty} p_n = I$ in $A$ such that $Ap_n$ is smooth in $p_nM$, for each $n$ then $A$ is smooth. 2°. If there exist two von Neumann algebras $M_1$ and $M_2$ and their smooth abelian subalgebras $A_1$ and $A_2$ such that $M = M_1 \otimes M_2$ and $A = A_1 \otimes A_2$, then $A$ is smooth under the additional assumption $M = M' = M_2'$.

**Proof.** 1°. Let $A = L^\infty(\Gamma, \mu)$. Let $P_n$ be the Borel set in $\Gamma$ associated with $p_n$. Then we have $Ap_n = L^\infty(P_n, \mu)$. By eliminating a Borel null set we

3) When the one of $M_1$ and $M_2$ has the part of type III and the other is not of type I, the question whether $M = M_1 \otimes M_2$ does or does not hold remains open up to now (cf. [2: p. 30 and p. 102]).
may assume \( \bigcup_{n=1}^{\infty} P_n = \Gamma \). Let \( \mathcal{A} \) and \( \Phi \) be the couple as in the preceding arguments. Putting \( M_n = p_n M p_n, \ A_n = A p_n, \ \mathcal{A}_n = \mathcal{A} p_n \) and \( \Phi_n = \{ \varphi \in \Phi ; \gamma \in P_n \} \), the equivalence relation \( \mathcal{R}_{M_n, \mathcal{A}_n}^{\mathcal{A}_n} = \mathcal{R}_n \) in \( P_n \) becomes the restriction of the original equivalence relation \( \mathcal{R}_{M, \mathcal{A}}^{\mathcal{A}} = \mathcal{R} \) to \( P_n \). It follows from Lemma 2.1 that there exist a Borel set \( N_n \subset P_n \) and a Borel set \( E_n \subset P_n \) for each \( n \) such that \( \mu(N_n) = 0 \) and such that \( E_n \) contains one and only one element in common with each \( \mathcal{R}_n \)-equivalence class in \( P_n - N_n \). Let \( Q_n \) be the \( \mathcal{R} \)-saturation of \( E_n \).

Then we have \( Q_n \supset P_n - N_n \). Putting \( E = \bigcup_{n=1}^{\infty} (E_n - \bigcup_{k=1}^{n-1} Q_k) \), \( E \) is an analytic subset of \( \Gamma \) whose saturation becomes \( \bigcup_{n=1}^{\infty} Q_n \) and it has one and only one element in common with each \( \mathcal{R} \)-equivalence class in \( \bigcup_{n=1}^{\infty} Q_n \). Putting \( N = \Gamma - \bigcup_{n=1}^{\infty} Q_n \), we have \( N \subset \bigcup_{n=1}^{\infty} N_n \), so that \( N \) is a null subset of \( \Gamma \). Therefore \( A \) becomes smooth by Lemma 2.1.

Then we have \( A = L^\infty(\Gamma, \mu). \) Let \( N_1 \subset \Gamma_1 \) and \( E_1 \subset \Gamma_1 \) be the couple satisfying the condition of Lemma 2.1, \( i = 1, 2 \). Let \( \mathcal{A}_1, \Phi_1 \) and \( \mathcal{A}_2, \Phi_2 \) be the couples as in the preceeding discussion for \( M_1, A_1 \) and \( M_2, A_2 \) respectively. Then \( \mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \) becomes a uniformly separable weakly dense \( C^* \)-subalgebra of \( \mathcal{M} \) by our assumption. Putting \( \Phi = \Phi_1 \otimes \Phi_2 = \{ \phi_{(\gamma_1, \gamma_2)} = \phi_{\gamma_1} \otimes \phi_{\gamma_2} : \phi_{\gamma_1} \in \Phi_1, \phi_{\gamma_2} \in \Phi_2, (\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2 \} \), \( \Phi \) is a family of representations of \( \mathcal{A} \) associated with the decomposition of \( \hat{\mathcal{A}} = \hat{\mathcal{A}}_1 \otimes \hat{\mathcal{A}}_2 \) with respect to \( A = A_1 \otimes A_2 \), where \( \hat{\mathcal{A}}_1 \) and \( \hat{\mathcal{A}}_2 \) are the underlying Hilbert spaces of \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) respectively. It is clear that the equivalence relation \( \mathcal{R}_{M_1, \mathcal{A}_1}^{\mathcal{A}_1, \mathcal{A}_2} \) in \( \Gamma \) is defined as the canonical product equivalence relation \( \mathcal{R}_{\mathcal{A}_1, \mathcal{A}_2}^{\mathcal{A}_1, \mathcal{A}_2} \) in \( \Gamma_1 \times \Gamma_2 \). Putting \( N = N_1 \times \Gamma_2 \cup \Gamma_1 \times N_2 \) and \( E = E_1 \times E_2 \), \( N \) and \( E \) satisfy the condition of Lemma 2.1. Hence \( A \) is smooth in \( \mathcal{M} \). This completes the proof.

**Theorem 2.1.** Let \( A \) be an abelian von Neumann subalgebra of a von Neumann algebra \( \mathcal{M} \). Then there exists a unique partition of unit \( e + f = I \) in \( A \) such that \( AE \) is smooth in \( eMf \) and such that \( Af \) is completely rough in \( fMf \).

**Proof.** Let \( \{ p_n \} \) be a maximal family of orthogonal non-zero projections in \( A \) such that \( A p_n \) is smooth in \( p_n M p_n \). By the separability of underlying

---

4) For any equivalence relation \( \mathcal{R} \) in \( \Gamma \) the \( \mathcal{R} \)-saturation of any subset \( S \subset \Gamma \) is the set of all elements of \( \Gamma \) that are \( \mathcal{R} \)-equivalent to some element of \( S \). If \( S \) contains every element that is \( \mathcal{R} \)-equivalent to some one of \( S \), \( S \) is called \( \mathcal{R} \)-saturated.
Hilbert space of $M$, $\{p_\alpha\}$ is at most countable. $e = \sum p_\alpha$ and $f = I - e$ are the desired projections in $A$ by Lemma 2.2 and by the maximality of $\{p_\alpha\}$. The unicity of $e$ and $f$ is clear from Definition 2.1. This completes the proof.

Theorem 2.1 reduces the study of abelian von Neumann subalgebras to that of smooth ones and that of completely rough ones.

3. Smooth maximal abelian subalgebras. In the present section we reduce the study of smooth maximal abelian subalgebras to that of simple ones.

In the following if a maximal abelian subalgebra $A = L^\infty(\Gamma, \mu)$ of a von Neumann subalgebra is smooth, then we assume that the quotient space $\Gamma/\mathbb{R} = \bar{\Gamma}$ of $\Gamma$ is standard by eliminating a null set from the whole space $\Gamma$.

**Lemma 3.1.** Let $A_1 = L^\infty(\Gamma_1, \mu_1)$ and $A_2 = L^\infty(\Gamma_2, \mu_2)$ be two abelian von Neumann subalgebras of the von Neumann algebra $M$. Let $\mathfrak{A}$, $\Phi^1$ and $\Phi^2$ be a triad as in §1. Let $E_1 \subset \Gamma_1$ and $E_2 \subset \Gamma_2$ be Borel subsets respectively. If there exists a Borel mapping $\Theta$ from $E_1$ to $E_2$ such that $\mathfrak{H}^{\mathfrak{A}, \Phi^1, \Phi^2}(\gamma_1, \Theta(\gamma_1))$ holds for almost every $\gamma_1 \in E_1$, then for almost every $\gamma_1 \in E_1$ there exists a unitary $u(\gamma_1)$ from $\Phi^1(\Theta(\gamma_1))$ onto $\Phi^2(\gamma_1)$ such that $u(\gamma_1)^{-1}x^1(\gamma_1)u(\gamma_1) = x^2(\Theta(\gamma_1))$ for every $x \in \mathfrak{A}$ and such that $u(\gamma_1)$ is a measurable vector field over $E_1$ if $\xi(\cdot)$ is so over $E_2$. If $\Theta$ is a Borel isomorphism such that $\Theta(\mu_1) \approx \mu_2$, then the operator $u$ defined by

$$u \xi = \int_{E_1} \xi(e_1) u(\gamma_1) \xi(\Theta(\gamma_1)) \sqrt{d(\Theta^{-1}(\mu_2))}(\gamma_1) d\mu_1(\gamma_1)$$

for $\xi = \int_{E_2} \xi(\gamma_2) d\mu_2(\gamma_2) \in \mathfrak{H}$ is a partial isometry of $M$ which carries $e_2$ onto $e_1$ where $e_1$ and $e_2$ mean the projections of $A_1$ and $A_2$ associated with $E_1$ and $E_2$ respectively.

**Proof.** We use the notation in the proof of Theorem 1.3. As in the proof of Theorem 1.3 we may assume that there exists a fixed Hilbert space $\mathfrak{H}_0$ such that $\mathfrak{H}_i(\gamma_i) = \mathfrak{H}_0$ for each $\gamma_i \in \Gamma_i$, $i = 1, 2$. Putting $B = \{(\gamma_i, u) \in E_i \times U; u^{-1}x^1(\gamma_1)u = x^2(\Theta(\gamma_1))$ for every $x \in \mathfrak{A}\}$, $B$ is a Borel subset of $E_i \times U$ whose projection to $E_i$ covers $E_i$. Indeed, $B$ is the projection of the intersection of $A$ and the product of the graph $R_\Theta$ of $\Theta$ in $E_i \times E_2$ and $U$ to $E_i \times U$. $R_\Theta$ is a Borel set in $E_i \times E_2$ by [1: § 6 Ex. 17] and the projection of $E_i \times E_2$ to $E_i$ is a one-to-one Borel mapping on $R_\Theta$. Hence the projection of $A \cap (R_\Theta \times U)$ onto $B$ is a one-to-one Borel mapping of the standard Borel space $A \cap (R_\Theta \times U)$ into the standard Borel space $E_i \times U$, so that $B$ becomes a Borel subset of $E_i \times U$ by [9: Theorem 3.2]. Applying [9: Theorem 6.3] to $E_i$, $U$ and $B$, there exist a null set $N_i \subset E_i$ and a Borel mapping $u(\gamma_i)$ from $E_i - N_i$ to $U$ such that $(\gamma_i, u(\gamma_i)) \in B$ for evry $\gamma_i \in E_i - N_i$. This $u(\cdot)$ is desired one. Suppose that $\Theta$ is a Borel isomorphism of $(E_i, \mu_i)$ onto $(E_2, \mu_2)$.
For $\xi = \int_{\mathbb{G}} \xi_2 d\mu_2(\gamma_2) \in \mathcal{H}$

we have

$$\|ux\xi\|^2 = \int_{\mathbb{G}} \|u(\gamma_1) \xi_2(\Theta(\gamma_1))\|^2 \frac{d\Theta^{-1}(\mu_2)}{d\mu_1}(\gamma_1) \; d\mu_1(\gamma_1)$$

$$= \int_{\mathbb{G}} \|\xi_2(\Theta(\gamma_1))\|^2 d(\Theta^{-1}(\mu_2))(\gamma_1)$$

