ON THE UNITARY EQUIVALENCE AMONG THE COMPONENTS OF DECOMPOSITIONS OF REPRESENTATIONS OF INVOLUTIVE BANACH ALGEBRAS AND THE ASSOCIATED DIAGONAL ALGEBRAS

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(Received September 2, 1963)

Introduction. Let $\varphi = \int_\Gamma^\oplus \varphi(\gamma) d\mu(\gamma)$ be an irreducible decomposition of a representation φ of an involutive Banach algebra $\mathfrak{B}$ over a measure space (Γ, μ). As shown by several authors in [4], [8], [9], [13] etc., this decomposition cannot be regarded as a decomposition of the unitary equivalence class of φ into the unitary equivalence classes of φ (γ) except for some fairly nice cases, whereas this decompositon is determined only up to unitary equivalence. For instance, some representations can be decomposed in two ways that have no common components as in [8] and some two representations of quite different types can be decomposed into the direct integrals of the same components as in [13]. Therefore it comes into considerations what determines the unitary equivalence relation among the components $\{\varphi(\gamma) : \gamma \in \Gamma\}$ of the decomposition $\varphi = \int_\Gamma^\oplus \varphi(\gamma) d\mu(\gamma)$. For this question we shall answer in § 1 that the algebraic relation between the commutant $\varphi(\mathfrak{R})' = \boldsymbol{M}$ of $\varphi(\mathfrak{B})$ and the associated diagonal algebra $\boldsymbol{A}$ determines completely the unitary equivalence relation $\mathfrak{R}$ among the components $\{\varphi(\gamma) : \gamma \in \Gamma\}$. So we can regard $\mathfrak{R}$ as an algebraic invariant of the couple $(\boldsymbol{M}, \boldsymbol{A})$. A. Guichardet used $\mathfrak{R}$ for characterization of discrete von Neumann algebras in [5]. We study the behavior of $\mathfrak{R}$ in more general situation. In § 2 we shall give the definitions of simplicity, smoothness and complete roughness of $\boldsymbol{A}$ in $\boldsymbol{M}$ using $\mathfrak{R}$. In § 3 we shall reduce the study of smooth maximal abelian subalgebras to that of simple ones. § 4 is devoted to show some relations between simple or completely rough maximal abelian subalgebras and regular, semi-regular or singular ones defined in [3]. Finally in § 5 we shall give some examples of factors of type II and type III with simple maximal abelian subalgebras and completely rough ones simultaneously respectively.

1. Unitary equivalence relation. Let Γ be a standard Borel space[1] and μ a Borel measure on Γ. Let $\boldsymbol{A} = L^\infty(\Gamma, \mu)$ be the commutative von Neumann

1) If a Borel space $(\Gamma, \mathscr{B})$ is Borel isomorphic to some separable complete metric space equipped with the Borel structure generated by closed sets, then we call it standard according to Mackey[9]. Calling the member of $\mathscr{B}$ Borel set, we shall omit the letter $\mathscr{B}$.

algebra consisting of all essentially bounded measurable functions over the measure space (Γ, μ). Suppose that $\boldsymbol{A}$ is imbedded in a von Neumann algebra $\boldsymbol{M}$ as a von Neumann subalgebra and that $\boldsymbol{M}$ has a faithful representation on a separable Hilbert space. Let π be a normal faithful representation of $\boldsymbol{M}$ onto a separable Hilbert space $\mathfrak{H}_\pi$ and let $\boldsymbol{M}'_\pi$ be the commutant algebra of $\pi(\boldsymbol{M})$. Then wg eet a decomposition

$$\mathfrak{H}_\pi = \int_\Gamma^{\oplus} \mathfrak{H}_\pi(\gamma)\, d\mu(\gamma)$$

of $\mathfrak{H}_\pi$ over the measure space (Γ, μ) relative to $\pi(\boldsymbol{A})$. $\pi(\boldsymbol{A})$ becomes the algebra of all diagonalizable operators which is called the (associated) diagonal algebra and each operator in $\pi(\boldsymbol{A})'$ is decomposable. Let $\mathfrak{A}$ be a uniformly separable C^*-algebra which is weakly dense. in $\boldsymbol{M}'_\pi$. Then for $\mathfrak{A}$ we can associate a family $\{\varphi_\gamma\}$ of representations of $\mathfrak{A}$ in $\mathfrak{H}_\pi(\gamma)$ such that

$$x = \int_\Gamma^{\oplus} \varphi_\gamma(x) d\mu(\gamma) \qquad \text{for every } x \in \mathfrak{A}\,.$$

Besides, we can choose the family $\{\varphi_\gamma\}$ as follows; the function $\gamma \to (\varphi_\gamma(x)\,\xi(\gamma), \eta(\gamma))$ is Borel measurable over Γ for every $x \in \mathfrak{A}$ and for every pair of $\xi = \int_\Gamma^{\oplus} \xi(\gamma) d\mu(\gamma)$, $\eta = \int_\Gamma^{\oplus} \eta(\gamma) d\mu(\gamma) \in \mathfrak{H}_\pi$. We denote such family $\{\varphi_\gamma\}$ by Φ. The family$\{\mathfrak{H}_\pi(\gamma) : \gamma \in \Gamma\}$ of Hilbert spaces and the family Φ are determined almost everywhere by the $\mathfrak{H}_\pi$ and the diagonal algebra $\pi(\boldsymbol{A})$. Indeed, if $\mathfrak{H}_\pi$ is represented by a decomposition $\mathfrak{H}_\pi = \int_\Gamma^{\oplus} \mathfrak{H}'_\pi(\gamma)\ d\mu(\gamma)$ with respect to $\pi(\boldsymbol{A})$ and if Φ' is another associated family of representations of $\mathfrak{A}$, then there exists a null set $N \subset \Gamma$ and a family $\{u_\gamma : \gamma \in \complement N\}$[2)] of unitary operators of $\mathfrak{H}_\pi(\gamma)$ onto $\mathfrak{H}_\pi(\gamma')$ such that $u_\gamma \varphi_\gamma u_\gamma^{-1} = \varphi'_\gamma$ for every $\gamma \in \complement N$.

Suppose that $\boldsymbol{A}_i = L^\infty(\Gamma_i, \mu_i)$ $(i = 1, 2)$ is a von Neumann subalgebra of $\boldsymbol{M}$, where (Γ_i, μ_i) $(i = 1, 2)$ are measure spaces as well as (Γ, μ). Then we get two decompositions

$$\mathfrak{H}_\pi = \int_{\Gamma_1}^{\oplus} \mathfrak{H}_\pi^1(\gamma_1)\ d\mu_1(\gamma_1) \quad \text{and} \quad \mathfrak{H}_\pi = \int_{\Gamma_2}^{\oplus} \mathfrak{H}_\pi^2(\gamma_2) d\mu_2(\gamma_2)$$

of $\mathfrak{H}_\pi$ over (Γ_1, μ_1) and (Γ_2, μ_2) relative to $\pi(\boldsymbol{A}_1)$ and $\pi(\boldsymbol{A}_2)$ respectively and we fix these decompositions of $\mathfrak{H}_\pi$. Let $\Phi^1 = \{\varphi^1_{\gamma_1}\}$ and $\Phi^2 = \{\varphi^2_{\gamma_2}\}$ be families of representations of the C^*-algebra $\mathfrak{A}$ which are given by the decompositions of $\mathfrak{H}_\pi$ as well as Φ. We define a relation $\mathfrak{R}^{M,\pi,\mathfrak{A},\Phi^1,\Phi^2}_{A_1,A_2}$ between the points of Γ_1 and Γ_2 as follows; $\mathfrak{R}^{M,\pi,\mathfrak{A},\Phi^1,\Phi^2}_{A_1,A_2}(\gamma_1, \gamma_2)$ holds if and only if the representations $\varphi^1_{\gamma_1}$ and $\varphi^2_{\gamma_2}$ of $\mathfrak{A}$ are unitarily equivalent.

Now let $\mathfrak{R}$ and $\mathfrak{R}'$ be two relations between the points of Γ_1 and Γ_2. We

2) $\complement N$ denotes the Complement of N in Γ.

define a relation $\mathfrak{R} \equiv \mathfrak{R}'$ by the fact that there exist subsets E_i of Γ_i $(i = 1, 2)$ with null complements such that $\mathfrak{R}(\gamma_1, \gamma_2)$ is equivalent to $\mathfrak{R}'(\gamma_1, \gamma_2)$ for every $(\gamma_1, \gamma_2) \in E_1 \times E_2$. Clearly this relation" $\equiv$ " is an equivalence relation. We denoe the equivalence class of $\mathfrak{R}$ under the relation " $\equiv$ " by $\hat{\mathfrak{R}}$.

LEMMA 1.1. *$\hat{\mathfrak{R}}^{M,\pi,\mathfrak{A},\Phi^1,\Phi^2}_{A_1,A_2}$ depends on neither Φ^1 and Φ^2 nor weakly dense uniformly separable C^*-subalgebra $\mathfrak{A}$ of M'_π. That is, for two weakly dense uniformly separable C^*-sublalgebras $\mathfrak{A}$ and $\mathfrak{B}$ of M'_π and for associated familis Φ^i and Ψ^i $(i = 1, 2)$ of representations of $\mathfrak{A}$ and $\mathfrak{A}_i$ $(i = 1, 2)$, there exist Borel subsets E_i of Γ_i $(l = 1, 2)$ with null complements such that $\mathfrak{R}^{M,\pi,\mathfrak{B},\Psi^1\Psi^2}_{A_1,A_2}$ (γ_1, γ_2) and $\mathfrak{R}^{M,\pi,\mathfrak{A},\Phi^1,\Phi^2}_{A_1,A_2}$ (γ_1, γ_2) are equivalent for every $(\gamma_1, \gamma_2) \in E_1 \times E_2$.*

PROOF. Let $\mathfrak{A}_0$ be countable uniformly dense subalgebra of $\mathfrak{A}$ over the rational complex number field $\boldsymbol{C}_0$. Then $\mathfrak{R}^{M,\pi,\mathfrak{A},\Phi^1,\Phi^2}_{A_1,A_2}$ (γ_1, γ_2) is equivalent to the fact that there exists a unitary u of $\mathfrak{H}^1_\pi(\gamma_1)$ onto $\mathfrak{H}^2_\pi(\gamma_2)$ such that $u\ \varphi^1_{\gamma_1}(x)u^{-1} = \varphi^2_{\gamma_2}(x)$ for every $x \in \mathfrak{A}_0$. Let $\{x_n\}$ be an enumeration of $\mathfrak{A}_0$. For each n there exists a sequence $\{y_{n,m}\}$ in $\mathfrak{B}$ that converges strongly to x_n. By [2 : Chap. II, § 2, no 3 Prop. 4], there exist a subsequence $\{y_{n,m_k}\}$ and null sets $N_i^n \subset \Gamma_i$ $(i = 1, 2)$ such that

$$\varphi^i_{\gamma_i}(x_n) = \text{strong-}\lim_{k\to\infty} \psi^i_{\gamma_i}(y_{n,m_k}) \quad \text{for every } \gamma_i \in \complement N_i^n \ (i = 1, 2).$$

Put $N_i = \bigcup_{n=1}^{\infty} N_i^n$ $(i = 1, 2)$. Suppose that $\mathfrak{R}^{M,\pi,\mathfrak{B},\Psi^1,\Psi^2}_{A_1,A_2}$ (γ_1, γ_2) holds for $(\gamma_1, \gamma_2) \in \complement N_1 \times \complement N_2$. Then there exirts a unitary u of $\mathfrak{H}^1_\pi(\gamma_1)$ onto $\mathfrak{H}^2_\pi(\gamma_2)$ such that $u\ \psi^1_{\gamma_1}(y_{n.m_k})u^{-1} = \psi^2_{\gamma_2}(y_{n,m_k})$ for all n and k, which implies $u\ \varphi^1_{\gamma_1}(x_n)u^{-1} = \varphi^2_{\gamma_2}(x_n)$ for all n. Hence $\mathfrak{R}^{M,\pi,\mathfrak{A},\Phi^1,\Phi^2}_{A_1,A_2}$ (γ_1, γ_2) holds. By symmetry $\mathfrak{R}^{M,\pi,\mathfrak{A},\Phi^1,\Phi^2}_{A_1,A_2}$ and $\mathfrak{R}^{M,\pi,\mathfrak{B},\Psi^1,\Psi^2}_{A_1,A_2}$ are equivalent on $E_1 \times E_1$ for some subsets $E_1 \subset \Gamma_1$ and $E_2 \subset \Gamma_2$ with null complements.

According to Lemma 1.1, we can denote $\mathfrak{R}^{M,\pi,\mathfrak{A},\Phi^1,\Phi,^2}_{A_1,A_2}$ by $\mathfrak{R}^{M,\pi}_{A_1,A_2}$ and $\varphi^1_{\gamma_1}(x)$ by $x^1(\gamma_1)$ without the indication of the family Φ^1.