$$= \int_{\mathbb{G}} \|\xi_2(\gamma_1)\|^2 d\mu_2(\gamma_2) = \|e_2\xi\|^2.$$

For each $x \in \mathbb{A}$ we have

$$ux \xi = \int_{\mathbb{G}} u(\gamma_1) x(\Theta(\gamma_1)) \xi_2(\Theta(\gamma_1)) \sqrt{\frac{d\Theta^{-1}(\mu_2)}{d\mu_1}}(\gamma_1) \; d\mu_1(\gamma_1)$$

$$= \int_{\mathbb{G}} u(\gamma_1) x(\Theta(\gamma_1)) \xi_2(\Theta(\gamma_1)) \sqrt{\frac{d\Theta^{-1}(\mu_2)}{d\mu_1}}(\gamma_1) \; d\mu_1(\gamma_1)$$

$$= \int_{\mathbb{G}} x(\gamma_1) u(\gamma_1) \xi_2(\Theta(\gamma_1)) \sqrt{\frac{d\Theta^{-1}(\mu_2)}{d\mu_1}}(\gamma_1) \; d\mu_1(\gamma_1)$$

$$= x \int_{\mathbb{G}} u(\gamma_1) \xi_2(\Theta(\gamma_1)) \sqrt{\frac{d\Theta^{-1}(\mu_2)}{d\mu_1}}(\gamma_1) \; d\mu_1(\gamma_1)$$

$$= xu \xi \quad \text{for every} \quad \xi = \int_{\mathbb{G}} \xi_2(\gamma_2) d\mu_2(\gamma_2) \in \mathcal{H},$$

so that $u$ is a partial isometry of $M$ mentioned above.

**Theorem 3.1.** For a maximal abelian subalgebra $A$ of a von Neumann algebra $M$, acting on a Hilbert space $\mathcal{H}$, to be smooth, it is necessary and sufficient that there exists a simple maximal abelian subalgebra $A$ of some von Neumann algebra $\hat{M}$ on a Hilbert space $\hat{\mathcal{H}}$ and a normal isomorphism $\theta$ of $\hat{A}$ into $A'$ such that $\theta(A) \subset A$ and $\theta(\hat{M}) = M$. If $A$ is smooth, then $(\hat{M}, A, \theta)$ is unique.

**Proof. Necessity:** Let $\mathcal{R} = \mathcal{R}^{\mu, \nu} \ast$ and $(\hat{\mathcal{R}}, \hat{\mathcal{A}}) = (\hat{\Gamma}, \hat{\mathcal{A}}) / \mathcal{R}$. Let $r$ be the canonical mapping of $\Gamma$ onto $\hat{\Gamma}$. Then there exists a Borel mapping $\phi$ of $\hat{\Gamma}$ into $\Gamma$ such that $\phi(\gamma) \in r^{-1}(\hat{\gamma})$ for every $\hat{\gamma} \in \hat{\Gamma}$ by Lemma 2.1, eliminating a null set from $\Gamma$. Putting $\hat{\mathcal{A}}(\hat{\gamma}) = \hat{\mathcal{A}}(\phi(\hat{\gamma}))$ for each $\gamma \in \hat{\Gamma}$, where $\hat{\mathcal{A}}(\gamma)$ means the component of the decomposition $\mathcal{A} = \int_{\mathbb{G}} \hat{\mathcal{A}}(\gamma) \; d\mu(\gamma)$ of $\mathcal{A}$ with respect to $A$, we get a measurable Hilbert space field $\{\mathcal{R}(\hat{\gamma})\}$ over $\hat{\Gamma}$, so that we can define a Hilbert space $\mathcal{R}$ by $\mathcal{R} = \int_{\hat{\mathcal{A}}} \mathcal{R}(\hat{\gamma}) \; d\hat{\mu}(\hat{\gamma})$. The diagonal algebra of this decomposition of $\mathcal{R}$ becomes $\hat{A} = L^\infty(\hat{\Gamma}, \hat{\mu})$. Since the mapping $\Theta = \phi r$ of $\Gamma$ onto $\phi(\hat{\Gamma})$
is a Borel mapping such that $\mathcal{M}(\gamma, \Theta(\gamma))$ holds for every $\gamma \in \Gamma$, there exists a family $\{u(\gamma)\}$ of unitaries from $\{\phi(\gamma)\}$ onto $\{\phi(\Theta(\gamma))\}$ as in the conclusion of Lemma 3.1. Since each operator $x \in \hat{A}'$ is decomposable with respect to the decomposition of $\mathcal{F}$, there exists a measurable operator field $\{x(\gamma)\}$ over $\hat{\Gamma}$ such that $x = \int_{\hat{\Gamma}} x(\gamma) d\hat{\mu}(\gamma)$. Putting $\theta(x) = \int_{\Gamma} u(\gamma)^{-1} x(r(\gamma)) u(\gamma) \, d\mu(\gamma)$ for $x \in \hat{A}'$, $\theta$ becomes a normal isomorphism of $\hat{A}'$ into $A'$. In fact, if there exists another measurable operator field $x_{1}(\gamma)$ over $\Gamma$ for $x \in \hat{A}'$ such that $\int_{\Gamma} x_{1}(\gamma) d\hat{\mu}(\gamma) = \int_{\Gamma} x(\gamma) d\hat{\mu}(\gamma) = x$, then $\hat{E} = \{\gamma \in \Gamma; x_{1}(\gamma) \neq x(\gamma)\}$ is a Borel null set in $\Gamma$. Since $[\gamma; x(\gamma) \neq x_{1}(\gamma)] = r^{-1}(\hat{E})$ is a Borel null subset of $\Gamma$, we have

$$
\int_{\Gamma} u(\gamma)^{-1} x(\gamma) u(\gamma) \, d\mu(\gamma) = \int_{\Gamma} u(\gamma)^{-1} x_{1}(\gamma) u(\gamma) \, d\mu(\gamma),
$$

so that $\theta$ is well defined. Similarly it is easily verified that $\theta$ preserves the algebraic operations. Let $\{x_{n}\}$ be a sequence in the unit sphere of $\hat{A}'$ converging strongly to zero. Then there exists a subsequence $\{x_{n_{j}}\}$ and a null subset $\hat{N}$ of $\hat{\Gamma}$ such that $[x_{n_{j}}(\gamma)]$ converges strongly to zero for every $\gamma \in \hat{\Gamma} \setminus \hat{N}$ by [2: Chap. II, §2 Prop. 4]. Since $x_{n_{j}}(r(\gamma))$ converges to zero for every $\gamma \in r^{-1}(N)$ and $r^{-1}(\hat{N})$ is a null subset of $\Gamma$, $\theta(x_{n_{j}})$ converges strongly to zero in $\hat{\mathcal{F}}$. Therefore any subsequence of $\{\theta(x_{n})\}$ contains a subsequence converging strongly to zero, which implies the strong convergence of $\{\theta(x_{n})\}$ to zero. Hence $\theta$ is strongly continuous on the unit sphere of $\hat{A}'$. It is clear that $\theta(\hat{A}) \subset A'$, and $\theta(A') \subset A'$, since each operator of $\theta(A')$ is diagonalizable and each one of $\theta(A')$ is decomposable.

Putting $\hat{x} = \int_{\Gamma} x(\phi(\gamma)) d\hat{\mu}(\gamma)$ for each $x \in \mathcal{F}$, we have

$$
\theta(\hat{x}) = \int_{\Gamma} u(\gamma)^{-1} \hat{x} (r(\gamma)) u(\gamma) \, d\mu(\gamma) = \int_{\Gamma} u(\gamma)^{-1} x(\Theta(\gamma)) u(\gamma) \, d\mu(\gamma) = \int_{\Gamma} x(\gamma) \, d\mu(\gamma) = x.
$$

Hence $\theta(\hat{A})$ covers $M'$. Putting $\theta^{-1}(\mathcal{M}) = \hat{\mathcal{M}}$ and $\mathcal{M}' = \hat{\mathcal{M}}$, we have $\theta(\hat{M}) = M'$. Since each $\gamma$-component of $\mathcal{M}$ coincides with $\phi(\gamma)$-component of $\mathcal{M}$ as the operator algebra over $\hat{\mathcal{F}}(\gamma) = \mathcal{M}(\phi(\gamma))$, almost every $\gamma$-component of $\hat{\mathcal{M}}$ is irreducibly acting on $\hat{\mathcal{F}}(\gamma)$. Because, putting $E = \{\gamma; \gamma$-component of $\mathcal{M}$ is not irreducible}, $E$ is saturated and $E$ is a null set by the maximality of $A$ in $M$. Hence $\hat{\mu}(r(E)) = 0$. On the other hand, we have
which implies that almost every \( \hat{\gamma} \)-component of \( \hat{\mathfrak{A}} \) is irreducible. Hence \( \hat{\mathfrak{A}} \) is a maximal abelian subalgebra of \( \hat{M} \).

Finally we shall show that \( \mathfrak{A} \) is simple in \( M \). Putting \( \psi_f(x) = \theta(x) \) for each \( x \in \mathfrak{A} \), suppose that \( \mathfrak{M} \mathfrak{a} \mathfrak{b} \mathfrak{c} (\gamma_1, \gamma_2) \) holds for \( \gamma_1, \gamma_2 \in \Gamma \). That is, there exists a unitary \( u \) of \( \mathfrak{F}(\mathfrak{A}) \) onto \( \mathfrak{F}(\mathfrak{A}) \) such that \( u\psi_{\gamma_1}(\hat{x})u^{-1} = \psi_{\gamma_2}(\hat{x}) \) for each \( \hat{x} \in \hat{\mathfrak{A}} \), which implies that \( \gamma_1(\hat{x}) = \gamma_2(\hat{x}) \) for each \( \hat{x} \in \hat{\mathfrak{A}} \), which implies \( \gamma_1(\hat{\gamma}) = \gamma_2(\hat{\gamma}) \) for each \( \hat{\gamma} \in \hat{\mathfrak{A}} \), hence \( \gamma_1 = \gamma_2 \), so that \( \hat{\mathfrak{A}} \) is simple in \( \mathfrak{A} \). After all, the triad \( (\mathfrak{M}, \hat{\mathfrak{A}}, \theta) \) is the desired one.

**SUFFICIENCY:** Suppose that there exists a triad \( (\mathfrak{M}, \hat{\mathfrak{A}}, \theta) \) satisfying the condition in our theorem. Let \( \mathfrak{A} = L^\infty(\Gamma, \mu) \). Let \( \hat{\mathfrak{A}} \) and \( \hat{\Phi} = \{\phi_\gamma\} \) be a couple as in § 1 for \( (\mathfrak{M}, \hat{\mathfrak{A}}) \). Let \( \mathfrak{N} = \int_\Gamma \mathfrak{F}(\mathfrak{A}) d\hat{\mu}(\hat{\gamma}) \) be the decomposition of \( \mathfrak{A} \) with respect to \( \hat{\mathfrak{A}} \), which induces the central decomposition \( \hat{\mathfrak{A}} = \int_\Gamma \hat{\mathfrak{A}}(\mathfrak{A}) d\hat{\mu}(\hat{\gamma}) \) of the von Neumann algebra \( \hat{\mathfrak{A}} \). Since almost every component \( \hat{\mathfrak{A}}(\mathfrak{A}) \) becomes the algebra \( \mathfrak{B}(\mathfrak{F}(\hat{\gamma})) \) of all bounded operators on \( \mathfrak{F}(\hat{\gamma}) \) and almost every \( \phi_\gamma \) is irreducible, almost every \( \hat{\mathfrak{A}}(\mathfrak{A}) \) is the weak closure of \( \phi_\gamma(\mathfrak{A}) \). By [2 : Chap. II, § 3 Prop. 11] there exists a decomposition \( \mathfrak{N} = \int_\Gamma \mathfrak{F}(\mathfrak{A}) d\hat{\mu}(\hat{\gamma}) \) of \( \mathfrak{A} \) over \( \Gamma \) with respect to \( \theta(\hat{\mathfrak{A}}) \) which induces the decomposition \( \theta(\hat{\mathfrak{A}}) = \int_\Gamma \theta(\mathfrak{A}) d\hat{\mu}(\hat{\gamma}) \) of \( \theta(\mathfrak{A}) \) and there exists a measurable field \( \{\theta_\gamma: \gamma \in \Gamma\} \) of normal isomorphisms of \( \mathfrak{A}(\mathfrak{A}) \) onto \( \theta(\hat{\mathfrak{A}}(\mathfrak{A})) \) such that \( \theta(x) = \int_\Gamma \theta_\gamma(x(\gamma)) d\hat{\mu}(\hat{\gamma}) \) for each \( x \in \mathfrak{A} \), that is, \( \theta = \int_\Gamma \theta_\gamma d\hat{\mu}(\hat{\gamma}) \). Putting \( \theta(\hat{\mathfrak{A}}) = \mathfrak{A} \) and \( \psi_\gamma(x) = \theta_\gamma(\phi_\gamma(x(\gamma)) \) for each \( x \in \mathfrak{A} \), almost every \( \psi_\gamma(\mathfrak{A}) \) is weakly dense in \( \mathfrak{A} \) and there exists \( \psi_\gamma(\mathfrak{A}) \) weakly dense in \( \mathfrak{A} \) by the continuity of almost every \( \theta_\gamma \). Hence almost every \( \psi_\gamma(\mathfrak{A}) \) is a representation of type I which is quasi-equivalent to irreducible representation \( \phi_\gamma \circ \theta^{-1} \) of \( \mathfrak{A} \). Modifying \( \hat{\Phi} \) on a null subset of \( \Gamma \), we can assume from the assumption for \( \hat{\mathfrak{A}} \) that each distinct members of \( \hat{\Phi} \) are disjoint. Besides, eliminating null set, \( \psi_\gamma \) is quasi-equivalent to \( \phi_\gamma \circ \theta^{-1} \) for every \( \gamma \in \Gamma \). After all, we conclude that there exists a von Neumann subalgebra \( \mathfrak{B} = \theta(\hat{\mathfrak{A}}) \cong L^\infty(\Gamma, \mu) \) and
a decomposition \( \hat{\phi} = \int_{\Gamma} \hat{\phi}(\gamma) d\hat{\mu}(\gamma) \) of \( \hat{\phi} \) with respect to \( B \) which induces a family \( \Psi = \{ \psi_\gamma \} \) of mutually disjoint factor representations of type I of \( \mathfrak{U} \) such that \( x = \int_{\Gamma} \psi_\gamma(x) d\hat{\mu}(\gamma) \) for each \( x \in \mathfrak{U} \).

By [5: § 5, Prop.2] we get the following:

1°. there exist null subsets \( N \subset \Gamma \) and \( \tilde{N} \subset \tilde{\Gamma} \) and a Borel mapping \( \Theta \) of \( \Gamma - N \) onto \( \tilde{\Gamma} - \tilde{N} \) such that for each \( a \in \tilde{A} \) \( \theta(a)(\gamma) = a(\Theta(\gamma)) \) for almost every \( \gamma \in \Gamma - N \) and \( \Theta(\mu) \approx \hat{\mu} \).

2°. there exists a decomposition \( \mu = \int_{\tilde{\Gamma}} \mu^\Lambda d\hat{\mu}(\gamma) \) of \( \mu \) such that \( \mu^\Lambda \) is concentrated on \( \Theta^{-1}(\gamma) \) for every \( \gamma \in \Gamma - N \).

3°. there exist a null set \( \tilde{N}_1 \subset \tilde{\Gamma} \) and a unitary of \( \hat{\phi}(\gamma) \) onto \( \int_{a^{-1}(\gamma)} \hat{\phi}(\gamma) d\mu^\Lambda(\gamma) \) for every \( \gamma \in \tilde{\Gamma} - \tilde{N}_1 \) which carries \( \psi_{\gamma}(x) \) onto \( \int_{a^{-1}(\gamma)} \psi_{\gamma}(x) d\mu^\Lambda(\gamma) \) for every \( \gamma \in \tilde{\Gamma} - \tilde{N}_1 \).

Since \( \psi_\gamma \) is a factor-representation of type I, \( \varphi_\gamma \) is quasi-equivalent to \( \psi_{\gamma} \) for \( \mu^\Lambda \)-almost every \( \gamma \in \Theta^{-1}(\gamma) \) by [8: p. 103, Lemma]. Putting \( N' = \{ \gamma \in \Gamma; \varphi_\gamma \) is not quasi-equivalent to \( \psi_{\gamma(\gamma)} \} \), we have \( \mu(N') = \int_{\tilde{\Gamma}} \mu^\Lambda(N') d\hat{\mu}(\gamma) = 0 \). For each pair \( (\gamma, \gamma') \in \{ \Gamma - (N \cup N') \} \times \{ \Gamma - (N \cup N') \} \), \( \mathfrak{B}(\gamma, \gamma') \) holds if and only if \( \Theta(\gamma') = \Theta(\gamma) \), so that \( (\Gamma - (N \cup N'), \mu)/\mathfrak{B} \) is isomorphic to the standard measure space \( (\tilde{\Gamma} - \tilde{N}, \hat{\mu}) \), which implies the smoothness of \( A \) in \( M \).

**Unicity:** Suppose that there exists another triad \( (\tilde{M}_1, \tilde{A}_1, \theta_1) \). For \( (\tilde{M}_1, \tilde{A}_1, \theta_1) \) we shall use the corresponding notations in the proof of sufficiency adding the suffix 1 (For instance, let \( A_1 = L^\infty(\tilde{\Gamma}_1, \hat{\mu}_1) \) and so on.) Suppose that \( \theta(A) = \theta_1(A_1) \) is proved. Since \( \tilde{A} \) (resp. \( \tilde{A}_1 \)) is generated by \( \tilde{M} \) and \( \tilde{A} \) (resp. \( \tilde{M}_1 \) and \( \tilde{A}_1 \)) by the maximality of \( \tilde{A} \) (resp. \( \tilde{A}_1 \)) in \( \tilde{M} \) (resp. \( \tilde{M}_1 \)), \( \theta(A) \) (resp. \( \theta_1(A_1) \)) is generated by \( M' = \theta(\tilde{M}') \) and \( \theta(A) \) (resp. \( M_1' = \theta(\tilde{M}_1') \) and \( \theta_1(A_1) \)), which implies \( \theta(A) = \theta_1(A_1) \). Hence \( \theta^{-1} \circ \theta_1 \) becomes an isomorphism of \( \tilde{A}_1 \) onto \( A \), which is a spatial isomorphism by [2: Chap. III, §3 Cor. of Prop. 3]. Therefore it remains only to prove \( \theta(A) = \theta_1(A_1) \). Each element \( a \in A = L^\infty(\Gamma, \mu) \) belongs to \( \theta(A) \) (resp. \( \theta_1(A_1) \)) if and only if \( a(\gamma) \) is constant on the coset \( \Theta^{-1}(\gamma) \) (resp. \( \Theta_1^{-1}(\gamma') \)) for almost every \( \gamma \in \tilde{\Gamma} \) (resp. \( \gamma' \in \tilde{\Gamma}_1 \)). As seen in the proof of sufficiency,
almost every coset $\Theta\sim \gamma (\gamma \in \mathcal{G})$ (resp. $\Theta'\sim \gamma (\gamma \in \mathcal{G}')$) becomes $\mathcal{H}$-equivalence class in $\Gamma$, which implies $\theta(\mathcal{A}) = \theta_1(\mathcal{A}_1)$. This completes the proof.

**Definition 3.1.** For each smooth maximal abelian subalgebra $A$ of a von Neumann algebra $M$, we call the triad $(M, A, \theta)$, appeared in Theorem 3.1, the simplification of the pair $(M, A)$.

**Definition 3.2.** Let $A_1, A_2, (\Gamma_1, \mu_1), (\Gamma_2, \mu_2)$ and $M$ be as in § 1. Let $p_1$ and $p_2$ be non-zero projections of $A_1$ and $A_2$ associated with Borel subsets $P_1 \subset \Gamma_1$ and $P_2 \subset \Gamma_2$ respectively. Let $E_1$ and $E_2$ be the projections of the graph of $\mathcal{H}^{u_1 , u_2} = \mathcal{H}$ in $P_1 \times P_2$ into $\Gamma_1$ and $\Gamma_2$ respectively. If there exist partitions of $E_1$ and $E_2$ such that $E_1 = F_1 \cup F_2$, $E_2 = F_3 \cup F_4$, $F_1 \cap F_3 = F_2 \cap F_4 = \emptyset$, $F_1, \cdots, F_4$ are measurable, $\mu_1(F_1) = \mu_2(F_2) = 0$ and $F_i$ contains every $\gamma_i \in E_i$ satisfying the condition $\mathcal{H}(\gamma_i, \gamma_2)$ for some $\gamma_j \in E_j$ such that $i \neq j$, $i, j = 1, 2$, then we say that $A_1p_1$ and $A_2p_2$ are unrelated. That is, $A_1p_1$ and $A_2p_2$ are unrelated if and only if $\mathcal{H}(\gamma_1, \gamma_2)$ does not hold for every pair $(\gamma_1, \gamma_2) \in P_1 \times P_2$, eliminating null sets from $P_1$ and $P_2$. Otherwise, we say that $A_1p_1$ and $A_2p_2$ are related. Moreover if for each non-negligible subset $E_i \subset P_i (i = 1, 2)$ the set $F_i$ of all $\gamma_i$'s of $P_i$ satisfying the condition $\mathcal{H}(\gamma_1, \gamma_2)$ for some $\gamma_i \in E_i (i \neq j, i, j = 1, 2)$ is not negligible, then we say that $A_1p_1$ and $A_2p_2$ are similar.