Let ι be the trivial representation of the scalar field $\boldsymbol{C}$ onto countably infinite dimensional Hilbert space $\mathfrak{H}_\infty$. For a representation π of $\boldsymbol{M}$ we define a representation $\pi \otimes \iota$ onto $\mathfrak{H}_\pi \otimes \mathfrak{H}_\infty$ by $(\pi \otimes \iota)(x)(\xi \otimes \eta) = (\pi(x)\xi) \otimes \eta$ for $x \in \boldsymbol{M}$, $\xi \in \mathfrak{H}_\pi$ and $\eta \in \mathfrak{H}_\infty$.

LEMMA 1.2. $$\widehat{\mathfrak{R}^{M,\pi}_{A_1,A_2}} = \widehat{\mathfrak{R}^{M,\pi\otimes\iota}_{A_1,A_2}}.$$

PROOF. Let $\mathfrak{A}$ and $\mathfrak{B}$ be uniformly separable weakly dense C^*-subalgebras of M'_π and$\boldsymbol{B}(\mathfrak{H}_\infty)$ with units respectively. Then the uniform closure $\mathfrak{C}$ of the set consiting of all $\sum_{i=1}^n x_i \otimes y_i$'s $x_i \in \mathfrak{A}$,, $y_i \in \mathfrak{B}$, which is $\mathfrak{A}\hat{\otimes}_\alpha\mathfrak{B}$ in the sense of Turumaru [14], is also uniformly separable weakly dense C^*-subalgebra of $M'_{\pi\otimes\iota} = M'_\pi \otimes \boldsymbol{B}(\mathfrak{H}_\infty)$.From $\mathfrak{H}_{\pi\otimes\iota} = \mathfrak{H}_\pi \otimes \mathfrak{H}_\iota$ and $(\pi\otimes\iota)(\boldsymbol{M}) = \pi(\boldsymbol{M}) \otimes \boldsymbol{C}$, we have $\mathfrak{H}^i_{\pi\otimes\iota}(\gamma_i) = \mathfrak{H}^i_\pi(\gamma_i) \otimes \mathfrak{H}_\infty$ and $(x \otimes y)^i(\gamma_i) = x^i(\gamma_i) \otimes y$ for almost every $\gamma_i \in \Gamma_i$

$(i=1,2)$ and for every $x\in\mathfrak{A}$ and $y\in\mathfrak{B}$. Suppose that $\mathfrak{R}^{M,\pi,\mathfrak{A},\Phi^1,\Phi^2}_{A_1,A_2}(\gamma_1,\gamma_2)$ holds, that is, there exists a unitary u of $\mathfrak{H}^1_\pi(\gamma_1)$ onto $\mathfrak{H}^2_\pi(\gamma_2)$ such that $ux^1(\gamma_1)u^{-1}=x^2(\gamma_2)$ for all $x\in\mathfrak{A}$. Putting $v=u\otimes I$, we have $v(x\otimes y)^1(\gamma)v^{-1}=(x\otimes y)^2(\gamma_2)$ for every $x\in\mathfrak{A}$ and $y\in\mathfrak{B}$, which implies $\mathfrak{R}^{M,\pi\otimes\mathfrak{l},\mathfrak{C},\Psi^1,\Psi^2}_{A_1,A_2}(\gamma_1,\gamma_2)$, where Ψ^i is the family of representations $\psi^i_{\gamma_i}$ of $\mathfrak{C}$ defined by $\psi^i_{\gamma_i}(x\otimes y)=\varphi^i_{\gamma_i}(x)\otimes y$ for $x\in\mathfrak{A}$ and $y\in\mathfrak{B}$.

Conversely suppose that $\mathfrak{R}^{M,\pi\otimes\mathfrak{l},\mathfrak{C}\Psi^1\Psi^2}_{A_1,A_2}(\gamma_1,\gamma_2)$ holds. There exists a unitary v of $\mathfrak{H}^1_{\pi\otimes\mathfrak{l}}(\gamma_1)$ onto $\mathfrak{H}^2_{\pi\otimes\mathfrak{l}}(\gamma_2)$ such that $v(x\otimes y)^1(\gamma_1)v^{-1}=(x\otimes y)^2(\gamma_2)$ for all $x\in\mathfrak{A}$ and $y\in\mathfrak{B}$. Taking $x=I$, $v(I\otimes y)v^{-1}=I\otimes y$ for every $y\in\mathfrak{B}$. Hence there exists a unitary u of $\mathfrak{H}^1_\pi(\gamma_1)$ onto $\mathfrak{H}^2_\pi(\gamma_2)$ such that $v=u\otimes I$ by [7 : p. 114, Lemma]. Since $(u\otimes I)(x^1(\gamma_1)\otimes y)(u\otimes I)^{-1}=x^2(\gamma_2)\otimes y$, we have $ux^1(\gamma_1)u^{-1}=x^2(\gamma_2)$. Hence $\mathfrak{R}^{M,\pi,\mathfrak{A},\Phi_1,\Phi^2}_{A_1,A_2}(\gamma_1,\gamma_2)$ holds.

THEOREM 1.1. *Equivalence class of* $\mathfrak{R}^{M,\pi,\mathfrak{A},\Phi^1,\Phi^2}_{A_1,A_2}$ *under the relation "$\equiv$" depends on neither* $\mathfrak{A}$ *nor* π. *That is, for normal faithfull representations* π *and* ρ *of* $\boldsymbol{M}$ *onto* $\mathfrak{H}_\pi$ *and* $\mathfrak{H}_\rho$, *for uniformly separable weakly dense* C^*-*subalgebras* $\mathfrak{A}$ *and* $\mathfrak{B}$ *of* $\boldsymbol{M}'_\pi$ *and* $\boldsymbol{M}'_\rho$ *respectively and for families* Φ^i *and* Ψ^i *of representations of* $\mathfrak{A}$ *and* $\mathfrak{B}$ *associated with the decomposition of* $\mathfrak{H}_\pi$ *and* $\mathfrak{H}_\rho$ *respectively* $(i=1,2)$, *there exist subsets* $E_1\subset\Gamma_1$ *and* $E_2\subset\Gamma_2$ *with null complements such that* $\mathfrak{R}^{M,\pi,\mathfrak{A},\Phi^1,\Phi^2}_{A_1,A_2}(\gamma_1,\gamma_2)$ *and* $\mathfrak{R}^{M,\rho,\mathfrak{B},\Psi^1,\Psi^2}_{A_1,A_2}(\gamma_1,\gamma_2)$ *are equivalent each other for every* $(\gamma_1,\gamma_2)\in E_1\times E_2$.

PROOF. If π and ρ are unitarily equivalent, then Lemma 1.1 assures our mentions. By [8 : p. 22, Lemma], we have $\pi\otimes\mathfrak{l}\simeq\rho\otimes\mathfrak{l}$. Hence Lemma 1.2 assures our theorem.

According to Theorem 1.1, in the notation $\mathfrak{R}^{M,\pi,\mathfrak{A},\Phi^1,\Phi^2}_{A_1,A_2}$ the letter π does not have essential meaning. So we assume the von Neumann algebra $\boldsymbol{M}$ to act on a fixed Hilbert space $\mathfrak{H}$ from the beginning and we can denote $\hat{\mathfrak{R}}^{M,\pi}_{A_1,A_2}$ by $\hat{\mathfrak{R}}^{M}_{A_1,A_2}$. In the following, we denote $\mathfrak{R}^{M,\pi,\mathfrak{A}_2,\Phi_1,\Phi_1}_{A_1,A_2}$ by $\mathfrak{R}^{M,\mathfrak{A},\Phi^1,\Phi^2}_{A_1,A_2}$, where π means the identical representation of $\boldsymbol{M}$. When we consider only one subalgebra $\boldsymbol{A}=L^\infty(\Gamma,\mu)$ of $\boldsymbol{M}$, $\mathfrak{R}^{M,\mathfrak{A},\Phi,\Phi}_{A,A}$ becomes an equivalence relation defined in the measure space (Γ,μ) which is simply denoted by $\mathfrak{R}^{M,\mathfrak{A},\Phi}_{A}$.

Now we shall give the interpretations of Theorem 1.1 to the decomposition theory of representations of involutive Banach algebras. In [13], in order to describe the structure of decompositions of some representations of certain C^*-algebras, at first we have studied the behavior of some special representation φ_0 of some special C^*-algebra $\mathfrak{A}_0$ and next we have investigated the representation φ of C^*-algebra $\mathfrak{A}$ such that $\varphi_0(\mathfrak{A}_0)'\cong\varphi(\mathfrak{A})'$ by comparing the decompositions of φ_0 and φ with respect to the diagonal algebras which are isomorphic under the isomorphism between $\varphi_0(\mathfrak{A}_0)'$ and $\varphi(\mathfrak{A})'$. According to Theorem 1.1, we can see the theoretical back ground of these arguments in [13]. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two separable involutive Banach algebras and let φ and ψ be representations of $\mathfrak{A}$ and $\mathfrak{B}$ onto separable Hilbert spaces $\mathfrak{H}_\varphi$ and $\mathfrak{H}_\psi$ respectively. Suppose

$\varphi(\mathfrak{A})' \simeq \varphi(\mathfrak{B})'$ under an isomorphism θ. That is, there exist a von Neumann algebra $\boldsymbol{M}$ and two normal faithful representations π and ρ such that $\pi(\boldsymbol{M}) = \varphi(\mathfrak{A})'$, $\rho(\boldsymbol{M}) = \psi(\mathfrak{B})'$ and $\sigma = \theta\circ\pi$. Let (Γ_1, μ_1), (Γ_2, μ_2), $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ be as in Theorem 1.1.

Then φ(resp. ψ) is decomposed with respect to $\pi(A_1)$ and $\pi(A_2)$ (resp. $\sigma(A_1)$ and $\rho(A_2)$) as follows;

$$\varphi = \int_{\Gamma_1}^{\oplus} \varphi^1(\gamma_1)d\mu_1(\gamma_1) \quad \text{and} \quad \varphi = \int_{\Gamma_2}^{\oplus} \varphi^2(\gamma_2)d\mu_2(\gamma_2)$$

$$\left(\text{resp. } \psi = \int_{\Gamma_1}^{\oplus} \psi^1(\gamma_1)d\mu_1(\gamma_1) \quad \text{and} \quad \psi = \int_{\Gamma_2}^{\oplus} \psi^2(\gamma_2)d\mu_2(\gamma_2).\right)$$

Then we get the folowing

COROLLARY 1. *There exist null sets $N_1\subset\Gamma_1$ and $N_2\subset\Gamma_2$ such that $\varphi^1(\gamma_1) \cong\varphi^2(\gamma_2)$ is equivalent to $\psi^1(\gamma_1) \cong \psi^2(\gamma_2)$ for every $(\gamma_1,\gamma_2) \in \complement N_1 \times \complement N_2$.*

PROOF. Putting $\mathfrak{A}_0 = \varphi(\mathfrak{A})$ and $\mathfrak{B}^0 = \psi(\mathfrak{B})$, the decomposition $\varphi = \int_{\Gamma_i}^{\oplus} \varphi^i(\gamma_i) d\mu_i(\gamma_i)$ and $\psi = \int_{\Gamma_i}^{\oplus} \psi^i(\gamma_i)d\mu_i(\gamma_i)$ $(i = 1,2)$ give the associated families Φ^i and Ψ^i $(i = 1,2)$ of $\mathfrak{A}_0$ and $\mathfrak{B}_0$ respectively. Then the relations "$\varphi^1(\gamma_1) \cong \varphi^2(\gamma_2)$" and "$\psi^1(\gamma_1) \cong \psi^2(\gamma_2)$" are equivalent to $\mathfrak{R}^{M,\pi,\mathfrak{A}_0,\Phi^1,\Phi^2}_{A_1,A_2}(\gamma_1, \gamma_2)$ and $\mathfrak{R}^{M,\rho,\mathfrak{B}_0,\Psi^1,\Psi^2}_{A_1,A_2}(\gamma_1, \gamma_2)$ respectively. Hence Theorem 1.1 implies our mention.

Corollary 1 states that the unitary equivalence among the components of representations is completely determined by the algebraic relation between the commutant algebra and the associated diagonal algebra.

COROLLARY 2. *Let $\boldsymbol{A}_i = L^\infty(\Gamma_i, \mu_i)$ be imbedded in a von Neumann algebra $\boldsymbol{M}_i$ acting on a Hilbert space $\mathfrak{H}_i$ $(i = 1,2)$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two separable involutive Banach algebras and let φ_1 and φ_2 (resp. ψ_1 and ψ_2) be two representations of $\mathfrak{A}$ (resp. $\mathfrak{B}$) onto $\mathfrak{H}_1$ nnd $\mathfrak{H}_2$ respectively such that $\varphi_1(\mathfrak{A}_1)' = \boldsymbol{M}_1$ and $\varphi_2(\mathfrak{A}_2)' = \boldsymbol{M}_2$ (resp. $\psi_1(\mathfrak{B})' = \boldsymbol{M}_2$ and $\psi_2(\mathfrak{B})' = \boldsymbol{M}_2$). Then φ_i and ψ_i are decomposed with respect to $\boldsymbol{A}_i$ as follows $(i = 1,2)$;*

$$\varphi_i = \int_{\Gamma_i}^{\oplus} \varphi_i(\gamma_i)\, d\mu_i(\gamma_i) \text{ and } \quad \psi_i = \int_{\Gamma_i}^{\oplus} \psi_i(\gamma_i)\, d\mu_i(\gamma_i) \quad (i = 1,2).$$

If $(\varphi_1\oplus\varphi_2)(\mathfrak{A})' = (\psi_1\oplus\psi_2)(\mathfrak{B})'$, then the relation "$\varphi_1(\gamma_1) \cong \varphi_2(\gamma_2)$" of γ_1 and γ_2 is equivalent to "$\psi_1(\gamma_1) \cong \psi_2(\gamma_2)$" except for some negligible part.