**Theorem 3.2.** For a maximal abelian subalgebra $A$ of a von Neumann algebra $M$ to be smooth, it is necessary and sufficient that there exists a partition of unit $p_0 + \sum_{n=1}^{\infty} p_n + p_\omega = I$ in $A$ satisfying the following conditions:

1°. For each $1 \leq n \leq \infty$ $p_0 M p_n$ and $A p_n$ can be represented such as $p_0 M p_n = M_n \otimes B_n$ and $A p_n = A_n \otimes L^\infty([1, 2, \cdots, n])$ by some von Neumann algebra $M_n$ and its simple maximal abelian subalgebra $A_n$, where $B_n$ means the full operator algebra over the $n$-dimensional Hilbert space $l^2([1, 2, \cdots, n])$. Besides $A p_n$ and $A p_m$ are unrelated if $n \neq m, n, m \geq 1$.

2°. $p_0 M p_0$ and $A p_0$ can be represented such as $p_0 M p_0 = M_0 \otimes B_0$ and $A p_0 = A_0 \otimes L^\infty(0, 1)$ by some von Neumann algebra $M_0$ and its simple maximal abelian subalgebra $A_0$, where $B_0$ means the full operator algebra over the Hilbert space $L^2(0, 1)$.

If $A$ is smooth, then the above decomposition of $M$ and $A$ is unique. If $M$ is of finite type, then $p_0 = p_\omega = 0$.

**Proof.** The sufficiency is a direct conclusion of Lemma 2.2, so we shall prove only the necessity. Suppose that $A$ is smooth. Let $A = L^\infty(\Gamma, \mu)$ and $\mathcal{H} = \mathcal{H}^{u_1, u_2}$. Let $(\hat{\Gamma}, \hat{\mu})$ be the quotient measure space of $(\Gamma, \mu)$ by $\mathcal{H}$ and let $r$ be the canonical mapping of $\Gamma$ onto $\hat{\Gamma}$. Identifying $f$ and $f \circ r$ for each $f \in L^\infty(\hat{\Gamma}, \hat{\mu})$
DECOMPOSITIONS OF REPRESENTATIONS 381

\[
\mathcal{A} \text{ becomes a von Neumann subalgebra of } \mathcal{A}. \text{ By [5: §6, Prop. 7] there exist unique orthogonal projections } p_0 \text{ and } q_0 \text{ in } \mathcal{A} \text{ such that } p_0 + q_0 = 1, \text{ } p_0 \text{ is the greatest relatively continuous projection with respect to } \mathcal{A} \text{ and } q_0 \text{ is the greatest relatively discrete projection with respect to } \mathcal{A}. \text{ So we shall study } (p_0, Mq_0, Aq_0) \text{ and } (p_0, Mp_0, Ap_0) \text{ separately.}
\]

1°. Case of \( q_0 = 1 \). For each non-zero projection \( e \in \mathcal{A} \) there exists the smallest projection \( \tilde{e} \) in \( \mathcal{A} \) majorizing \( e \), which is called \( \mathcal{A} \)-carrier of \( e \). Let \( e \) and \( f \) be two relatively minimal projections in \( \mathcal{A} \) with respect to \( \mathcal{A} \) with the same \( \mathcal{A} \)-carrier. Let \( E \) and \( F \) be the Borel subsets of \( \Gamma \) associated with \( e \) and \( f \) respectively. Since \( Ae = Af \) and \( Af = Af \), both the \( r|E \) and \( r|F \), the restrictions of \( r \), are one-to-one mappings except for negligible parts. Since \( \tilde{e} \) and \( \tilde{f} \) are the projections of \( \mathcal{A} \) associated with \( r(E) \) and \( r(F) \) respectively, we have \( r(E) = r(F) \). Hence there exists a one-to-one Borel isomorphism \( \phi \) from \( E \) onto \( F \) such \( \mathbb{R}(\gamma, \phi(\gamma)) \) for almost every \( \gamma \in E \). Since \( Ae = \tilde{e} \equiv \tilde{e} \) under the canonical correspondence, \( r \) transforms the class of all null sets in \( E \) onto the one in \( r(E) \). Hence \( \phi \) is an isomorphism of the measure space \( (E, \mu|E) \) onto \( (F, \mu|F) \). By Lemma 3.1 there exists a partial isometry \( u \) of \( M \) defined by a family \( \{u(\gamma), \gamma \in E\} \) of unitaries from \( \mathbb{D}(\phi(\gamma)) \) onto \( \mathbb{D}(\gamma) \) such that \( uu^* = e \) and \( u^* u = f \). Hence, for each pair of relatively minimal projections \( e, f \) of \( \mathcal{A} \) with respect to \( \mathcal{A} \), there exist orthogonal projections \( g, h, k \) in \( \mathcal{A} \) such \( e = g + h + k = 1 \), \( (ge)^e = (gf)^e = g, g + h \geq e, g + k \geq f \) and \( ge \sim gf \).

For each non-zero projection \( e \in \mathcal{A} \) there exists a relatively minimal projection \( f \) of \( \mathcal{A} \) with respect to \( \mathcal{A} \) such that \( f \leq e \) and \( f = \tilde{e} \). Indeed, let \( \{f_a\} \) be a maximal family of relatively minimal orthogonal projections in \( \mathcal{A} \) such that \( f_a \leq e \) and the \( f_a \)'s are orthogonal each other. Then \( f = \sum_a f_a \) is required one.

Let \( \{e_a\} \) be a maximal family of relatively minimal orthogonal projections in \( \mathcal{A} \) with \( \mathcal{A} \)-carrier \( I \). If \( I \neq \sum_a e_a \), then \( (I - \sum_a e_a)^e \neq I \) by the maximality of \( \{e_a\} \). Putting \( p = I - (I - \sum_a e_a)^e \in \mathcal{A} \), we have \( p = \sum_a pe_a \) and \( p = (pe_a)^e \).

If the cardinal of \( \{e_a\} \) is finite, then we repeat this argument for \( \mathcal{A}(I - p) \) and \( \mathcal{A}(I - p) \). If it is infinite, there exists a family \( \{f_a\} \) of relatively minimal orthogonal projections such that \( \sum_a f_a = I \) and \( f_a = I \). Indeed, let \( \{g_b\} \) be a maximal family of relatively minimal orthogonal subprojections of \( I - \sum_a e_a \) in \( \mathcal{A} \).

Since \( \mathcal{A} \) is discrete over \( \mathcal{A} \), we have \( I - \sum e_a = \sum g_b \). Since \( M \) is acting on a separable Hilbert space, both \( \{e_a\} \) and \( \{g_b\} \) are at most countable. Let \( \{e_a\} \) and \( \{g_b\} \) be their enumerations respectively. Let \( E_a \) and \( G_a \) be Borel subsets of
Γ associated with $e_n$ and $g_n$ respectively. Since $e_n = I \geq g_n$ and $e_n$ and $g_n$ are relatively minimal, there exists a one-to-one Borel mapping $\phi$ from $\Gamma$ into $\bigcup_{n=1}^{\infty} E_n$ such that $\phi(G_n) \subseteq E_{2n+1}$, $\phi(E_n) = E_{2n}$, $\mathfrak{H}(\gamma, \phi(\gamma))$ for almost every $\gamma \in \Gamma$ and $\phi(\mu) \approx \mu | \phi(\Gamma)$. By Bernstein’s method it is easily shown that there exists a Borel one-to-one mapping $\psi$ from $\Gamma$ onto $\bigcup_{n=1}^{\infty} E_n$ such that $\mathfrak{H}(\gamma, \psi(\gamma))$ for almost every $\gamma \in \Gamma$ and $\psi(\mu) \approx \mu$. By Bernstein’s method it is easily shown that there exists a Borel one-to-one mapping $\psi$ from $\Gamma$ onto $\bigcup_{n=1}^{\infty} E_n$ such that $\mathfrak{H}(\gamma, \psi(\gamma))$ for almost every $\gamma \in \Gamma$ and $\psi(\mu) \approx \mu$. The family $\{f_n\}$ of projections associated with $\psi^{-1}(E_n)$ is the required one.

After all, there exists a family $\{p_n\}_{n=1,2,\ldots,m}$ of orthogonal projections of $\Lambda$ and for each $n$ there exists a family $\{e_n, k : 1 \leq k \leq n\}$ of relatively minimal orthogonal projections of $\Lambda$ such that $e_n, k = p_n$ for $k = 1, 2, \ldots, n$ and $p_n = \sum_{k=1}^{n} e_n, k$. Besides for each $n$ and $k$ there exists a partial isometry $u$ of $M$ such that $uw = u$ and $uAe_n, u = Ae_n, k$. Since $Ae_{n,1} \equiv Ap_n$ under the natural correspondence, $p_n M p_n = e_{n,1} M \otimes B_n$ and $Ap_n = Ae_{n,1} \otimes l^\infty(1, 2, \ldots, n)$. Now it is clear that $Ae_{n,1}$ is a simple maximal abelian subalgebra of $e_{n,1} M e_{n,1}$ and that $Ap_n$ and $Ap_m$ are unrelated if $n \neq m$.

2°. Case of $\mu = I$. Replacing $\mu$ by an equivalent finite measure, we may assume the finiteness of $\mu$. By the smoothness of $\Lambda$ we get a decomposition $\mu = \int \mu^2 \, d\mu(\gamma)$ of $\mu$ over the measure space $(\Gamma, \mathfrak{H})$ with respect to the mapping $r$. By [5: §5, Prop. 1] we can define

$$\mathfrak{H}(\gamma) = \int_{r^{-1}(\gamma)} \mathfrak{H}(\gamma) \, d\mu^2(\gamma) \text{ and } x(\gamma) = \int_{r^{-1}(\gamma)} x(\gamma) \, d\mu^2(\gamma)$$

for almost every $\gamma \in \Gamma$ and for $x \in \mathfrak{A}$ and we get a decomposition

$$\mathfrak{H} = \int_{\Gamma} \mathfrak{H}(\gamma) \, d\mu(\gamma) \text{ and } x = \int_{\Gamma} x(\gamma) \, d\mu(\gamma)$$

under suitable identification. $\Lambda$ becomes the diagonal algebra in this new decomposition. Since there exists a unitary $u$ of $\mathfrak{H}(\gamma)$ onto $\mathfrak{H}(\gamma)$ for each $\gamma \in r^{-1}(\gamma)$ such $ux(\gamma) u^{-1} = x(\gamma)$, we get

$$\mathfrak{H}(\gamma) = \mathfrak{H}(\gamma) \otimes L^\infty(r^{-1}(\gamma), \mu^2) \text{ and } x(\gamma) = x(\gamma) \otimes I$$

for almost every $\gamma \in \Gamma$ by [2: Chap. II, §2 Theorem 2]. Moreover, $\Lambda$ is decomposable with respect to this new decomposition, whose almost every component $\Lambda(\gamma)$ is represented by $\Lambda(\gamma) = C \otimes L^\infty(r^{-1}(\gamma), \mu^2)$, where $C$ means the complex number field. By [5: §6, Prop. 10] $\Lambda(\gamma)$ is relatively continuous with respect to $\Lambda(\gamma) = C$ for almost every $\gamma \in \Gamma$, so that almost every measure space $(r^{-1}(\gamma), \mu^2)$ has no discrete summand. Since almost every
\( \tau^{-1}(\tilde{\gamma}) \) is a Borel subset of the standard Borel space \( \Gamma \), almost every measure space \((r^{-1}(\tilde{\gamma}), \mu^\circ)\) is isomorphic to \((0,1)\)-interval equipped with Lebesgue measure. Hence we get

\[
\mathcal{H}(\tilde{\gamma}) = \mathcal{H}(\gamma) \otimes L^2(0,1) \quad \text{and} \quad A(\tilde{\gamma}) = C \otimes L^\omega(0,1)
\]

for almost every \( \gamma \in \Gamma \).