PROOF. Putting $\varphi = \varphi_1 \oplus\varphi_2$, $\psi = \psi_1\oplus\psi_2$, $\varphi(\mathfrak{A})' = \psi(\mathfrak{B})' = \boldsymbol{M}$, $(\Gamma,\mu) = (\Gamma_1, \mu_1)\oplus(\Gamma_2, \mu_2)$ and $\boldsymbol{A} = \boldsymbol{A}_1\oplus\boldsymbol{A}_2$, we have $\boldsymbol{M}\supset \boldsymbol{M}_1\oplus\boldsymbol{M}_2 \supset \boldsymbol{A}_1\oplus\boldsymbol{A}_2 = \boldsymbol{A} = L^\infty(\Gamma, \mu)$ Application of Corollary 1 to φ, ψ, $\boldsymbol{M}$ and $\boldsymbol{A}$ assures our mention.

REMARK. If φ_1 and φ_2(resp. ψ_1 and ψ_2) are disjoint representations, then our assumption $(\varphi_1 \oplus \varphi_2)(\mathfrak{A})' = (\psi_1 \oplus \psi_2)(\mathfrak{B})'$ is automatically satisfied. Indeed, we have $(\varphi_1 \oplus \varphi_2)(\mathfrak{A})' = \boldsymbol{M}_1 \oplus \boldsymbol{M}_2$. Even if φ_1 and φ_2 are representations of quite different types, it may happen that there exists a Borel isomorphism Θ of Γ_1 onto Γ_2 such that $\varphi_1(\gamma_1) \cong \varphi_2(\Theta(\gamma_1))$ for all $\gamma_1 \in \Gamma_1$ (cf. [13]), though, of course, μ_2 and $\Theta(\mu_1)$ are disjoint.

Suppose that there is an isomorphism θ of $\boldsymbol{A}_1 = L^\infty(\Gamma_1, \mu_1)$ onto $\boldsymbol{A}_2 = L^\infty(\Gamma_2, \mu_2)$. By [5 : § 1, Prop. 1], there exist null sets $N_1 \subset \Gamma_1$ and $N_2 \subset \Gamma_2$ respectively and a one-to-one measurable mapping Θ of $\complement N_2$ onto $\complement N_1$ such that $\theta(a)(\gamma_2) = a(\Theta(\gamma_2))$ for every $a \in \boldsymbol{A}_1$ and for every $\gamma_2 \in \complement N_2$ and $\Theta(\mu_2)$ is equivalentto μ_1.

THEOREM 1. 2. *Suppose that there exists a unitary u of $\boldsymbol{M}$ such that $u\boldsymbol{A}_1u^{-1} = \boldsymbol{A}_2$. Let Θ be the measurable mapping of $\complement N_2$ onto $\complement N_1$ associated with the isomorphism θ of A_1 onto A_2 induced by u, where N_1 and N_2 are the null subsets of Γ_1 and Γ_2 defined above respectively. Then $\mathfrak{R}^{M,\mathfrak{A},\varphi^1,\varphi^2}_{A_1,A_2}(\gamma_1, \Theta^{-1}(\gamma_1))$ holds for almost every $\gamma_1 \in \complement N_1$.*

PROOF. Let $\boldsymbol{M}$ act on a Hilbert space $\mathfrak{H}$. Let $\mathfrak{H} = \int_{\Gamma_1}^{\oplus} \mathfrak{H}^1(\gamma_1)\, d\mu_1(\gamma_1)$ and $\mathfrak{H} = \int_{\Gamma_2}^{\oplus} \mathfrak{H}^2(\gamma_2)\, d\mu_2(\gamma_2)$ be the decompositions of $\mathfrak{H}$ with respect to $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ respectively. Applying [2 : Chap. II, § 6 Theorem 4] to u and Θ, there exist a null set $N'_1 \subset \Gamma_1$ and a unitary $u(\gamma_1)$ of $\mathfrak{H}^1(\gamma_1)$ onto $\mathfrak{H}^2(\Theta^{-1}(\gamma_1))$ for every $\gamma_1 \in \complement N'_1$ such that the unitary of $\int_{\Gamma_1}^{\oplus} \mathfrak{H}^1(\gamma_1)\, d\mu_1(\gamma_1)$ onto $\int_{\Gamma 2}^{\oplus} \mathfrak{H}^2(\gamma_2) d\mu_2(\gamma_2)$ naturally induced by $\{u(\gamma_1)\}$ coincides with the original unitary u. For $x \in \mathfrak{A}$, putting $x'(\Theta^{-1}(\gamma_1)) = u(\gamma_1)\varphi^1_{\gamma_1}(x)u(\gamma_1)^{-1}$ for $\gamma_1 \in \complement N'_1$ and $x'(\gamma_2) = 0$ for $\gamma_2 \notin \Theta^{-1}(\complement N'_1)$, $x'(\cdot)$ becomes a bounded measurable operator field over Γ_2 which defines an operator x' on $\mathfrak{H}$. Let $\mathfrak{A}_0$ be a countable dense $*$-subalgebra of $\mathfrak{H}$ over the rational complex number field $\boldsymbol{C}_0$. Let $\mathfrak{F}$ be a countable dense linear subspace of $\mathfrak{H}$ over $\boldsymbol{C}_0$ that is invariant under the actions of $\mathfrak{A}_0$ and u. For each $\xi \in \mathfrak{F}$ there exists a null set $N_\xi \subset \Gamma_1$ such that $(u\xi)(\Theta^{-1}(\gamma_1)) = u(\gamma_1)\xi(\gamma_1)$ for $\gamma \in \complement N_\xi$. Putting $N = \bigcup_{\xi \in \mathfrak{F}} N_\xi$, we have $(u\xi)(\Theta^{-1}(\gamma_1)) = u(\gamma_1)\xi(\gamma_1)$ for every $\xi \in \mathfrak{F}$ and for every $\gamma_1 \in \complement N$. For $x \in \mathfrak{A}_0$, $\xi \in \mathfrak{F}$ and $\gamma_1 \in \complement N$ we have

$$
\begin{aligned}
(x'\xi)(\Theta^{-1}(\gamma_1)) &= x'(\Theta^{-1}(\gamma_1))\,\xi(\Theta^{-1}(\gamma_1)) \\
&= u(\gamma_1)\varphi^1_{\gamma_1}(x)u(\gamma_1)^{-1}\,\xi(\Theta^{-1}(\gamma_1)) \\
&= u(\gamma_1)\varphi^1_{\gamma_1}(x)(u^{-1}\xi)(\gamma_1) \\
&= u(\gamma_1)(xu^{-1})(\gamma_1) = (uxu^{-1}\xi)(\Theta^{-1}(\gamma_1)),
\end{aligned}
$$

so that $x'\xi = uxu^{-1}\xi$ for every $x \in \mathfrak{A}_0$ and for every $\xi \in \mathfrak{F}$. Continuity of x'

and uxu^{-1} implies $x' = uxu^{-1}$. On the other hand, we have $uxu^{-1} = x$ for every $x \in \mathfrak{A}$. Hence we have $x = x'$ for every $x \in \mathfrak{A}$. Then for $x \in \mathfrak{A}_0$ there exists a null set $N_x \subset \Gamma_2$ such that $x'(\gamma_2) = \varphi^2_{\gamma_2}(x)$ for every $\gamma_2 \in \complement N_x$. Putting $N_2 = \bigcup_{x \in \mathfrak{A}_0} N_x$, N_2 is a null set and we have

$$u(\gamma_1)\varphi^1_{\gamma_1}(x)\,u(\gamma_1)^{-1} = x'(\Theta^{-1}(\gamma_1)) = \varphi^2_{\Theta^{-1}(\gamma_1)}(x)$$

for every $x \in \mathfrak{A}_0$ and for every $\gamma_1 \in \complement\,(\Theta(N_2) \cap N)$. By the continuity of $u(\gamma_1)\varphi^1_{\gamma_1} u(\gamma_1)^{-1}$ and $\varphi^2_{\Theta^{-1}(\gamma_1)}$ we have $\varphi^1_{\gamma_1} \cong \varphi^2_{\Theta^{-1}(\gamma_1)}$ for almost every $\gamma_1 \in \Gamma_1$, that is, $\mathfrak{R}^{M,\mathfrak{A},\Phi^1,\Phi^2}_{A_1,A_2}(\gamma_1, \Theta^{-1}(\gamma_1))$ holds for almos every $\gamma_1 \in \Gamma_1$.

REMARK. When an abelian subalgebra $\boldsymbol{A}$ of $\boldsymbol{M}$ is represented in two ways as $\boldsymbol{A} \cong L^\infty(\Gamma_1, \mu_1)$ and $\boldsymbol{A} \cong L^\infty(\Gamma_2, \mu_2)$, taking $\boldsymbol{A}_1 = \boldsymbol{A}_2 = \boldsymbol{A}$ and $u = I$ in Theorem1. 2, there exis null sets $N_1 \subset \Gamma_1$, $N_2 \subset \Gamma_2$ and a one-to-one measurable mapping Θ from $\complement N_2$ onto $\complement N_1$ such that $\Theta(\mu_2) \approx \mu_1$ and $\mathfrak{R}^{M,\mathfrak{A},\Phi^1,\Phi^2}_{A_1,A_2}(\gamma_1, \Theta^{-1}(\gamma_1))$ holds for every $\gamma_1 \in \complement N_1$. Hence the behaviors of the equivalence relations $\mathfrak{R}^{M,\mathfrak{A},\Phi 1}_{A}$ and $\mathfrak{R}^{M,\mathfrak{A},\Phi^2}_{A}$ in the measure spaces (Γ_1, μ_1) and (Γ_2, μ_2) are almost isomorphic. That is, we can say that the equivalence relation $\mathfrak{R}^{M,\mathfrak{A},\Phi}_{A}$ depends only on the algebraic relation of $\boldsymbol{M}$ and $\boldsymbol{A}$.

In order to study the behavior of $\hat{\mathfrak{R}}^{M}_{A_1,A_2}$, we set the following.

THEOREM 1. 3. *Let* $\boldsymbol{A}_1 = L^\infty(\Gamma_1, \mu_1)$ *and* $\boldsymbol{A}_2 = L^\infty(\Gamma_2, \mu_2)$ *be two abelian von Neumann subalgebras of a von Neumann algebra* $\boldsymbol{M}$ *acting on a Hlbert space* $\mathfrak{H}$. *Let* $\mathfrak{H} = \int^{\oplus}_{\Gamma_1} \mathfrak{H}^1(\gamma_1)d\mu_1(\gamma_1)$ *and* $\mathfrak{H} = \int^{\oplus}_{\Gamma_2} \mathfrak{H}^2(\gamma_2)\,d\mu_2(\gamma_2)$ *be the decompositions of* $\mathfrak{H}$ *with respect to* $\boldsymbol{A}_1$ *and* $\boldsymbol{A}_2$ *respectively. Let* $\mathfrak{A}$ *be a uniformly separable weakly dense* C^**-subalgebra of* $\boldsymbol{M}'$ *and let* $\Phi^1 = \{\varphi^1_{\gamma_1} : \gamma_1 \in \Gamma_1\}$ *and* $\Phi^2 = \{\varphi^2_{\gamma_2} : \gamma_2 \in \Gamma_2\}$ *be families of representations of* $\mathfrak{A}$ *associated with the decompositions of* $\mathfrak{H}$. *Then the graph of* $\mathfrak{R}^{M,\mathfrak{A},\Phi^1,\Phi^2}_{A_1,A_2}$ *in* $\Gamma_1 \times \Gamma_2$ *is an analytic subset of* $\Gamma_1 \times \Gamma_2$. *Besides, if* $\boldsymbol{A}_1$ *and* $\boldsymbol{A}_2$ *are maximal abelian in* $\boldsymbol{M}$ *then there exist null sets* $N_1 \subset \Gamma_1$ *and* $N_2 \subset \Gamma_2$ *such that the graph of* $\mathfrak{R}^{M,\mathfrak{A},\Phi^1,\Phi^2}_{A_1,A_2}$ *in* $(\Gamma_1 - N_1) \times (\Gamma_2 - N_2)$ *is a Borel subset of* $(\Gamma_1 - N_1) \times (\Gamma_2 - N_2)$.