By Lemma 2.1 there exists a measurable mapping \( \phi \) from \( \tilde{\Gamma} \) to \( \Gamma \) such that \( \phi(\gamma) \subseteq r^{-1}(\tilde{\gamma}) \) for almost every \( \gamma \). Since

\[
\mathcal{H}(\tilde{\gamma}) = \mathcal{H}(\phi(\tilde{\gamma})) \otimes L^2(0,1) \quad \text{and} \quad A(\tilde{\gamma}) = C \otimes L^\omega(0,1)
\]

for almost every \( \gamma \), we get

\[
\mathcal{H} = \left\{ \int_\Gamma \mathcal{H}(\phi(\tilde{\gamma})) \, d\mu(\tilde{\gamma}) \right\} \otimes L^2(0,1)
\]

and

\[
x = \left\{ \int_\Gamma x(\phi(\tilde{\gamma})) \, d\mu(\tilde{\gamma}) \right\} \otimes I \quad \text{for each} \quad x \in \mathfrak{A}.
\]

Putting \( \mathfrak{A} = \int_\Gamma \mathcal{H}(\phi(\tilde{\gamma})) \, d\mu(\tilde{\gamma}) \) and \( x_\mathfrak{A} = \int_\Gamma x(\phi(\tilde{\gamma})) \, d\mu(\tilde{\gamma}) \) for \( x \in \mathfrak{A} \), we have \( \tilde{\gamma} = \mathfrak{A} \otimes L^2(0,1) \) and \( x = x_\mathfrak{A} \otimes I \). It is clear that the diagonal algebra \( A_0 \) in the decomposition of \( \mathfrak{A} \) is isomorphic to \( A \) under the canonical correspondence and that \( \{x_\mathfrak{A}(\gamma) ; x \in \mathfrak{A}\} = \{x(\phi(\gamma)) ; x \in \mathfrak{A}\} \) acts on \( \mathfrak{A}(\gamma) = \mathcal{H}(\phi(\gamma)) \) irreducibly for almost every \( \gamma \in \Gamma \). Besides the representations \( x_\mathfrak{A} \rightarrow x_\mathfrak{A}(\gamma) \) of the \( C^* \)-algebra \( \mathfrak{B} = \{x_\mathfrak{A} ; x \in \mathfrak{A}\} \) are mutually disjoint. Hence \( A_0 \) is a simple maximal abelian subalgebra of \( M_0 = \mathfrak{B} \otimes B_1 \). Since \( x = x_\mathfrak{A} \otimes I \) for every \( x \in \mathfrak{A} \), we have \( M = \mathfrak{B} = M_0 \otimes B_1 \). And we get \( A = A_0 \otimes L^\omega(0,1) \).

The unicity of \( \{A_\mathfrak{A}\}_{\mathfrak{A} = \mathfrak{A}_1, \ldots, \mathfrak{A}_n} \) is almost clear from its construction. This completes the proof.

**THEOREM 3.3.** Let \( A_1 \) and \( A_2 \) be two maximal abelian subalgebras of a von Neumann algebra \( M \). Let \( e_1 \) and \( e_2 \) be non-zero projections of \( A_1 \) and \( A_2 \) such that \( A_1 e_1 \) and \( A_2 e_2 \) are smooth in \( e_1 M e_1 \) and \( e_2 M e_2 \) respectively. \( A_1 e_1 \) and \( A_2 e_2 \) are similar if the simplifications \( (e_1 M e_1), (A_1 e_1, \theta_1) \) and \( (e_2 M e_2), (A_2 e_2, \theta_2) \) are unitarily equivalent in the sense that there exists a unitary \( u \) of the underlying Hilbert space \( \mathfrak{H}_1 \) of \( (e_1 M e_1) \) onto the one \( \mathfrak{H}_2 \) of \( (e_2 M e_2) \) such that \( u (A_1 e_1) u^{-1} = (A_2 e_2) \), \( u(e_1 M e_1) u^{-1} = (e_2 M e_2) \) and \( \theta_2 (u x u^{-1}) = \theta_1 (x) \) for \( x \in (e_1 M e_1) \).

**PROOF.** Suppose that \( A_1 e_1 \) and \( A_2 e_2 \) are similar. Let \( E_1 \) and \( E_2 \) be the Borel sets in \( \Gamma_1 \) and \( \Gamma_2 \) associated with \( e_1 \) and \( e_2 \) respectively. Putting \( \mathfrak{R}_1 = \mathfrak{R}_{e_1} \) and \( \mathfrak{R}_2 = \mathfrak{R}_{e_2} \) and \( \Phi e_i = \{ \phi_{i \alpha} \in \Phi^i ; \gamma_{i \alpha} \geq E_i \} \) \((i=1,2)\), \( \mathfrak{R}_{e_{i \alpha} \gamma_{i \alpha}} = \mathfrak{R}_i \) and \( \mathfrak{R}_{e_{i \alpha} \gamma_{i \alpha} \psi_{i \alpha}} = \mathfrak{R}_2 \) are the restrictions of \( \mathfrak{R}^{e_{1 \alpha} \gamma_{1 \alpha}} \) and \( \mathfrak{R}^{e_{2 \alpha} \gamma_{2 \alpha}} \) to \( E_1 \) and \( E_2 \) respectively. Let \( (\tilde{E}_1, \tilde{\mu}_1) \) and \( (\tilde{E}_2, \tilde{\mu}_2) \) be the quotient measure spaces of \( (E_1, \mu_1) \) and \( (E_2, \mu_2) \) by \( \mathfrak{R}_1 \) and \( \mathfrak{R}_2 \) respectively.
Let $r_1$ and $r_2$ be the associated canonical mappings of $E_1$ and $E_2$ onto $\hat{E}_1$ and $\hat{E}_2$ respectively. Since $\mathcal{M}_{\hat{E}_1}^\times,\hat{\gamma}_1,\hat{\gamma}_2$, $\mathcal{M}_{\hat{E}_2}^\times,\hat{\gamma}_1,\hat{\gamma}_2$ imply $\mathcal{M}_{\hat{E}_1}^\times,\hat{\gamma}_1,\hat{\gamma}_2$ for $\gamma_1$, $\gamma'_1 \in \Gamma_1$ and for $\gamma_2$, $\gamma'_2 \in \Gamma_2$, the mapping $e$ of $\hat{E}_1$ to $\hat{E}_2$ defined by

$$\hat{\varphi}(\hat{x}) = r_2 \circ p_2 ((r_1^{-1}(\hat{x})) \times E_2) \cap \hat{E}_2$$

for $\hat{x} \in \hat{E}_1$ is a one-to-one mapping, where $R$ is the graph of $\mathcal{M}_{\hat{E}_1}^\times,\hat{\gamma}_1,\hat{\gamma}_2$, and it has the range with null complement in $\hat{E}_2$. Eliminating null sets from $\hat{E}_1$ and $\hat{E}_2$ respectively, we may assume that $\hat{\varphi}$ is a one-to-one mapping of $\hat{E}_1$ onto $\hat{E}_2$. By the similarity of $\mathcal{M}_{\hat{E}_1}^\times,\hat{\gamma}_1,\hat{\gamma}_2$ and $\mathcal{M}_{\hat{E}_2}^\times,\hat{\gamma}_2$, $\hat{x}$ is defined almost everywhere in $\hat{E}_1$ and it has the range with null complement in $\hat{E}_2$. By the proof of necessity in Theorem 3.1, there exists a unitary $u$ of $\mathcal{M}_{\hat{E}_1}^\times,\hat{\gamma}_1,\hat{\gamma}_2$ onto $\mathcal{M}_{\hat{E}_2}^\times,\hat{\gamma}_2,\hat{\gamma}_2$ such that $u_\xi = \int_{\hat{E}_1} u(\hat{x}) \xi(x) \, d\hat{\gamma}(x)$ for $\xi \in \mathfrak{S}_2$. It is clear that $u$ carries $(A_1 e_1)$ onto $(A_1 e_1)$. Since $u \theta^{-1}(x) u^{-1} = \theta^{-1}(x)$ for each $x \in \mathfrak{A}$, $u^{-1}$ induces the desired spatial isomorphism of $(e_1 \mathcal{M}_{e_1})$ onto $(e_1 \mathcal{M}_{e_1})$.

Conversely suppose that there exists a unitary $u$ of $\mathfrak{S}_2$ onto $\mathfrak{S}_2$ satisfying the condition of our theorem. By [8 : Theorem 2.7] and Lemma 3.1 there exists a one-to-one mapping $r$ from a Borel subset of $E_1$ onto a Borel subset of $E_2$ with null complements such that $\hat{\varphi}(\hat{\mu}_2) \approx \hat{\mu}_2$ and there exists a family $\{u(\hat{y}_1); \hat{y}_1 \in \hat{E}_1 \}$ of unitaries from $\mathfrak{S}_2(\hat{\varphi}(\hat{y}_1))$ onto $\mathfrak{S}_2(\hat{\varphi}(\hat{y}_1))$ such that $u(\hat{y}_1)^{-1} x(\hat{y}_1) = x(\hat{y}_2)$ for each $x \in \mathfrak{A}$. By Lemma 3.1 there exists a family $\{u(\hat{y}_1); \hat{y}_1 \in \hat{E}_1 \}$ of unitaries from $\mathfrak{S}_2(\hat{\varphi}(\hat{y}_1))$ onto $\mathfrak{S}_2(\hat{\varphi}(\hat{y}_1))$ such that $u(\hat{y}_1)^{-1} x(\hat{y}_1) = x(\hat{y}_2)$ for each $x \in \mathfrak{A}$, which defines a unitary $u$ of $\mathfrak{S}_2$ onto $\mathfrak{S}_2$ by

$$u(\hat{y}_1) \xi(\hat{\varphi}(\hat{y}_1)) \sqrt{\frac{d\hat{\varphi}^{-1} \hat{\mu}_2(\hat{y}_1)}{d\hat{\mu}_2(\hat{y}_1)}} \, d\hat{\mu}_2(\hat{y}_1)$$

for $\xi \in \mathfrak{S}_2$ and $u(\hat{y}_1)^{-1} \theta^{-1}(x, e_2) \hat{y}_1 u(\hat{y}_1) = \theta^{-1}(x, e_2) \hat{y}_2$ for almost every $\hat{y}_1 \in \hat{E}_1$ and for each $x \in \mathfrak{A}$. By the proof of necessity in Theorem 3.1 there exist unitaries $u_1$ and $u_2$ of $\mathfrak{S}_2(\hat{\gamma}_1)$ and $\mathfrak{S}_2(\hat{\gamma}_2)$ onto $\mathfrak{S}_2(\hat{\gamma}_1)$ and $\mathfrak{S}_2(\hat{\gamma}_2)$ respectively such that