PROOF. Let R be the graph of $\mathfrak{R}^{M,\mathfrak{A},\Phi^1,\Phi^2}_{A_1,A_2}$. Putting $\Gamma^n_i = \{\gamma_i \in \Gamma_i\,;\ \dim. \mathfrak{H}^i(\gamma_i) = n\}$, $i = 1, 2$, Γ^n_i becomes a Borel subset of Γ_i for each n, $i = 1, 2$, and we have $\left(\bigcup_{n=1}^{\infty} \Gamma^n_i\right) \cup \Gamma^\infty_i = \Gamma_i$, $i = 1, 2$, and $R \subset \bigcup_{n=1}^{\infty} (\Gamma^n_1 \times \Gamma^n_2) \cup (\Gamma^\infty_1 \times \Gamma^\infty_2)$. So we may assume that there exists a fixd Hilbert space $\mathfrak{H}_0$ such that $\mathfrak{H}^i(\gamma_i) = \mathfrak{H}_0$ for each $\gamma_i \in \Gamma_i$, $i = 1, 2$. Let $\boldsymbol{B} = \boldsymbol{B}(\mathfrak{H}_0)$ be the algebra of all bounded operators on $\mathfrak{H}_0$ equipped with the Borel structure induced by the weak topology. Then $\boldsymbol{B}$ is a standard Borel space, since $\boldsymbol{B}$ is covered by countably many metrizable compact subsets. For each $x \in \mathfrak{A}$ the function $(\gamma_1, \gamma_2, u) \in \Gamma_1 \times \Gamma_2 \times \boldsymbol{U} \to (x^1(\gamma_1), x^2(\gamma_2), u) \in \boldsymbol{B} \times \boldsymbol{B} \times \boldsymbol{U}$ becomes a Borel function, where $\boldsymbol{U}$ means the unitary group of $\boldsymbol{B}$. Besides the function $(x, y, u) \in \boldsymbol{B} \times \boldsymbol{B} \times \boldsymbol{U} \rightarrow ux - yu \in \boldsymbol{B}$ is a Borel function. Indeed, let $\{\xi_n\}$ be a complete normalized orthogonal system of $\mathfrak{H}_0$, then we have

$$([ux - yu]\xi_n, \xi_m) = (ux\,\xi_n, \xi_m) - (yu\,\xi_m, \xi_m)$$

$$= \sum_{k=1}^{\infty} (x\xi_n, \xi_k)(u\xi_k, \xi_m) - \sum_{k=1}^{\infty} (u\xi_n, \xi_k)(y\xi_k, \xi_m).$$

Since each member of summands is a Borel function of $(x, y, u) \in \boldsymbol{B} \times \boldsymbol{B} \times \boldsymbol{U}$, $([ux - yu]\xi_n, \xi_m)$ is a Borel function of (x, y, u). For each $\xi, \eta \in \mathfrak{H}_0$

$$([ux - yu]\xi, \eta) = \sum_{n,m} (\xi, \xi_n)\overline{(\eta, \xi_m)}([ux - yu]\xi_n, \xi_m)$$

is a Borel function of (x, y, u). Hence the function $(x, y, u) \rightarrow ux - yu$ is a Borel function. After all, the set

$$A = \{(\gamma_1, \gamma_2, u) \in \Gamma_1 \times \Gamma_2 \times U : \quad ux^1(\gamma_1) = x^2(\gamma_2)u \quad \text{for each} \quad x \in \mathfrak{A}\}$$

is a Borel subset of a standard Borel space $\Gamma_1 \times \Gamma_2 \times U$. R is the projection of A to $\Gamma_1 \times \Gamma_2$, so that R is analytic.

If $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are maximal abelian in $\boldsymbol{M}$, then there exist null sets $N_1 \subset \Gamma_1$ and $N_2 \subset \Gamma_2$ such that $\varphi^1_{\gamma_1}$ and $\varphi^2_{\gamma_2}$ are irreducible representations for every $\gamma_1 \in \Gamma_1 - N_1$ and $\gamma_2 \in \Gamma_2 - N_2$. Hence $(\gamma_1, \gamma_2) \in R \cap (\Gamma_1 - N_1) \times (\Gamma_2 - N_2)$ is equivalent to $\mathfrak{J}(\varphi^1_{\gamma_1}, \varphi^2_{\gamma_2}) > 0$, where $\mathfrak{J}(\varphi^1_{\gamma_1}, \varphi^2_{\gamma_2})$ means the linear dimension of the space of all bounded operators u such that $u\;\varphi^1_{\gamma_1}(x) = \varphi^2_{\gamma_1}(x)u$ for all $x \in \mathfrak{A}$. But $\mathfrak{J}(\varphi^1_{\gamma_1}, \varphi^2_{\gamma_2})$ is a Borel function of (γ_1, γ_2) by [9; Theorem 8.2]. Thus. $R \cap (\varphi_1 - N_1) \times (\Gamma_2 - N_2)$ is a Borel subset of $(\Gamma_1 - N_1) \times (\Gamma_2 - N_2)$.

2. Classification of abelian von Neumann subalgebras. Let $\boldsymbol{A} = L^\infty(\Gamma, \mu)$ be an abelian von Neumann subalgebra of $\boldsymbol{M}$. Then $\mathfrak{R}^{\boldsymbol{M}, \mathfrak{A}, \Phi}_{\boldsymbol{A}} = \mathfrak{R}$ is an equivalence relation associated with $\boldsymbol{M}$ and $\boldsymbol{A}$ defined in the measure space (Γ, μ). Let $\widehat{\Gamma}$ be the Borel space of all $\mathfrak{R}$-equivalence classes in Γ equipped with the quotient Borel structure of the Borel structure of Γ under $\mathfrak{R}$. If $\widehat{\Gamma}$ is countably separated Borel space, then for each Borel set $S \subset \Gamma$ the space $\widehat{S}$ of all $\mathfrak{R}$-equivalence classes in S equipped with the quotient Borel structure of the Borel structure of S is so. Hence we can set the following definition by Theorem 1.1 and Theorem 1.2.

DEFINITION 2.1. If there exists a Borel null set $N \subset \Gamma$ for any $\mathfrak{R}$ associated with $\boldsymbol{M}$ and $\boldsymbol{A}$ such that $(\Gamma - N)/\mathfrak{R}$ is countably separated, then we call $\boldsymbol{A}$ *smooth* in M. If Ae is not smooth in $\boldsymbol{eMe}$ for each nonzero projection e of $\boldsymbol{A}$, we call $\boldsymbol{A}$ *completely rough* in $\boldsymbol{M}$. If for any $\mathfrak{R}$ there exists a Borel null set $N \subset \Gamma$ such that $\mathfrak{R}(\gamma, \gamma')$ implies $\gamma = \gamma'$ for each $(\gamma, \gamma') \in (\Gamma - N) \times (\Gamma - N)$, then we call $\boldsymbol{A}$ simple in $\boldsymbol{M}$. Of course, simple subalgebra is also smooth.

LEMMA 2.1. *An abelian subalgebra $\boldsymbol{A} = L^\infty(\Gamma, \mu)$ of $\boldsymbol{M}$ is smooth if and only if for any $\mathfrak{R}^{\boldsymbol{M}, \mathfrak{A}, \Phi}_{\boldsymbol{A}}$ there exists a Borel subset $N \subset \Gamma$ and an analytic* *subset E of Γ such that $\mu(N)=0$ and such that E contains one and only one element in common with each $\mathfrak{R}^{M,\mathfrak{A},\Phi}_{A}$-equivalence class in $\Gamma-N$. Besides if $\boldsymbol{A}$ is smooth, then we can choose E to be a Borel subset of Γ.*

PROOF. Denote $\mathfrak{R}^{M,\mathfrak{A},\Phi}_{A}=\mathfrak{R}$. Suppose that $\boldsymbol{A}$ is smooth. Eliminating a Borel null set from Γ, $\hat{\Gamma}=\Gamma/\mathfrak{R}$ is countably separated, so that $\hat{\Gamma}$ is an analytic Borel space by [9: Cor. of Theorem 5.1]. Hence there exists a Borel $\hat{\mu}$-null set $N\subset\Gamma$ such that $\Gamma-N$ is standard by [Theorem 6.1], wehre $\hat{\mu}$ is the quotient measure of μ in $\hat{\Gamma}$. Let r be the natural mapping of Γ onto $\hat{\Gamma}$. Then r is a Borel mapping from the standard Borel space $\Gamma-r^{-1}(\widehat{N})$ onto the standard Borel space $(\hat{\Gamma}-\widehat{N})$, so that it follows from [1: §6, Ex. 17] that the graph of r in $\{\Gamma-r^{-1}(\widehat{N})\}\times(\hat{\Gamma}-\widehat{N})$ is its Borel subset. From [9: Theorem 6.3] we conclude the existence of a Borel null set $\widehat{N_1}\subset\hat{\Gamma}$ and a Borel mapping ϕ from $\hat{\Gamma}-\widehat{N_1}$ to Γ such that $r\circ\phi(\hat{\gamma})=\hat{\gamma}$ for every $\hat{\gamma}\in\hat{\Gamma}-\widehat{N_1}$. Since ϕ is one-to-one, its image E is a required subset of Γ by [9: Theorem 3.2].

Conversely, suppose that there exist an analytic set $E\subset\Gamma$ and a Borel null set $N\subset\Gamma$ as in the statement of our Lemma. Then r isa one-to-one Borel mapping of E onto $(\Gamma-N)/\mathfrak{R}=(\Gamma-N)^{\wedge}$Hence if r is a Borel isomorphism then $(\Gamma-N)^{\wedge}$ is analytic Borel space, so that $(\Gamma-N)/\mathfrak{R}$ is countably separated. So it suffices to show that r is a Borel isomorphism, that is, to show that $r(F)$ is a Borel subset of $(\Gamma-N)^{\wedge}$. for every relative Borel subset F of E. Hence we shall show that $r^{-1}r(F)$ is a Borel subset of $\Gamma-N$. Let R be the graph of $\mathfrak{R}$ in $(\Gamma-N)^{\wedge}\times(\Gamma-N)$. Then we have $r^{-1}r(F)=pr_2(F\times(\Gamma-N)\cap R)$, where pr_2 is defined by $\mathrm{pr}_2(\gamma,\gamma')=\gamma'$ for $(\gamma,\gamma')\in\Gamma\times\Gamma$. Since F is a relative Borel subset of the analytic set E, F is analytic. Hence $r^{-1}r(F)$ is analytic by Theorem 1.3. Similarly $r^{-1}r(E-F)$ is also analytic. Since $r^{-1}r(F)$ and $r^{-1}r(E-F)$ are complementary subsets of $\Gamma-N$, they are both Borel sets. This completes the proof.

LEMMA 2.2. *Let $\boldsymbol{A}$ be an abelian subalgebra of a von Neumann algebra $\boldsymbol{M}$. 1°. If there exists a partition of unit $\sum_{n=1}^{\infty}p_n=I$ in $\boldsymbol{A}$ such that $\boldsymbol{A}p_n$ is smooth in $p_n\boldsymbol{M}p_n$ for each n, then A is smooth. 2°. If there exist two von Neumann algebras $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ and their smooth abelian subalgebras $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ such that $\boldsymbol{M}=\boldsymbol{M}_1\otimes\boldsymbol{M}_2$ and $\boldsymbol{A}=\boldsymbol{A}_1\otimes\boldsymbol{A}_2$, then $\boldsymbol{A}$ is smooth under the additional assumption $\boldsymbol{M}'=\boldsymbol{M}_1'\otimes\boldsymbol{M}_2'$*[3].

PROOF. 1°. Let $\boldsymbol{A}=L^{\infty}(\Gamma,\mu)$. Let P_n be the Borel set in Γ associated with p_n. Then we have $\boldsymbol{A}p_n=L^{\infty}(P_n,\mu)$. By eliminating a Borel null set we

3) When the one of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ has the part of type III and the other is not of type I, the question whether $\boldsymbol{M}'=\boldsymbol{M}_1'\otimes\boldsymbol{M}_2'$ does or does not hold remains open up to now (cf. [2: p. 30 and p. 102]).

may assume $\bigcup_{n=1}^{\infty} P_n = \Gamma$. Let $\mathfrak{A}$ and Φ be the couple as in the preceeding arguments. Putting $\boldsymbol{M}_n = p_n \boldsymbol{M} p_n$, $\boldsymbol{A}_n = \boldsymbol{A} p_n$, $\mathfrak{A}_n = \mathfrak{A} p_n$ and $\Phi_n = \{\varphi_\gamma \in \Phi ; \gamma \in P_n\}$, the equivalence relation $\mathfrak{R}^{\boldsymbol{M}_n, \mathfrak{A}_n, \Phi_n}_{\boldsymbol{A}_n} = \mathfrak{R}_n$ in P_n becomes the restriction of the original equivalence relation $\mathfrak{R}^{\boldsymbol{M}, \mathfrak{A}, \Phi}_{\boldsymbol{A}} = \mathfrak{R}$ to P_n. It follows from Lemma 2.1 that there exist a Borel set $N_n \subset P_n$ and a Borel set $E_n \subset P_n$ for each n such that $\mu(N_n) = 0$ and such that E_n contains one and only one element in common with each $\mathfrak{R}_n$-equivalence class in $P_n - N_n$. Let Q_n be the $\mathfrak{R}$-saturation of E_n[4]. Then we have $Q_n \supset P_n - N_n$. Putting $E = \bigcup_{n=1}^{\infty} (E_n - \bigcup_{k=1}^{n-1} Q_k)$, E is an analytic subset of Γ whose saturation becomes $\bigcup_{n=1}^{\infty} Q_n$ and it has one and only one element in common with each $\mathfrak{R}$-equivalence class in $\bigcup_{n=1}^{\infty} Q_n$. Putting $N = \Gamma - \bigcup_{n=1}^{\infty} Q_n$, we have $N \subset \bigcup_{n=1}^{\infty} N_n$, so that N is a null subset of Γ. Therefore $\boldsymbol{A}$ becomes smooth by Lemma 2.1.