$$u_1 x(\hat{\gamma}_1) u_1^{-1} = \theta^{-1}(x, e_1) (r_1(\hat{\gamma}_1))$$

and

$$u_2 x(\hat{\gamma}_2) u_2^{-1} = \theta^{-1}(x, e_2) (r_2(\hat{\gamma}_2))$$
for every \( x \in \mathbb{A} \). Therefore we have \( \mathbb{M}_{\mathfrak{m}, \mathfrak{n}}(\gamma_1, \gamma_2) \) if \( \mathfrak{r} \circ r_1(\gamma_1) = r_2(\gamma_2) \) for almost every \( \gamma_1 \in E_1 \) and \( \gamma_2 \in E_2 \) that is,

\[
pr_1((S_1 \times E_2) \cap R) \supset r_1^*pr_1(S_1) \quad \text{and} \quad pr_2((E_1 \times S_2) \cap R) \supset r_2^*pr_2(S_2)
\]

for each subset \( S_1 \subset E_1 \) and \( S_2 \subset E_2 \) respectively. Since \( r_1(\mu_1) = \mu_2 \) and \( r_2(\mu_2) = \mu_3 \), we have

\[
\mu_1(pr_1((S_1 \times E_2) \cap R)) > 0 \quad \text{and} \quad \mu_2(pr_2((E_1 \times S_2) \cap R)) > 0
\]

for each non-negligible subsets \( S_1 \subset E_1 \) and \( S_2 \subset E_2 \), which implies the similarity of \( A_1e_1 \) and \( A_2e_2 \). This completes the proof.

Then we get the following

**Corollary.** Similar simple maximal abelian subalgebras of a von Neumann algebra are unitarily equivalent.

**Theorem 3.4.** Let \( A_1 \) and \( A_2 \) be two maximal abelian subalgebras of a von Neumann algebra \( M \). Let \( e_1 \) and \( e_2 \) be non-zero projections of \( A_1 \) and \( A_2 \) respectively. If \( A_1e_1 \) and \( A_2e_2 \) are both smooth in \( e_1Me_1 \) and \( e_2Me_2 \) respectively, then there exist unique projections \( p_1 \) and \( p_2 \) of \( A_1 \) and \( A_2 \) majorized by \( e_1 \) and \( e_2 \) respectively such that \( A_1p_1 \) and \( A_2p_2 \) are similar and \( A_1(e_1 - p_1) \) and \( A_2(e_2 - p_2) \) are unrelated.

**Proof.** If \( A_1e_1 \) and \( A_2e_2 \) are unrelated, then our mention is trivial. So we assume \( A_1e_1 \) and \( A_2e_2 \) are related. We use the notation in the proof of Theorem 3.3.

As in the proof of Theorem 3.3, there exists a one-to-one measurable mapping \( \mathfrak{r} \) from subset of \( E_1 \) into \( E_2 \), whose definition domain \( F_1 \) and range \( F_2 \) are given by

\[
F_1 = r_1 \circ pr_1((E_1 \times E_2) \cap R) \quad \text{and} \quad F_2 = r_2 \circ pr_2((E_1 \times E_2) \cap R)
\]

respectively. By the relatedness of \( A_1e_1 \) and \( A_2e_2 \), the measures \( \mathfrak{r}(\mu_1|F_1) \) and \( \mu_2|F_2 \) are not disjoint. Hence there exists a unique subset \( \tilde{P}_2 \subset \tilde{F}_2 \) up to \( \mu_2 \)-null set such that the measures \( \mathfrak{r}(\mu_1|F_1)|\tilde{P}_2 \) and \( \mu_2|\tilde{P}_2 \) are equivalent and the measures \( \mathfrak{r}(\mu_1|F_1)|\tilde{P}_2 \) and \( \mu_2|\tilde{P}_2 \) are disjoint. Putting \( r_1^{-1}o\mathfrak{r}^{-1}(\tilde{P}_2) = P_1 \subset E_1 \) and \( r_2^{-1}(\tilde{P}_2) = P_2 \subset E_2 \), the projections \( p_1 \) and \( p_2 \), associated with \( P_1 \) and \( P_2 \) respectively, are required ones.

In closing this section we state the following interpretation in the representation theory.

**Corollary.** Let \( \varphi_1 \) and \( \varphi_2 \) be two representations of an involutive separable Banach algebra \( \mathfrak{B} \) over separable Hilbert spaces \( \mathfrak{H}_1 \) and \( \mathfrak{H}_2 \) respectively. Let \( A_1 = L^\infty(\Gamma_1, \mu_1) \) and \( A_2 = L^\infty(\Gamma_2, \mu_2) \) be smooth maximal abelian subalgebras of \( \varphi_1(\mathfrak{B}) = M_1 \) and \( \varphi_2(\mathfrak{B}) = M_2 \) respectively. Decompose \( \varphi_1 \) and \( \varphi_2 \) into direct integrals of irreducible representations over \( \Gamma_1 \) and \( \Gamma_2 \) with respect to \( A_1 \) and \( A_2 \) as follows respectively;
Then there exist Borel subsets \( P_1 \subset \Gamma_1 \) and \( P_2 \subset \Gamma_2 \) such that \( \varphi_1(\gamma_1) \approx \varphi_2(\gamma_2) \) for every \((\gamma_1, \gamma_2) \in \mathcal{C}P_1 \times \mathcal{C}P_2 \) and if \( P_1 \) and \( P_2 \) are non-negligible then for each non-negligible subset \( S_i \subset P_i \) we have

\[
\mu_i(\{\gamma_i \in P_i; \varphi_i(\gamma_i) \approx \varphi_2(\gamma_2) \text{ for some } \gamma_1 \in S_i \}) > 0
\]

\( i, j = 1, 2, i \neq j \). Besides if \( p_1 \) and \( p_2 \) are the projections of \( A_e \) and \( A_{e_2} \) associated with \( P_1 \) and \( P_2 \), then \( \varphi^p_1 \) and \( \varphi^p_2 \) are quasi-equivalent\(^5\). Hence if \( \varphi_1 \) and \( \varphi_2 \) are disjoint, then \( \mu_1(P_1) = \mu_2(P_2) = 0 \).

**Proof.** Put

\[
\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2, \varphi = \varphi_1 \oplus \varphi_2 \text{ and } A = A_1 \oplus A_2 = L^\infty(\Gamma_1 \oplus \Gamma_2, \mu_1 \oplus \mu_2).
\]

Then \( A \subset M_1 \oplus M_2 \subset \varphi(\mathcal{B})' = M \) and \( A \) becomes a maximal abelian subalgebra of \( M \). Let \( e_1 \) and \( e_2 \) be the projections of \( \mathcal{B} \) onto \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) respectively. Then \( e_1 \) and \( e_2 \) belong to \( A \). Application of Theorem 3.3 and 3.4 to \( M, A_{e_1} \) and \( A_{e_2} \) yields our mention.

**Remark.** Unfortunately \( \varphi^{(I-e)}_1 \) and \( \varphi^{(I-e)}_2 \) need not be disjoint. Indeed, if \( e_1 - p_1 \sim e_2 - p_2 \) in \( M \) then \( \varphi^{(I-e)}_1 \) and \( \varphi^{(I-e)}_2 \) are unitarily equivalent. Such case often occurs if \( M \) is of continuous type. For instance, let \( A \) be a simple maximal abelian subalgebra of \( M \) and let \( e \) be a projection of \( A \) such \( e \sim I - e \). Then \( \varphi^e \) and \( \varphi^{(I-e)} \) are unitarily equivalent, though the decompositions of \( \varphi^e \) and \( \varphi^{(I-e)} \) with respect to \( Ae \) and \( A(I-e) \) has no common component.

### 4. Simple maximal abelian subalgebras and completely rough maximal abelian subalgebras.

In [3] Dixmier introduced the notions of regularity, semi-regularity and singularity for the maximal abelian subalgebra of factor. This section is devoted to the study of the relation between these algebraic properties and rather analytic properties: simplicity and complete roughness, of maximal subalgebras. First we shall slightly generalize the notion defined by Dixmier.

**Definition 4.1.** Let \( M \) be a von Neumann algebra, \( Z \) its center and \( A \) a maximal abelian subalgebra. Let \( P \) be the von Neumann subalgebra of \( M \) generated by all unitaries of \( M \) satisfying the condition \( uAu^{-1} = A \). We call \( A \) regular, semi-regular or singular according to \( P = M, P \cap M = Z \) or \( P = A \).

\(^5\) For a representation \( \varphi \) of \( \mathcal{A} \) and a projection \( e \) of \( \varphi(\mathcal{A}) \), \( \varphi^e \) is the representation of \( \mathcal{A} \) to \( e \mathcal{A} \) defined by \( \varphi^e(x)A = \varphi(x)A \) for every \( A \in \mathcal{A} \) and \( x \in \mathcal{A} \).
**Theorem 4.1.** A simple maximal abelian subalgebra is singular.

**Proof.** Suppose that a maximal abelian subalgebra $A = L^\infty(\Gamma, \mu)$ of a von Neumann algebra $M$ is not singular. Then there exists a unitary $u$ of $M$ such that $uAu^{-1} = A$ and $u \notin A$. By the maximality of $A$, $u$ does not commute with some element of $A$, so that $u$ induces a nontrivial automorphism $\theta$ of $A$. Hence the associated mapping $\Theta$ of $\Gamma$ onto $\Gamma'$ is not trivial, that is, there exists a Borel set $E$ such that $\mu(E) > 0$ and $\Theta(\gamma) \neq \gamma$ for every $\gamma \in E$. By Theorem 1.2 we have $\mathcal{H}_{\Sigma, A}^u(\gamma, \Theta(\gamma))$ for almost every $\gamma \in E$, so that $\mathcal{H}_{\Sigma, A}^u(\gamma, \gamma')$ does not imply $\gamma = \gamma'$. Hence $A$ is not simple.

**Theorem 4.2.** If there exists a semi-regular smooth maximal abelian subalgebra in a von Neumann algebra $M$, then $M$ is of type $I$.

**Proof.** Let $A = L^\infty(\Gamma, \mu)$. Let $G$ be a countable group of unitaries satisfying the condition $uAu^{-1} = A$ which generates $P$. The existence of such group is guaranteed by the separability condition for $M$. By the countability of $G$ we may assume that the associated automorphism $\Theta_u$ in $(\Gamma, \mu)$ is defined over the whole space $\Gamma$ for all $u \in G$ by elimination of some null Borel set from $\Gamma$.