2°. Let $\boldsymbol{A}_1 = L^\infty(\Gamma_1, \mu_1)$, $\boldsymbol{A}_2 = L^\infty(\Gamma_1, \mu_2)$ and $(\Gamma, \mu) = (\Gamma_1 \times \Gamma_2, \ \mu_1 \times \mu_2)$. Then we have $\boldsymbol{A} = L^\infty(\Gamma, \mu)$. Let $N_i \subset \Gamma_i$ and $E_i \subset \Gamma_i$ be the couple statisfying the condition of Lemma 2.1, $i = 1, 2$. Let $\mathfrak{A}_1, \Phi_1$ and $\mathfrak{A}_2, \Phi_2$ be the couples as in the preceeding discussion for $\boldsymbol{M}_1$, $\boldsymbol{A}_1$ and $\boldsymbol{M}_2$, $\boldsymbol{A}_2$ respectively. Then $\mathfrak{A} = \mathfrak{A}_1 \hat{\otimes}_\alpha \mathfrak{A}_2$ becomes a uniformly separable weakly dense C^*-subalgebra of $\boldsymbol{M}'$ by our assumption. Putting $\Phi = \Phi_1 \otimes \Phi_2 = \{\varphi_{(\gamma_1, \gamma_2)} = \varphi^1_{\gamma_1} \otimes \varphi^2_{\gamma_2} : \varphi^1_{\gamma_1} \in \Phi_1, \varphi^{2}_{\gamma_2} \in \Phi_2, \ (\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2\}$, Φ is a family of representations of $\mathfrak{A}$ associated with the decomposition of $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$ with respect to $\boldsymbol{A} = \boldsymbol{A}_1 \otimes \boldsymbol{A}_2$, where $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are the underlying Hilbert spaces of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ respectively. It is clear that the equivalence relation $\mathfrak{R}^{\boldsymbol{M}, \mathfrak{A}, \Phi}_{\boldsymbol{A}}$ in Γ is defined as the canonical product equivalence relation $\mathfrak{R}^{\boldsymbol{M}_1, \mathfrak{A}_1, \Phi_1}_{\boldsymbol{A}_1} \times {}^{\boldsymbol{M}_2, \mathfrak{A}_2, \Phi_2}_{\boldsymbol{A}}$ in $\Gamma_1 \times \Gamma_2$. Putting $N = N_1 \times \Gamma_2 \cup \Gamma_1 \times N_2$ and $E = E_1 \times E_2$, N and E satisfy the condition of Lemma 2.1. Hence $\boldsymbol{A}$ is smooth in $\boldsymbol{M}$. This completes the proof.

THEOREM 2.1. *Let $\boldsymbol{A}$ be an abelian von Neumann subalgebra of a von Neumann algebra $\boldsymbol{M}$. Then there exists a unique partition of unit $e + f = I$ in $\boldsymbol{A}$ such that $\boldsymbol{A}e$ is smooth in $e\boldsymbol{M}e$ and such that $\boldsymbol{A}f$ is completely rough in $f\boldsymbol{M}f$.*

PROOF. Let $\{p_\alpha\}$ be a maximal family of orthogonal non-zero projections in $\boldsymbol{A}$ such that $\boldsymbol{A}p_\alpha$ is smooth in $p_\alpha \boldsymbol{M} p_\alpha$. By the separability of underlying

4) For any equivalence relation $\mathfrak{R}$ in Γ the $\mathfrak{R}$-saturation of any subset $S \subset \Gamma$ is the set of all elements of Γ that are $\mathfrak{R}$-equivalent to some element of S. If S contains every element that is $\mathfrak{R}$-equivalent to some one of S, S is called $\mathfrak{R}$-saturated.

Hilbert space of $\boldsymbol{M}$, $\{p_\alpha\}$ is at most countable. $e = \sum_\alpha p_\alpha$ and $f = I - e$ are the desired projections in $\boldsymbol{A}$ by Lemma 2. 2 and by the maximality of $\{p_\alpha\}$. The unicity of e and f is clear from Definition 2. 1. This completes the proof.

Theorem 2. 1 reduces the study of abelian von Neumann subalgebras to that of smooth ones and that of completely rough ones.

3. Smooth maximal abelian subalgebras. In the present section we reduce the study of smooth maximal abelian subalgebras to that of simple ones. In the following if a maximal abelian subalgebra $\boldsymbol{A} = L^\infty(\Gamma, \mu)$ of a von Neumann subalgebra is smooth, then we assume that the quotient space $\Gamma/\mathfrak{R} = \hat{\Gamma}$ of Γ is standard by eliminating a null set from the whole space Γ.

LEMMA 3. 1. *Let $\boldsymbol{A}_1 = L^\infty(\Gamma_1, \mu_1)$ and $\boldsymbol{A}_2 = L^\infty(\Gamma_2, \mu_2)$ be two abelian von Neumann subalgebras of* the *von Neumann algebra $\boldsymbol{M}$. Let $\mathfrak{A}$, Φ^1 and Φ^2 be a triard as in* §1. *Let $E_1 \subset \Gamma_1$ and $E_2 \subset \Gamma_2$ be Borel subsets respectively. If there exists a Borel mapping Θ from E_1 to E_2 such that $\mathfrak{R}^{M,\mathfrak{A},\Phi^1,\Phi^2}_{A_1 A_2}(\gamma_1, \Theta(\gamma_1))$ holds for almost every $\gamma_1 \in E_1$, then for almost every $\gamma_1 \in E_1$ there exists a unitary $u(\gamma_1)$ from $\mathfrak{H}^2(\Theta(\gamma_1))$ onto $\mathfrak{H}^1(\gamma_1)$ such that $u(\gamma_1)^{-1}x^1(\gamma_1)u(\gamma_1) = x^2(\Theta(\gamma_1))$ for every $x \in \mathfrak{A}$ and such that $u(\gamma_1)\,\xi(\Theta(\gamma_1))$ is a measurable vector field over E_1 if $\xi(\cdot)$ is so over E_2. If Θ is a Borel isoamorphism such that $\Theta(\mu_1) \approx \mu_2$, then the operator u defined by*

$$u\,\xi = \int_{\Gamma_1}^{\oplus} e_1(\gamma_1)\, u(\gamma_1)\, \xi^2(\Theta(\gamma_1)) \sqrt{\frac{d(\Theta^{-1}(\mu_2))}{d\mu_1}}\,(\gamma_1) d\mu_1(\gamma_1)$$

for $\xi = \int_{\Gamma}^{\oplus} \xi^2(\gamma_2) d\mu_2(\gamma_2) \in \mathfrak{H}$ is a partial isometry of $\boldsymbol{M}$ which carries e_2 onto e_1 where e_1 and e_2 mean the projections of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ associated with E_1 and E_2 respectively.

PROOF. We use the notation in the proof of Theorem 1. 3. As in the proof of Theorem 1.3 we may assume that there exists a fixed Hilbert space $\mathfrak{H}_0$ such that $\mathfrak{H}^i(\gamma_i) = \mathfrak{H}_0$ for each $\gamma_i \in \Gamma_i$, $i = 1, 2$. Putting $B = \{(\gamma_1, u) \in E_1 \times \boldsymbol{U};\ u^{-1}x^1(\gamma_1)u = x^2(\Theta(\gamma_1))$ for every $x \in \mathfrak{A}\}$, B is a Borel subset of $E_1 \times \boldsymbol{U}$ whose projection to E_1 covers E_1. Indeed, B is the projection of the intersection of A and the product of the graph R_Θ of Θ in $E_1 \times E_2$ and $\boldsymbol{U}$ to $E_1 \times \boldsymbol{U}$. R_Θ is a Borel set in $E_1 \times E_2$ by [1 : §6 Ex. 17] and the projection of $E_1 \times E_2$ to E_1 is a one-to-one Borel mapping on R_Θ. Hence the projection of $A \cap (R_\Theta \times U)$ onto B is a one-to-one Borel mapping of the standard Borel space $A \cap (R_\Theta \times U)$ into the standard Borel space $E_1 \times \boldsymbol{U}$, so that B becomes a Borel subset of $E_1 \times \boldsymbol{U}$ by [9: Theorem 3. 2]. Applying [9: Theorem 6. 3] to E_1, U and B, there exist a null set $N_1 \subset E_1$ and a Borel mapping $u(\gamma_1)$ from $E_1 - N_1$ to $\boldsymbol{U}$ such that $(\gamma_1, u(\gamma_1)) \in B$ for evry $\gamma_1 \in E_1 - N_1$. This $u(\cdot)$ is desired one. Suppose that Θ is a Borel isomorphism of (E_1, μ_1) onto (E_2, μ_2).

For $\xi = \int_{\Gamma_2}^{\oplus} \xi^2(\gamma_2) d\mu_2(\gamma_2) \in \mathfrak{H}$

we have

$$\|u\xi\|^2 = \int_{E_1} \|u(\gamma_1)\xi^2(\Theta(\gamma_1))\|^2 \frac{d\Theta^{-1}(\mu_2)}{d\mu_1}(\gamma_1) d\mu_1(\gamma_1)$$

$$= \int_{E_1} \|\xi^2(\Theta(\gamma_1))\|^2 d(\Theta^{-1}(\mu_2))(\gamma_1)$$

$$= \int_{\Theta(E_1)} \|\xi^2(\gamma_2)\|^2 d\mu_2(\gamma_2) = \|e_2\xi\|^2.$$

For each $x \in \mathfrak{A}$ we have

$$ux\xi = \int_{E_1}^{\oplus} u(\gamma_1)(x\xi)^2(\Theta(\gamma_1))\sqrt{\frac{d\Theta^{-1}(\mu_2)}{d\mu_1}(\gamma_1)} d\mu_1(\gamma_1)$$

$$= \int_{E_1}^{\oplus} u(\gamma_1)x^2(\Theta(\gamma_1))\xi^2(\Theta(\gamma_1))\sqrt{\frac{d\Theta^{-1}(\mu_2)}{d\mu_1}(\gamma_1)} d\mu_1(\gamma_1)$$

$$= \int_{E_1}^{\oplus} x^1(\gamma_1)u(\gamma_1)\xi^2(\Theta(\gamma_1))\sqrt{\frac{d\Theta^{-1}(\mu_2)}{d\mu_1}(\gamma_1)} d\mu_1(\gamma_1)$$

$$= x\int_{E_1}^{\oplus} u(\gamma_1)\xi^2(\Theta(\gamma_1))\sqrt{\frac{d\Theta^{-1}(\mu_2)}{d\mu_1}(\gamma_1)} d\mu_1(\gamma_1)$$

$$= xu\xi \quad \textit{for every } \xi = \int_{\Gamma_2}^{\oplus} \xi^2(\gamma_2) d\mu_2(\gamma_2) \in \mathfrak{H},$$

so that u is a partial isometry of $\boldsymbol{M}$ mentioned above.