Replacing the measure $\mu$ with an equivalent finite one, we assume that $\mu$ is a finite measure. We shall use the notations in the proof of 2° of Theorem 3.2. As in the proof of Theorem 3.2, we decompose $\mu$ over the measure space $(\Gamma, \mu)$ as follows; $\mu = \int \mu^x d\mu(\gamma)$ and $\mu^x$ is concentrated in $r^{-1}(\gamma)$ for almost every $\gamma$. Putting

$$\mathcal{D}(\mathfrak{A}) = \int_{r^{-1}(\gamma)} \mathcal{D}(\gamma) d\mu(\gamma) \quad \text{and} \quad x(\gamma) = \int_{r^{-1}(\gamma)} x(\gamma) d\mu(\gamma)$$

for each $x \in \mathfrak{A}$, we have

$$\mathcal{D}(\gamma) = \mathcal{D}(\gamma) \otimes L^\infty(\gamma^{-1}(\gamma), \mu^x), \quad x(\gamma) = x(\gamma) \otimes I$$

for some $\gamma \in r^{-1}(\gamma)$ and

$$\mathfrak{D} = \int_{r} \mathcal{D}(\gamma) d\mu(\gamma), \quad x = \int_{r} x(\gamma) d\mu(\gamma)$$

for each $x \in \mathfrak{A}$ as in the proof of Theorem 3.2. Since $A$ is maximal abelian in $M$, $\{x(\gamma) : x \in \mathfrak{A}\}$ acts irreducibly on $\mathcal{D}(\gamma)$ for almost every $\gamma \in \Gamma$, so that $\{x(\gamma) : x \in \mathfrak{A}\}$ generates a factor of type I on $\mathcal{D}(\gamma)$. The associated diagonal algebra in the new decomposition of $\mathcal{D}$ becomes $\hat{A} = L^\infty(\Gamma, \tilde{\mu})$. On the other hand, $\Theta_u, u \in G$, transforms each equivalence class onto itself, so that every element of $A$ commutes with $u \in G$. Hence we have $A = Z$, since $P \cap M = Z$. Therefore the new decomposition of $\mathcal{D}$ induces the central decomposition $M' = \int_{r} M(\gamma) d\mu(\gamma)$ of $M$. Since $M(\gamma)$ is generated by $\{x(\gamma) : x \in \mathfrak{A}\}$, $M(\gamma)$ is of
type I. Hence $M$ is of type I by [2: Chap. II, § 3 Prop. 3], so that $M$ is of type I. This completes the proof.

**COROLLARY.** Every semi-regular maximal abelian subalgebra $A$ of a von Neumann algebra $M$ of continuous type is completely rough.

**PROOF.** Let $A = L^∞(Γ, μ)$. Suppose that there exists a non-zero projection $e$ of $A$ such that $Ae$ is a smooth maximal abelian subalgebra of $eM$. Let $E$ be the Borel subset of $Γ$ associated with $e$. Then we have $Ae = L^∞(E, μ)$. Denote $R = R_{e, e}^{Γ, μ}$ and $R^* = R_{e, e}^{Γ, μ}$ in $Γ$ and $E$ respectively. By Lemma 2.1 there exists a Borel subset $S ⊂ E$ which has one and only one element in common with each $R^*$-equivalence class, eliminating a null set.

Now let $G$ and $\{Θ_u : u ∈ G\}$ be the groups of unitaries of $M$ and of transformations in $(Γ, μ)$ defined in the proof of Theorem 4.2 respectively. Putting $U = U_{e}^{Γ, μ}$ and $φ = φ_{e}^{Γ, μ}$ in $Γ$ and $E$ respectively. By Lemma 2.1 there exists a Borel subset $S ⊂ E$ which has one and only one element in common with each $R^*$-equivalence class, eliminating a null set.

Now let $G$ and $\{Θ_u : u ∈ G\}$ be the groups of unitaries of $M$ and of transformations in $(Γ, μ)$ defined in the proof of Theorem 4.2 respectively. Putting $U = U_{e}^{Γ, μ}$ and $φ = φ_{e}^{Γ, μ}$ in $Γ$ and $E$ respectively. By Lemma 2.1 there exists a Borel subset $S ⊂ E$ which has one and only one element in common with each $R^*$-equivalence class, eliminating a null set.

Now let $G$ and $\{Θ_u : u ∈ G\}$ be the groups of unitaries of $M$ and of transformations in $(Γ, μ)$ defined in the proof of Theorem 4.2 respectively. Putting $U = U_{e}^{Γ, μ}$ and $φ = φ_{e}^{Γ, μ}$ in $Γ$ and $E$ respectively. By Lemma 2.1 there exists a Borel subset $S ⊂ E$ which has one and only one element in common with each $R^*$-equivalence class, eliminating a null set.

Now let $G$ and $\{Θ_u : u ∈ G\}$ be the groups of unitaries of $M$ and of transformations in $(Γ, μ)$ defined in the proof of Theorem 4.2 respectively. Putting $U = U_{e}^{Γ, μ}$ and $φ = φ_{e}^{Γ, μ}$ in $Γ$ and $E$ respectively. By Lemma 2.1 there exists a Borel subset $S ⊂ E$ which has one and only one element in common with each $R^*$-equivalence class, eliminating a null set.

Now let $G$ and $\{Θ_u : u ∈ G\}$ be the groups of unitaries of $M$ and of transformations in $(Γ, μ)$ defined in the proof of Theorem 4.2 respectively. Putting $U = U_{e}^{Γ, μ}$ and $φ = φ_{e}^{Γ, μ}$ in $Γ$ and $E$ respectively. By Lemma 2.1 there exists a Borel subset $S ⊂ E$ which has one and only one element in common with each $R^*$-equivalence class, eliminating a null set.

Now let $G$ and $\{Θ_u : u ∈ G\}$ be the groups of unitaries of $M$ and of transformations in $(Γ, μ)$ defined in the proof of Theorem 4.2 respectively. Putting $U = U_{e}^{Γ, μ}$ and $φ = φ_{e}^{Γ, μ}$ in $Γ$ and $E$ respectively. By Lemma 2.1 there exists a Borel subset $S ⊂ E$ which has one and only one element in common with each $R^*$-equivalence class, eliminating a null set.
Borel mapping of $F_1/\mathfrak{H}_1 = \hat{F}_1$ onto $\hat{F}_2$ because of $r_i^{-1}\phi^{-1}(S_1) = f^{-1}(S_2)$ for each Borel set $S_1 \subseteq \hat{F}_1$, where $r_1$ means the canonical mapping of $F_1$ onto $\hat{F}_1$. For each Borel set $S_1 \subseteq \hat{F}_1$, $\phi(S_1) = f(r_i^{-1}(S_1))$ is analytic in $\hat{F}_2$ and $\phi(S_1)$ are complementary subsets of $\hat{F}_2$, so that $\phi$ becomes a Borel isomorphism of $\hat{F}_1$ onto $\hat{F}_2$. Hence $\hat{F}_1$ is a standard Borel space. Since $F_1$ is analytic, there exists a relatively Borel null set $N_1 \subseteq F_1$ such that $F_1 - N_1$ is standard, that is, $F_1 - N_1$ is a Borel subset of $\Gamma$, $(F_1 - N_1)/\mathfrak{H}_1 = (F_1 - N_1)$ is analytic since $(F_1 - N_1) = r_1(F_1 - N_1) \subseteq \hat{F}_1$. Hence if $f_1$ is the projection of $A_1$ associated with $F_1 - N_1$, then $A_1f_1$ becomes smooth in $f_1Mf_1$, but $f_1$ does not vanish by the definitions of $F_1$ and $F_2$. This contradicts to the complete roughness of $A_1$ in $M$.

Combining Corollary of Theorem 4.2 and Theorem 4.3 we assert the following

**Corollary.** In the von Neumann algebra of continuous type, a smooth maximal abelian subalgebra and a semi-regular one are unrelated.

**Theorem 4.4.** A smooth singular maximal abelian subalgebra is simple.

**Proof.** Let $A$ be a smooth singular maximal abelian subalgebra of a von Neumann algebra $M$. Let $\{p_n\}_{n=0,1,\ldots,\infty}$ be the family of projections appeared in Theorem 3.2. If $p_n \neq 0$ for some $n \neq 1$, it is clear that there exists a unitary $u_n$ of $p_nM_{p_n}$ such that $u_nA_pu_n^{-1} = A_{p_n}$ and $u_n \notin A_{p_n}$. Putting $u = u_n + (I - p_n)$, then $uAu^{-1} = A, u \in M$ and $u \notin A$. Hence $A$ is not singular. Therefore we have $p_n = 0$ for each $n \neq 1$, which implies the simplicity of $A$.

Throughout the discussion of § 3 and § 4 the following natural questions arise for us: Are there algebraic characterizations of simple, smooth or completely rough maximal abelian subalgebra? In particular, is any singular maximal abelian subalgebra simple? Indeed, as shown in the next §, every already known example of singular maximal abelian subalgebra is simple.

5. **Examples.** In [8, Chap. III § 5] Mackey gave an example of unrelated pair of maximal abelian subalgebras in a factor of type $\Pi_1$ which consists of simple one and regular. Besides his arguments show that the example of singular maximal abelian subalgebra of hyperfinite factor constructed by Dixmier [3] is simple. So in this section we shall give an example of simple maximal abelian subalgebra in a factor of type $\Pi_3$ by showing the example of singular one in a factor of type $\Pi_3$ constructed by Pukanszky [12] to be simple.

Let $G$ be an arbitrary countably infinite discrete abelian group. For each element $g \in G$ we associate the cyclic group $\Omega_g = \{0, 1\}$ of order 2. By $\Omega$ we denote the product compact group of $\{\Omega_g; g \in G\}$. $\Delta$ is the subgroup of $\Omega$ composed of the element $\alpha$ such that $\alpha(g) = 0$ except for a finite member of $g$'s. For $0 < p \leq 1/2$ we define the measure $\mu_\alpha$ in $\Omega_\alpha$ by $\mu_\alpha(\{0\}) = p$ and
\( \mu_s(\{1\}) = 1 - p \) and the measure \( \mu \) in \( \Omega \) by \( \mu = \prod_{g \in G} \mu_g \). For \( g_0 \in G \) we define an automorphism of \( \Omega \) by \( \omega_0(g) = \omega(g_0 g) \). Putting \( \Theta = G \times \Delta \), we define the product in \( \Theta \) by \( (g_1, \alpha_1)(g_2, \alpha_2) = (g_1 g_2, \alpha_1 + \alpha_2) \). We canonically identify \( G \) and \( \Delta \) with \( G \times \{0\} \) and \( \{e\} \times \Delta \) respectively. Next we define the action of \( \Theta \) on \( \Omega \) by \( \omega s = \omega s + \alpha \) for \( s = g \alpha \in \Theta \). Then the measure \( \mu \) becomes quasi-invariant under the action of \( \Theta \) by \[ \frac{d\mu}{d\mu_s}(\omega) = \rho(\omega, s), \]
where \( \mu_s \) means the measure defined by \( \mu_s(E) = \mu(Es) \), we have \( \rho(\omega, g \alpha) = \rho(\omega g, \alpha) \) for \( g \in G \) and \( \alpha \in \Delta \).