THEOREM. 3.1. *For a maximal abelian subalgebra* $\boldsymbol{A}$ *of a von Neumann algebra* $\boldsymbol{M}$, *acting on a Hilbert space* $\mathfrak{H}$, *to be smooth, it is necessary and sufficient that there exists a simple maximal abelian subalgebra* A *of some von Neumann algebra* $\widehat{\boldsymbol{M}}$ *on a Hilbert space* $\mathfrak{K}$ *and a normal isomorphism* θ *of* $\widehat{\boldsymbol{A}}'$ *into* $\boldsymbol{A}'$ *such that* $\theta(\boldsymbol{A}) \subset \boldsymbol{A}$ *and* $\theta(\widehat{\boldsymbol{M}}') = \boldsymbol{M}'$. *If* $\boldsymbol{A}$ *is smooth, then* $(\widehat{\boldsymbol{M}}, \widehat{\boldsymbol{A}}, \theta)$ *is unique.*

PROOF. NECESSITY: Let $\mathfrak{R} = \mathfrak{R}^{\boldsymbol{M}, \mathfrak{A}, \Phi}_{\boldsymbol{A}}$ and $(\hat{\Gamma}, \hat{\mu}) = (\Gamma, \mu)/\mathfrak{R}$. Let r be the canonical mapping of Γ onto Γ. Then there exists a Borel mapping ϕ of $\hat{\Gamma}$ into Γ such that $\phi(\gamma) \in r^{-1}(\hat{\gamma})$ for every $\hat{\gamma} \in \hat{\Gamma}$ by Lemma 2.1, eliminating a null set from Γ. Putting $\mathfrak{K}(\hat{\gamma}) = \mathfrak{H}(\phi(\hat{\gamma}))$ for each $\gamma \in \hat{\Gamma}$, where $\mathfrak{H}(\gamma)$ means the component of the decomposition $\mathfrak{H} = \int_{\Gamma}^{\oplus} \mathfrak{H}(\gamma)\, d\mu(\gamma)$ of $\mathfrak{H}$ with respect to $\boldsymbol{A}$, we get a measurable Hilbert space field $\{\mathfrak{K}(\hat{\gamma})\}$ over $\hat{\Gamma}$, so that we can define a Hilbert space $\mathfrak{K}$ by $\mathfrak{K} = \int_{\hat{\Gamma}}^{\oplus} \mathfrak{K}(\hat{\gamma}) d\hat{\mu}(\hat{\gamma})$. The diagonal algebra of this decomposition of $\mathfrak{K}$ becomes $\widehat{\boldsymbol{A}} = L^\infty(\hat{\Gamma}, \hat{\mu})$. Since the mapping $\Theta = \phi \circ r$ of Γ onto $\phi(\hat{\Gamma})$

is a Borel mapping such that $\mathfrak{R}(\gamma, \Theta(\gamma))$ holds for every $\gamma \in \Gamma$, there exists a family $\{u(\gamma)\}$ of unitaries from $\{\mathfrak{H}(\gamma)\}$ onto $\{\mathfrak{H}(\Theta(\gamma))\}$ as in the conclusion of Lemma 3.1. Since each operator $x \in \hat{A}'$ is decomposable with respect to the decomposition of $\mathfrak{K}$, there exists a measurable operator field $\{x(\hat{\gamma})\}$ over $\hat{\Gamma}$ such that $x = \int_{\hat{\Gamma}}^{\oplus} x(\hat{\gamma})d\hat{\mu}(\hat{\gamma})$. Putting $\theta(x) = \int_{\Gamma}^{\oplus} u(\gamma)^{-1}x(r(\gamma))\,u(\gamma)\;d\mu\;(\gamma)$ for $x \in \widehat{\boldsymbol{A}}'$, θ becomes a normal isomorphism of $\widehat{\boldsymbol{A}}'$ into $\boldsymbol{A}'$. In fact, if there exists another measurable operator field $x_1(\gamma)$ over Γ for $x \in \widehat{\boldsymbol{A}}'$ such that $\int_{\hat{\Gamma}}^{\oplus} x_1(\hat{\gamma})d\hat{\mu}(\hat{\gamma}) = \int_{\hat{\Gamma}}^{\oplus} x(\hat{\gamma})d\hat{\mu}(\hat{\gamma}) = x$, then $\widehat{E} = \{\hat{\gamma}\,;x_1(\hat{\gamma}) \neq x(\hat{\gamma})\}$ is a Borel null set in $\hat{\Gamma}$. Since $\{\gamma; x(r(\gamma)) \neq x_1(r(\gamma))\} = r^{-1}(\widehat{E})$ is a Borel null subset of Γ, we have

$$\int_{\Gamma}^{\oplus} u(\gamma)^{-1}x(r(\gamma))u(\gamma)d\mu(\gamma) = \int_{\Gamma}^{\oplus} u(\gamma)^{-1}x_1(r(\gamma))u(\gamma)d\mu(\gamma),$$

so that θ is well defined. Similarly it is easily verified that θ preserves the algebraic operations. Let $\{x_n\}$ be a sequence in the unit sphere of $\widehat{\boldsymbol{A}}'$ converging strongly to zero. Then there exists a subsequence $\{x_{n_j}\}$ and a null subset $\widehat{N}$ of $\hat{\Gamma}$ such that $\{x_j(\gamma)\}$ converges strongly to zero for every $\hat{\gamma} \in \complement\,\widehat{N}$ by [2: Chap. II, § 2 Prop. 4]. Since $x_{n_j}(r(\gamma))$ converges to zero for every $\gamma \notin r^{-1}(\widehat{N})$ and $r^{-1}(\widehat{N})$ is a null subset of Γ, $\theta(x_{n_j})$ converges strongly to zero in $\mathfrak{H}$. Therefore any subsequence of $\{\theta(x_n)\}$ contains a subsequence converging strongly to zero, which implies the strong convergence of $\{\theta(x_n)\}$ to zero. Hence θ is strongly continuous on the unit sphere of $\widehat{\boldsymbol{A}}'$. It is clear that $\theta(\widehat{\boldsymbol{A}}) \subset \boldsymbol{A}'$, and $\theta(\boldsymbol{A}') \subset \boldsymbol{A}'$, since each operator of $\theta(\widehat{\boldsymbol{A}})$ is diagonalizable and each one of $\theta(\boldsymbol{A}')$ is decomposable.

Putting
$$\hat{x} = \int_{\Gamma}^{\oplus} x(\phi(\hat{\gamma}))d\hat{\mu}\,(\hat{\gamma}) \quad \text{for each } x \in \mathfrak{A},$$

we have
$$\begin{aligned}\theta(\hat{x}) &= \int_{\Gamma}^{\oplus} u\,(\gamma)^{-1}\,\hat{x}\,(r\,(\gamma))\,u\,(\gamma)\,d\mu\,(\gamma)\\ &= \int_{\Gamma}^{\oplus} u\,(\gamma)^{-1}\,x\,(\Theta(\gamma))\,u\,(\gamma)\,d\mu\,(\gamma)\\ &= \int_{\Gamma}^{\oplus} x\,(\gamma)\,d\mu\,(\gamma) = x.\end{aligned}$$

Hence $\theta\,(\widehat{\boldsymbol{A}}')$ covers $\boldsymbol{M}'$. Putting $\theta^{-1}(\mathfrak{A}) = \widehat{\mathfrak{A}}$ and $\widehat{\mathfrak{A}}' = \widehat{\boldsymbol{M}}$, we have $\theta(\widehat{\boldsymbol{M}}') = \boldsymbol{M}'$. Since each γ-component of $\mathfrak{A}$ coincides with $\phi\,(\gamma)$-component of $\mathfrak{A}$ as the operator algebra over $\mathfrak{K}\,(\hat{\gamma}) = \mathfrak{K}\,(\phi(\hat{\gamma}))$, almost every γ-component of $\widehat{\mathfrak{A}}$ is irreducibly acting on $\mathfrak{K}\,(\hat{\gamma})$. Because, putting $E = \{\gamma\,;\gamma\text{-component of } \mathfrak{A} \text{ is not irreducible}\}$, E is saturated and E is a null set by the maximality of $\boldsymbol{A}$ in $\boldsymbol{M}$. Hence $\hat{\mu}\,(r\,(E)) \equiv 0$. On the other hand, we have

$$r(E) = \{\hat{\gamma}\,;\ \hat{\gamma}\text{-component of } \widehat{\mathfrak{A}} \text{ is not irreducible}\},$$

which implies that almost every $\hat{\gamma}$-component of $\widehat{\mathfrak{A}}$ is irreducible. Hence $\widehat{\boldsymbol{A}}$ is a maximal abelian subalgebra of $\widehat{\boldsymbol{M}}$.

Finally we shall show that $\boldsymbol{A}$ is simple in $\boldsymbol{M}$. Putting $\varphi_{\hat{\gamma}}(\hat{x}) = \theta(\hat{x})(\phi(\hat{\gamma})) = \hat{x}(\hat{\gamma})$ for each $\hat{x} \in \widehat{\mathfrak{A}}$ and $\Phi = \{\varphi_{\hat{\gamma}} : \hat{\gamma} \in \hat{\Gamma}\}$, suppose that $\mathfrak{R}^{M,\mathfrak{A},\Phi}_{A}(\gamma_1,\gamma_2)$ holds for $\gamma_1, \gamma_2 \in \Gamma$. That is, there exists a unitary u of $\mathfrak{K}(\hat{\gamma}_1)$ onto $\mathfrak{K}(\hat{\gamma}_2)$ such that $u\varphi_{\hat{\gamma}_1}(\hat{x})u^{-1} = \varphi_{\hat{\gamma}_2}(\hat{x})$ for each $\hat{x} \in \widehat{\mathfrak{A}}$, which implies that

$$u\varphi_{\hat{\gamma}_1}(\hat{x})\,u^{-1} = u\theta(\hat{x})(\phi(\hat{\gamma}_1))\,u^{-1} = \theta(\hat{x})(\phi(\hat{\gamma}_2))$$

for each $\hat{x} \in \widehat{\mathfrak{A}}$. Hence we have $ux(\phi(\hat{\gamma}_1))\,u^{-1} = x(\phi(\hat{\gamma}_2))$ for each $x \in \mathfrak{A}$, which implies $\mathfrak{R}(\phi(\hat{\gamma}_1), \phi(\hat{\gamma}_2))$. Hence we have $\hat{\gamma}_1 = \gamma_2$, so that $\widehat{\boldsymbol{A}}$ is simple in $\widehat{\boldsymbol{M}}$. After all, the triard $(\widehat{\boldsymbol{M}}, \widehat{\boldsymbol{A}}, \theta)$ is the desired one.

SUFFICIENCY: Suppose that there exists a triard $(\widetilde{\boldsymbol{M}}, \widetilde{\boldsymbol{A}}, \theta)$ satisfying the condition in our theorem. Let $\boldsymbol{A} = L^\infty(\widetilde{\Gamma}, \widetilde{\mu})$. Let $\widetilde{\mathfrak{A}}$ and $\widetilde{\Phi} = \{\varphi_{\tilde{\gamma}}\}$ be a couple as in §1 for $(\widetilde{\boldsymbol{M}}, \widetilde{\boldsymbol{A}})$. Let $\mathfrak{K} = \int_{\tilde{\Gamma}}^{\oplus} \mathfrak{K}(\tilde{\gamma})\,d\tilde{\mu}(\tilde{\gamma})$ be the decomposition of $\mathfrak{K}$ with respect to $\widetilde{\boldsymbol{A}}'$, which induces the central decomposition $\widetilde{\boldsymbol{A}}' = \int_{\tilde{\Gamma}}^{\oplus} \widetilde{\boldsymbol{A}}'(\tilde{\gamma})\,d\tilde{\mu}(\tilde{\gamma})$ of the von Neumann algebra $\widetilde{\boldsymbol{A}}'$. Since almost every component $\widetilde{\boldsymbol{A}}'(\tilde{\gamma})$ becomes the algebra $\boldsymbol{B}(\mathfrak{K}(\tilde{\gamma}))$ of all bounded opertors on $\widetilde{\mathfrak{K}}(\tilde{\gamma})$ and almost every φ_γ is irreducible, almost every $\widetilde{\boldsymbol{A}}'(\tilde{\gamma})$ is the weak closure of $\varphi_{\tilde{\gamma}}(\widetilde{\mathfrak{A}})$. By [2: Chap. II, §3 Prop. 11] there exists a decomposition $\mathfrak{H} = \int_{\tilde{\Gamma}}^{\oplus} \mathfrak{H}(\tilde{\gamma})\,d\tilde{\mu}(\tilde{\gamma})$ of $\mathfrak{H}$ over $\widetilde{\Gamma}$ with respect to $\theta(\widetilde{\boldsymbol{A}})$ which induces the decomposition $\theta(\widetilde{\boldsymbol{A}}') = \int_{\tilde{\Gamma}}^{\oplus} \theta(\widetilde{\boldsymbol{A}}')(\tilde{\gamma})\,d\tilde{\mu}(\tilde{\gamma})$ of $\theta(\widetilde{\boldsymbol{A}}')$ and there exists a measurable field $\{\theta_{\tilde{\gamma}} : \tilde{\gamma} \in \Gamma\}$ of normal isomorphisms of $\widetilde{\boldsymbol{A}}'(\tilde{\gamma})$ onto $\theta(\widetilde{\boldsymbol{A}}')(\tilde{\gamma})$ such that $\theta(x) = \int_{\tilde{\Gamma}}^{\oplus} \theta_{\tilde{\gamma}}(x(\tilde{\gamma}))\,d\tilde{\mu}(\tilde{\gamma})$ for each $x \in \widetilde{\boldsymbol{A}}'$, that is, $\theta = \int_{\tilde{\Gamma}}^{\oplus} \theta_{\tilde{\gamma}}\,d\tilde{\mu}(\tilde{\gamma})$. Putting $\theta(\widetilde{\mathfrak{A}}) = \mathfrak{A}$ and $\psi_{\tilde{\gamma}}(x) = \theta_{\tilde{\gamma}}\,\varphi_{\tilde{\gamma}}(\theta^{-1}(x))$ for each $x \in \mathfrak{A}$, $\mathfrak{A}$ is weakly dense in $\boldsymbol{M}'$ and almost every $\psi_{\tilde{\gamma}}(\mathfrak{A})$ is weakly dense in $\theta(\widetilde{\boldsymbol{A}}')(\tilde{\gamma})$ by the continuity of almost every $\theta_{\tilde{\gamma}}$. Hence almost every $\psi_{\tilde{\gamma}}$ is a representation of type I which is quasi-equivalent to irreducible representation $\varphi_{\tilde{\gamma}} \circ \theta^{-1}$ of $\mathfrak{A}$. Modifying $\widetilde{\Phi}$ on a null subset of $\widetilde{\Gamma}$, we can assume from the assumption for $\widetilde{\boldsymbol{A}}$ that each distinct members of $\widetilde{\Phi}$ are disjoint. Besides, eliminating null set, ψ_γ is quasi-equivalent to $\varphi_{\tilde{\gamma}} \circ \theta^{-1}$ for every $\tilde{\gamma} \in \widetilde{\Gamma}$. After all, we conclude that there exists a von Neumann subalgebra $\boldsymbol{B} = \theta(\widetilde{\boldsymbol{A}}) \cong L^\infty(\widetilde{\Gamma}, \widetilde{\mu})$ and a decomposition $\mathfrak{H} = \int_{\tilde{\Gamma}}^{\oplus} \mathfrak{H}(\tilde{\gamma})\, d\tilde{\mu}(\tilde{\gamma})$ of $\mathfrak{H}$ with respect to $\boldsymbol{B}$ which induces a family $\Psi = \{\psi_\gamma\}$ of mutually disjoint factor representations of type I of $\mathfrak{A}$ such that $x = \int_{\tilde{\Gamma}}^{\oplus} \psi_{\tilde{\gamma}}(x)\, d\tilde{\mu}(\tilde{\gamma})$ for each $x \in \mathfrak{A}$.

By [5 : § 5, Prop.2] we get the following:

1^0. there exist null subsets $N \subset \Gamma$ and $\tilde{N} \subset \tilde{\Gamma}$ and a Borel mapping Θ of $\Gamma - N$ onto $\tilde{\Gamma} - \tilde{N}$ such that for each $a \in \tilde{\boldsymbol{A}}$ $\theta(a)(\gamma) = a(\Theta(\gamma))$ for almost every $\gamma \in \Gamma - N$ and $\Theta(\mu) \approx \tilde{\mu}$.

2^0. there exists a decomposition $\mu = \int_{\tilde{\Gamma}}^{\oplus} \mu^{\tilde{\gamma}}\, d\tilde{\mu}(\tilde{\gamma})$ of μ such that $\mu^{\tilde{\gamma}}$ is concentrated on $\Theta^{-1}(\tilde{\gamma})$ for every $\tilde{\gamma} \in \tilde{\Gamma} - \tilde{N}$.

3^0. there exist a null set $\tilde{N}_1 \subset \tilde{\Gamma}$ and a unitary of $\mathfrak{H}(\gamma)$ onto $\int_{\Theta^{-1}(\tilde{\gamma})} \mathfrak{H}(\gamma)\, d\mu^{\tilde{\gamma}}(\gamma)$ for every $\tilde{\gamma} \in \tilde{\Gamma} - N_1$ which carries $\psi_{\tilde{\gamma}}(x)$ onto $\int_{\Theta^{-1}(\tilde{\gamma})} \varphi_{\tilde{\gamma}}(x)\, d\mu^{\tilde{\gamma}}(\gamma)$ for every $\tilde{x} \in \tilde{\mathfrak{A}}$.

Since $\psi_{\tilde{\gamma}}$ is a factor-representation of type I, φ_γ is quasi-equivalent to $\psi_{\tilde{\gamma}}$ for μ^{γ}-almost every $\gamma \in \Theta^{-1}(\tilde{\gamma})$ by [8 : p. 103, Lemma]. Putting $N' = \{\gamma \in \Gamma; \varphi_{\tilde{\gamma}}$ is not quasi-equivalent to $\psi_{\Theta(\gamma)}\}$, we have $\mu(N') = \int_{\tilde{\Gamma}} \mu^{\tilde{\gamma}}(N')\, d\tilde{\mu}(\tilde{\gamma}) = 0$. For each pair $(\gamma, \gamma') \in \{\Gamma - (N \cup N')\} \times \{\Gamma - (N \cup N')\}, \mathfrak{R}(\gamma, \gamma')$ holds if and only if $\Theta(\gamma) = \Theta(\gamma')$, so that $(\Gamma - (N \cup N'), \mu)/\mathfrak{R}$ is isomorphic to the standard measure space $(\tilde{\Gamma} - \tilde{N}_1, \tilde{\mu})$, which implies the smoothness of $\boldsymbol{A}$ in $\boldsymbol{M}$.

UNICITY : Suppose that there exists another triard $(\tilde{\boldsymbol{M}}_1, \tilde{\boldsymbol{A}}_1, \theta_1)$. For $(\tilde{\boldsymbol{M}}_1, \tilde{\boldsymbol{A}}_1, \theta_1)$ we shall use the corresponding notations in the proof of sufficiency adding the suffix 1 (For instance, let $\boldsymbol{A}_1 = L^\infty(\tilde{\Gamma}_1, \tilde{\mu}_1)$ and so on.) Suppose that $\theta(\tilde{\boldsymbol{A}}) = \theta_1(\tilde{\boldsymbol{A}}_1)$ is proved. Since $\tilde{\boldsymbol{A}}'$ (resp. $\tilde{\boldsymbol{A}}_1'$) is generated by $\tilde{\boldsymbol{M}}'$ and $\tilde{\boldsymbol{A}}$ (resp. $\tilde{\boldsymbol{M}}_1'$ and $\tilde{\boldsymbol{A}}_1$) by the maximality of $\tilde{\boldsymbol{A}}$ (resp. $\tilde{\boldsymbol{A}}_1$) in $\tilde{\boldsymbol{M}}$ (resp. $\tilde{\boldsymbol{M}}_1$), $\theta(\tilde{\boldsymbol{A}}')$ (resp. $\theta_1(\tilde{\boldsymbol{A}}_1')$) is generated by $\boldsymbol{M}' = \theta(\tilde{\boldsymbol{M}}')$ and $\theta(\tilde{\boldsymbol{A}})$ (resp. $\boldsymbol{M}' = \theta_1(\tilde{\boldsymbol{M}}_1')$ and $\theta(\tilde{\boldsymbol{A}}_1)$), which implies $\theta(\tilde{\boldsymbol{A}}') = \theta_1(\tilde{\boldsymbol{A}}_1')$. Hence $\theta^{-1} \circ \theta_1$ becomes an isomorphism of $\tilde{\boldsymbol{A}}_1'$ onto $\tilde{\boldsymbol{A}}'$, which is a spatial isomorphism by [2 : Chap. III, §3 Cor. of Prop. 3]. Therefore it remains only to prove $\theta(\tilde{\boldsymbol{A}}) = \theta_1(\tilde{\boldsymbol{A}}_1)$. Each element $a \in \boldsymbol{A} = L^\infty(\Gamma, \mu)$ belongs to $\theta(\tilde{\boldsymbol{A}})$ (resp. $\theta_1(\tilde{\boldsymbol{A}}_1)$) if and only if $a(\gamma)$ is constant on the coset $\Theta^{-1}(\tilde{\gamma})$ (resp. $\tilde{\Theta}_1^{-1}(\tilde{\gamma}_1)$) for almost every $\tilde{\gamma} \in \tilde{\Gamma}$ (resp. $\tilde{\gamma}_1 \in \tilde{\Gamma}_1$). As seen in the proof of sufficiency, almost every coset $\Theta^{-1}(\tilde{\gamma})$ (resp. $\Theta^{-1}(\tilde{\gamma}_1)$) becomes $\mathfrak{R}$-equivalence class in Γ, which implies $\theta(\widetilde{\boldsymbol{A}}) = \theta_1(\widetilde{\boldsymbol{A}}_1)$. This completes the proof.

DEFINITION 3.1. For each smooth maximal abelian subalgebra A of a von Neumann algebra $\boldsymbol{M}$, we call the triard $(\boldsymbol{M}, \boldsymbol{A}, \theta)$, appeared in Theorem 3.1, the *simplification* of the pair $(\boldsymbol{M}, \boldsymbol{A})$.

DEFINITION 3.2. Let $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, (Γ_1, μ_1), (Γ_2, μ_2) and $\boldsymbol{M}$ be as in § 1. Let p_1 and p_2 be non-zero projections of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ associated with Borel subsets $P_1 \subset \Gamma_1$ and $P_2 \subset \Gamma_2$ respectively. Let E_1 and E_2 be the projections of the graph of $\mathfrak{R}^{\boldsymbol{M}, \mathfrak{A}, \Phi^1, \Phi^2}_{\boldsymbol{A}_1, \boldsymbol{A}_2} = \mathfrak{R}$ in $P_1 \times P_2$ into Γ_1 and Γ_2 respectively. If there exist partitions of E_1 and E_2 such that $E_1 = F_1 \cup F_1'$, $E_2 = F_2 \cup F_2'$, $F_1 \cap F_1' = F_2 \cap F_2' = \phi$, $F_1, \cdots, F_2'$ are measurable, $\mu_1(F_1') = \mu_2(F_2') = 0$ and F_i' contains every $\gamma_i \in E_i$ satisfying the condition $\mathfrak{R}(\gamma_1, \gamma_2)$ for some $\gamma_j \in F_j$ $i \neq j$, $i, j = 1, 2$, then we say that $\boldsymbol{A}_1 p_1$ and $\boldsymbol{A}_2 p_2$ are *unrelated*. That is, $\boldsymbol{A}_1 p_1$ and $\boldsymbol{A}_2 p_2$ are *unrelated* if and only if $\mathfrak{R}(\gamma_1, \gamma_2)$ does not hold for every pair $(\gamma_1, \gamma_2) \in P_1 \times P_2$, eliminating null sets from P_1 and P_2. Otherwise, we say that $A_1 p_1$ and $A_2 p_2$ are related. Moreover if for each non-negligible subset $E_i \subset P_i$ $(i = 1, 2)$ the set F_j of all γ_j's of P_j satisfying the condition $\mathfrak{R}(\gamma_1, \gamma_2)$ for some $\gamma_i \in E_i$ $(i \neq j,\ i, j = 1, 2)$ is not negligible, then we say that $\boldsymbol{A}_1 p_1$ and $\boldsymbol{A}_2 p_2$ are *similar*

THEOREM 3.2. *For a maximal abelian subalgebra $\boldsymbol{A}$ of a von Neumann algebra $\boldsymbol{M}$ to be smooth, it is necessary and sufficient that there exists a partition of unit $p_0 + \sum_{n=1}^{\infty} p_n + p_\infty = I$ in $\boldsymbol{A}$ satisfying the following conditions*:

1°. *For each $1 \leqq n \leqq \infty$ $p_n \boldsymbol{M} p_n$ and $\boldsymbol{A} p_n$ can be represented such as $p_n \boldsymbol{M} p_n = \boldsymbol{M}_n \otimes \boldsymbol{B}_n$ and $\boldsymbol{A} p_n = \boldsymbol{A}_n \otimes l^\infty(\{1, 2, \cdots, n\})$ by some von Neumann algebra $\boldsymbol{M}_n$ and its simple maximal abelian subalgebra $\boldsymbol{A}_n$, where $\boldsymbol{B}_n$ means the full operator algebra over the n-dimensional Hilbert space $l^2(\{1, 2, \cdots, n\})$. Besides $\boldsymbol{A} p_n$ and $\boldsymbol{A} p_m$ are unrelated if $n \neq m, n, m \geqq 1$.*

2°. *$p_0 \boldsymbol{M} p_0$ and $\boldsymbol{A} p_0$ can be represented such as $p_0 \boldsymbol{M} p_0 = \boldsymbol{M}_0 \otimes \boldsymbol{B}_0$ and $\boldsymbol{A} p_0 = \boldsymbol{A}_0 \otimes L^\infty(0, 1)$ by some von Neumann algebra $\boldsymbol{M}_0$ and its simple maximal abelian subalgebra $\boldsymbol{A}_0$, where $\boldsymbol{B}_0$, means the full operator algebra over the Hilbert space $L^2(0, 1)$.*

If $\boldsymbol{A}$ is smooth, then the above decomposition of $\boldsymbol{M}$ and $\boldsymbol{A}$ is unique. If $\boldsymbol{M}$ is of finite type, then $p_0 = p_\infty = 0$.