Let \( \delta_\Delta = L^2(\Gamma \times \Delta, \mu \times \delta) \), where \( \delta \) is the discrete measure in \( \Delta \). Let \( \Gamma \) be the dual group of \( G \) with Haar measure \( \nu \). For each \( \gamma \in \Gamma \) and \( \xi \in \delta_\Delta \), defining
\[
(\mathcal{U}_\Delta(g \alpha) \xi)(\omega, \beta) = \gamma(g) \rho(\omega, \alpha)^{1/2} \xi(\omega^g + \alpha, \beta^g + \alpha) \quad \text{for } g \in G \text{ and } \alpha \in \Delta,
\]
\[
(\mathcal{I}_\Delta(a) \xi)(\omega, \beta) = a(\omega) \xi(\omega, \beta) \quad \text{for } a \in L^\infty(\Omega, \mu),
\]
and
\[
(m_\Delta(a) \xi)(\omega, \beta) = a(\omega - \beta) \xi(\omega, \beta) \quad \text{for } a \in L^\infty(\Omega, \mu),
\]
we get bounded operators \( \mathcal{U}_\Delta(s) \), \( \mathcal{I}_\Delta(a) \), \( \mathcal{I}_\Delta(\alpha) \) and \( m_\Delta(a) \) on \( \delta_\Delta \) for \( s \in \Theta \), \( \alpha \in \Delta \) and \( a \in L^\infty(\Omega, \mu) \). Besides \( \mathcal{U}_\Delta(s) \) becomes a strongly continuous operator valued function over \( \Gamma \) and \( \mathcal{I}_\Delta(a) \) becomes a constant function, so that we can define operators \( \vartheta(s) \) and \( l(a) \) on \( \delta = \delta_\Delta \otimes L^2(\Gamma, \nu) \) by
\[
\vartheta(s) = \int_\Gamma \mathcal{U}_\Delta(s) d\nu(\gamma) \quad \text{and} \quad l(a) = \int_\Gamma \mathcal{I}_\Delta(a) d\nu(\gamma).
\]
Of course, we have \( l(a) = \mathcal{I}_\Delta(a) \otimes I \).

Let \( U^\gamma \) be the unitary representation of \( \Theta \) induced by the one-dimensional representation \( \gamma \) of the subgroup \( G \) and let \( u \) be the unitary representation of \( \Theta \) on \( L^2(\Omega, \mu) \) defined by
\[
(u(s) \xi)(\omega) = \rho(\omega, s)^{1/2} \xi(\omega s) \quad \text{for } s \in \Theta \text{ and } \xi \in L^2(\Omega, \mu).
\]
Then we have, for \( s \in \Theta \) and \( a \in L^\infty(\Omega, \mu), \)
\[
u(s) = u(s) \otimes U^\gamma(s) \quad \text{and} \quad \mathcal{I}_\Delta(a) = a \otimes I \text{ on } \delta_\Delta = L^2(\Omega, \mu) \otimes L^2(\Delta).
\]
Since
\[
\delta = \delta_\Delta \otimes L^2(\Gamma, \nu) = \delta_\Delta \otimes L^2(G) = L^2(\Omega, \mu) \otimes L^2(\Theta)
\]
under the natural identification and the right regular representation \( R \) of \( \Theta \) is decomposed into the direct integral \( R = \int_\Gamma U^\gamma(s) d\nu(\gamma) \) by \[ 7 : \text{Cor. of Theorem} \]
10.1], we have
\[ u(s) = u(s) \otimes R(s) \] and \[ l(a) = a \otimes I \] on \( \mathfrak{G} = L^2(\Omega, \mu) \otimes P(\emptyset) \), for \( s \in \emptyset \) and \( a \in L^\infty(\Omega, \mu) \). Since the diagonal algebra in the decomposition \( R = \int_{\Gamma}^\emptyset U^* d\nu(y) \) is generated by the image of \( G \) under the left regular representation of \( \emptyset \), the diagonal algebra \( \mathcal{A} \) in the decomposition \( u(s) = \int_{\Gamma}^\emptyset u_\emptyset(s) \, d\nu(y) \) is generated by the image of \( G \) under the representation \( \nu \) of \( \emptyset \) defined by
\[ (\nu(s_\emptyset) \xi)(\omega, s) = \xi(\omega, s_\emptyset) \] for \( s_\emptyset \in \emptyset \) and \( \xi \in L^2(\Omega \times \emptyset, \mu \times \delta) \). Let \( M_\emptyset \) be the von Neumann algebra generated by \( \{ \nu_\emptyset(\alpha), m_\emptyset(\alpha) : \alpha \in \Delta, a \in L^\infty(\Omega, \mu) \} \). Then for every \( \gamma \in \Gamma \{ u_\emptyset^\emptyset(\alpha), l_\emptyset^\emptyset(\alpha) : \alpha \in \Delta \} \) generates \( M_\emptyset \). Let \( M \) be the von Neumann algebra acting on \( \mathfrak{G} \) generated by \( \{ \nu(s), m(a) : s \in \emptyset \) and \( a \in L^\infty(\Omega, \mu) \} \), where \( m(a) \) is defined by \( (m(a) \xi)(\omega, s) = a'(\omega s^{-1}) \xi(\omega, s) \) for \( \xi \in \mathfrak{G} = L^2(\Omega \times \emptyset, \mu \times \delta) \). Then \( M \) and \( M_\emptyset \) become a factor of type II or type III according to the choice of \( p \) and \( A \) is a singular maximal abelian subalgebra of \( M \) by [12]. We shall show that \( A \) is simple.

Let \( \mathfrak{B} \) be the \( C^\star \)-subalgebra of \( M \) generated by \( \{ u(s), l(a) : s \in \emptyset \) and \( a \in C(\Omega) \} \). Then \( \mathfrak{B} \) is a uniformly separable weakly dense subalgebra of \( M \) by [12]. For every \( \gamma_1 \) and \( \gamma_2 \) of \( \Gamma \) suppose that there exists a bounded operator \( x \) on \( \mathfrak{G} \), such that
\[ u_\emptyset^\emptyset(s)x = xu_\emptyset^\emptyset(s) \] and \[ l_\emptyset^\emptyset(a)x = xl_\emptyset^\emptyset(a) \]
for every \( s \in \emptyset \) and for every \( a \in C(\Omega) \). Then \( x \) belongs to \( M_\emptyset \), so that \( x \) can be expressed by \( x = \sum_{a \in \Delta} m_\emptyset(x_a) x(\alpha) \) in the strong operator topology. For each \( g \in G \) we have
\[ (u_\emptyset^\emptyset(g)x \xi)(\omega, \alpha) = \gamma_1(g) x \xi(\omega^g, \alpha^g) \]
\[ = \gamma_1(g) \sum_{\beta \in \Delta} x_\beta(\omega - \alpha) \xi(\omega^g, \alpha^g - \beta) \]
and
\[ (xu_\emptyset^\emptyset(g) \xi)(\omega, \alpha) = \sum_{\beta \in \Delta} x_\beta(\omega - \alpha) (u_\emptyset^\emptyset(g) \xi)(\omega, \alpha - \beta) \]
\[ = \gamma_1(g) \sum_{\beta \in \Delta} x_\beta(\omega - \alpha) \xi(\omega^g, \alpha^g - \beta^g) \]
for every \( \xi \in \mathfrak{G}_\emptyset \). Hence we have
\[ \gamma_1(g) \sum_{\beta \in \Delta} x_\beta(\omega^g - \alpha^g) \xi(\omega^g, \alpha^g - \beta) \]
for every $\xi \in \mathcal{D}_\Delta$. Putting $\xi_o(\omega, \alpha) = 1$ if $\alpha = 0 = 0$, if $\alpha \neq 0$ we have

$$\gamma_1(g)x_n^\alpha(\omega)^2 = \gamma_1(g)x_n(\omega)$$

for every $\alpha \in \Delta$, $g \in G$ and for almost every $\omega \in \Omega$. Hence

$$\gamma_1(g)x_n^\alpha(\omega)^2 = \gamma_1(g)x_n(\omega).$$

for every $\alpha \in \Delta$, $g \in G$ and for almost every $\omega \in \Omega$. It follows that

$$\int_\Omega |x_n^\alpha(\omega)|^2 d\mu(\omega) = \int_\Omega |x_n(\omega)|^2 d\mu(\omega) = \int_\Omega |x_n(\omega)|^2 d\mu(\omega).$$

Since $\sum_{\alpha \in \Delta} \int_\Omega |x_n(\omega)|^2 d\mu(\omega) = (xx^\#\xi_o, \xi_o) < +\infty$ and the set $\{\alpha^o : g \in G\}$ has infinitely many elements if $\alpha \neq 0$, we have $x_n(\omega) = 0$ almost everywhere for $\alpha \neq 0$. Putting

$$\xi_n(\omega) = (\mp 1)\sum_{g \in G}(p/(1-p))$$

for $\alpha \in \Delta$, $\{\xi_n; \alpha \in \Delta\}$ becomes a complete orthonormalized system of $L^2(\Omega, \mu)$ by [12: Lemma 4]. Putting

$$c_\alpha = (\varepsilon, \xi_n) = \int_\Omega x_n(\omega)\xi_n(\omega)d\mu(\omega),$$

we have

$$c_\alpha^o = \int_\Omega x_n(\omega)\xi_n^\#(\omega)d\mu(\omega) = \int_\Omega x_n(\omega)\xi_n(\omega)\xi_n(\omega)d\mu(\omega)$$

$$= \int_\Omega x_n(\omega)\xi_n(\omega)d\mu(\omega) = \int_\Omega \gamma_1(g) x_n(\omega)\xi_n(\omega)d\mu(\omega)$$

which implies $|c_\alpha^o| = |c_\alpha|$. Hence $c_\alpha = 0$ if $\alpha \neq 0$. This means that $x_n(\omega)$ is a constant. That is, $x$ becomes a scalar. Therefore, if $\gamma_1$ and $\gamma_2$ are different characters of $G$ then $x = 0$. If $\gamma_1 = \gamma_2$, say $\gamma$, then $\{u_\gamma(s)\}$ and $\{d_\gamma(a)(\alpha) : s \in \mathcal{S}$ and $a \in C(\Omega)\}$, which generates the $\gamma$-component of $\mathfrak{H}$, acts irreducibly on $\mathcal{D}_\Delta$. Hence $\mathfrak{H}_\gamma$ acts irreducibly on $\mathfrak{D}_\Delta$. Hence $\mathfrak{H}_\gamma^\Delta(\gamma_1, \gamma_2)$ implies $\gamma_1 = \gamma_2$ for $\gamma_1, \gamma_2 \in \Gamma$, that is, $\mathfrak{A}$ is a simple maximal abelian subalgebra of $\mathfrak{M}$, where $\Phi$ means the set of representations of $\mathfrak{H}$ defined by its $\gamma$'s-components.

After all, we get the following
Theorem 5.1. Hyperfinite factor has a simple maximal abelian subalgebra and a completely rough one simultaneously. There exists a factor of type \( \text{III} \) which has a simple maximal abelian subalgebra and a completely rough one simultaneously.

References