PROOF. The sufficiency is a direct conclusion of Lemma 2.2, so we shall prove only the necessity. Suppose that $\boldsymbol{A}$ is smooth. Let $\boldsymbol{A} = L^\infty(\Gamma, \mu)$ and $\mathfrak{R} = \mathfrak{R}^{\boldsymbol{M}, \mathfrak{A}, \Phi}_{\boldsymbol{A}}$. Let $(\hat{\Gamma}, \hat{\mu})$ be the quotient measure space of (Γ, μ) by $\mathfrak{R}$ and let r be the canonical mapping of Γ onto $\hat{\Gamma}$. Identifying f and $f \circ r$ for each $f \in L^\infty(\hat{\Gamma}, \hat{\mu})$

$= \hat{A}, \hat{A}$ becomes a von Neumann subalgebra of A. By [5: §6, Prop. 7] there exist unique orthogonal projections p_0 and q_0 in A such that $p_0 + q_0 = I$, p_0 is the greatest relatively continuous projection with respect to $\hat{A}$ and q_0 is the greatest relatively discrete projection with respect to $\hat{A}$. So we shall study $(p_0 M q_0, A q_0)$ and $(p_0 M p_0, A p_0)$ separately.

1^0. Case of $q_0 = I$. For each non-zero projection $e \in A$ there exists the smallest projection $\bar{e}$ in $\hat{A}$ majorizing e, which is called $\hat{A}$-carrier of e. Let e and f be two relatively minimal projections in A with respect to A with the same $\hat{A}$-carrier. Let E and F be the Borel subsets of Γ associted with e and f respectively. Since $Ae = Ae$ and $Af = Af$, both the $r|E$ and $r|F$, the restrictions of r, are one-to-one mappings except for negligible parts. Since $\bar{e}$ and $\bar{f}$ are the projections of A associated with $r(E)$ and $r(F)$ respectively, we have $r(E) = r(F)$. Hence there exists a one-to-one Borel isomorphism ϕ from E onto F such $\mathfrak{R}(\gamma, \phi(\gamma))$ for almost every $\gamma \in E$. Since $Ae = \hat{A}e \cong \hat{A}\bar{e}$ under the canonical correspondence, r transforms the class of all null sets in E onto the one in $r(E)$. Hence ϕ is an isomorphism of the measure space $(E, \mu|E)$ onto $(F, \mu|F)$. By Lemma 3.1 there exists a partial isometry u of M defined by a family $\{u(\gamma), \gamma \in E\}$ of unitaries from $\mathfrak{H}(\phi(\gamma))$ onto $\mathfrak{H}(\gamma)$ such that $uu^* = e$ and $u^* u = f$. Hence, for each pair of relatively minimal projections e, f of A with respect to A, there exist orthogonal projections g, h, k in A such $g + h + k = I$, $(ge)^- = (gf)^- = g, g + h \geqq e, g + k \geqq f$ and $ge \sim gf$.

For each non-zero projection $e \in A$ there exists a relatively minimal projection f of A with respect to $\hat{A}$ such that $f \leqq e$ and $\bar{f} = \bar{e}$. Indeed, let $\{f_\alpha\}$ be a maximal family of relatively minimal orthogonal projections in A such that $f_\alpha \leqq e$ and the $\bar{f}_\alpha$'s are orthogonal each other. Then $f = \sum_\alpha f_\alpha$ is requried one.

Let $\{e_\alpha\}$ be a maximal family of relatively minimal orthogonal projections in A with $\hat{A}$-carrier I. If $I \neq \sum_\alpha e_\alpha$, then $(I - \sum_\alpha e_\alpha)^- \neq I$ by the maximality of $\{e_\alpha\}$. Putting $p = I - (I - \sum_\alpha e_\alpha)^- \in \hat{A}$, we have $p = \sum_\alpha pe_\alpha$ and $p = (pe_\alpha)^-$ If the cardinal of $\{e_\alpha\}$ is finite, then we repeat this argument for $A(I - p)$ and $A(I - p)$. If it is infinite, there exits a family $\{f_\alpha\}$ of relatively minimal orthogonal projections such that $\sum_\alpha f_\alpha = I$ and $f_\alpha = I$. Indeed, let $\{g_\beta\}$ be a maximal family of relatively minimal orthogonal subprojections of $I - \sum_\alpha e_\alpha$ in A. Since A is discrete over A, we have $I - \sum e_\alpha = \sum g_\beta$. Since M is acting on a separable Hilbert space, both $\{e_\alpha\}$ and $\{g_\beta\}$ are at most countable. Let $\{e_n\}$ and $\{g_n\}$ be their enumerations respectively. Let E_n and G_n be Borel subsets of

Γ associated with e_n and g_n respectively. Since $e_n = I \geqq g_n$ and e_n and g_n are relatively minimal, there exists a one-to-one Borel mapping ϕ from Γ into $\bigcup_{n=1}^{\infty} E_n$ such that $\phi(G_n) \subset E_{2n+1}$, $\phi(E_n) = E_{2n}$, $\mathfrak{R}(\gamma, \phi(\gamma))$ for almost every $\gamma \in \Gamma$ and $\phi(\mu) \approx \mu | \phi(\Gamma)$. By Bernstein's method it is easily shown that there exists a Borel one-to-one mapping ψ from Γ onto $\bigcup_{n=1}^{\infty} E_n$ such that $\mathfrak{R}(\gamma, \psi(\gamma))$ for almost every $\gamma \in \Gamma$ and $\psi(\mu) \approx \mu \Big| \bigcup_{n=1}^{\infty} E_n$. The family $\{f_n\}$ of projections associated with $\psi^{-1}(E_n)$ is the required one.

After all, there exists a family $\{p_n\}_{n=1,2,\ldots,\infty}$ of orthogonal projections of $\hat{A}$ and for each n there exists a family $\{e_{n,k} : 1 \leqq k \leqq n\}$ of relatively minimal orthogonal projections of $\boldsymbol{A}$ such that $\bar{e}_{n,k} = p_n$ for $k = 1, 2, \cdots, n$ and $p_n = \sum_{k=1}^{n} e_{n,k}$. Besides for each n and k there exists a partial isometry u of $\boldsymbol{M}$ such that $u^* u = e_{n,1}$, $uu^* = e_{n,k}$ and $u\boldsymbol{A}e_{n,1}u^* = \boldsymbol{A}e_{n,k}$. Since $\boldsymbol{A}e_{n,1} \cong \boldsymbol{A}p_n$ under the natural correspondence, $p_n\boldsymbol{M}p_n = e_{n,1}\boldsymbol{M} \otimes \boldsymbol{B}_n$ and $\boldsymbol{A}p_n = \boldsymbol{A}e_{n,1} \otimes l^\infty(1, 2, \cdots, n)$. Now it is clear that $\boldsymbol{A}e_{n,1}$ is a simple maximal abelian subalgebra of $e_{n,1}\boldsymbol{M}e_{n,1}$ and that $\boldsymbol{A}p_n$ and $\boldsymbol{A}p_m$ are unrelated if $n \neq m$.

2°. Case of $p_0 = I$. Replacing μ by an equivalent finite measure, we may assume the finiteness of μ. By the smoothness of $\boldsymbol{A}$ we get a decomposition $\mu = \int_{\hat{\Gamma}} \mu^{\hat{\gamma}} \, d\hat{\mu}(\hat{\gamma})$ of μ over the measure space $(\hat{\Gamma}, \hat{\mu})$ with respect to the mapping r. By [5 : § 5, Prop. 1] we can define

$$\mathfrak{H}(\hat{\gamma}) = \int_{r^{-1}(\hat{\gamma})}^{\oplus} \mathfrak{H}(\gamma) \, d\mu^{\hat{\gamma}}(\gamma) \text{ and } x(\hat{\gamma}) = \int_{r^{-1}(\hat{\gamma})}^{\oplus} x(\gamma) \, d\mu^{\gamma}(\gamma)$$

for almost every $\gamma \in \Gamma$ and for $x \in \mathfrak{A}$ and we get a decompostion

$$\mathfrak{H} = \int_{\hat{\Gamma}}^{\oplus} \mathfrak{H}(\hat{\gamma}) \, d\hat{\mu}(\gamma) \text{ and } x = \int_{\hat{\Gamma}}^{\oplus} x(\hat{r}) \, d\hat{\mu}(\hat{r})$$

under suitable identification. $\boldsymbol{A}$ becomes the diagonal algebra in this new decomposition. Since there exists a unitary u of $\mathfrak{H}(\gamma_0)$ onto $\mathfrak{H}(\gamma)$ for each $\gamma \in r^{-1}(\hat{\gamma}_0)$ such $ux(\gamma_0)u^{-1} = x(\gamma)$, we get

$$\mathfrak{H}(\hat{\gamma}_0) = \mathfrak{H}(\gamma_0) \otimes L^2(r^{-1}(\hat{\gamma}), \mu^{\hat{\gamma}}) \text{ and } x(\hat{\gamma}_0) = x(\gamma_0) \otimes I$$

for almost every $\hat{\gamma}_0 \in \hat{\Gamma}$ by [2 : Chap. II, § 2 Theorem 2]. Moreover, $\boldsymbol{A}$ is decomposable with respect to this new decomposition, whose almost every component $\boldsymbol{A}(\hat{\gamma})$ is represented by $\boldsymbol{A}(\hat{\gamma}) = \boldsymbol{C} \otimes L^\infty(r^{-1}(\hat{\gamma}), \mu^{\hat{\gamma}})$, where $\boldsymbol{C}$ means the complex number field. By [5 : § 6, Prop. 10] $\boldsymbol{A}(\gamma)$ is relatively continuous with respect to $\hat{A}(\hat{\gamma}) = \boldsymbol{C}$ for almost every $\hat{\gamma} \in \hat{\Gamma}$, so that almost every measure space $(r^{-1}(\hat{\gamma}), \mu^{\hat{\gamma}})$ has no discrete summand. Since almost every

$\hat{r}^{-1}(\hat{\gamma})$ is a Borel subset of the standard Borel space Γ, almost every measure space $(r^{-1}(\hat{\gamma}), \mu^{\gamma})$ is isomorphic to $(0,1)$-interval equipped with Lebesgue measure. Hence we get

$$\mathfrak{H}(\hat{\gamma}) = \mathfrak{H}(\gamma) \otimes L^2(0,1) \text{ and } \boldsymbol{A}(\hat{\gamma}) = \boldsymbol{C} \otimes L^\infty(0,1)$$

for almost every $\gamma \in \Gamma$.

By Lemma 2.1 there exists a measurable mapping ϕ from $\hat{\Gamma}$ to Γ such that $\phi(\gamma) \in r^{-1}(\hat{\gamma})$ for almost every γ. Since

$$\mathfrak{H}(\hat{\gamma}) = \mathfrak{H}(\phi(\hat{\gamma})) \otimes L^2(0,1) \text{ and } x(\gamma) = x(\phi(\hat{\gamma})) \otimes I$$

for almost every γ, we get

$$\mathfrak{H} = \left\{\int_{\hat{\Gamma}}^{\oplus} \mathfrak{H}(\phi(\hat{\gamma}))\, d\mu(\hat{\gamma})\right\} \otimes L^2(0,1)$$

and

$$x = \left\{\int_{\hat{\Gamma}} x(\phi(\hat{\gamma}))\, d\hat{u}(\hat{\gamma})\right\} \otimes I \qquad \text{for each } x \in \mathfrak{A}.$$

Putting $\mathfrak{K} = \int_{\Gamma}^{\oplus} \mathfrak{H}(\phi(\hat{\gamma}))\, d\hat{u}(\hat{\gamma})$ and $x_{\mathfrak{K}} = \int_{\Gamma}^{\oplus} x(\phi(\hat{\gamma}))\, d\hat{\mu}(\hat{\gamma})$ for $x \in \mathfrak{A}$, we have $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,1)$ and $x = x_{\mathfrak{K}} \otimes I$. It is clear that the diagonal algebra $\boldsymbol{A}_0$ in the decomposition of $\mathfrak{K}$ is isomorphic to $\boldsymbol{A}$ under the canonical correspondence and that $\{x_{\mathfrak{K}}(\gamma); x \in \mathfrak{K}\} = \{x(\phi(\gamma)); x \in \mathfrak{A}\}$ acts on $\mathfrak{K}(\gamma) = \mathfrak{H}(\phi(\gamma))$ irreducibly for almost every $\gamma \in \Gamma$. Besides the representations $x_{\mathfrak{K}} \to x_{\mathfrak{K}}(\gamma)$ of the C^*-algebra $\mathfrak{A}_{\mathfrak{K}} = \{x_{\mathfrak{K}}; x \in \mathfrak{A}\}$ are mutually disjoint. Hence $\boldsymbol{A}_0$ is a simple maximal abelian subalgebra of $\boldsymbol{M}_0 = \mathfrak{A}'_{\mathfrak{K}}$. Since $x = x_{\mathfrak{K}} \otimes I$ for every $x \in \mathfrak{A}$, we have $\boldsymbol{M} = \mathfrak{A}' = \boldsymbol{M}_0 \otimes \boldsymbol{B}_0$. And we get $\boldsymbol{A} = \boldsymbol{A}_0 \otimes L^\infty(0,1)$.

The unicity of $\{p_n\}_{n=0,1,\dots,\infty}$ is almost clear from its construction. This completes the proof.

THEOREM 3.3. *Let $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ be two maximal abelian subalgebras of a von Neumann algebra $\boldsymbol{M}$. Let e_1 and e_2 be non-zero projections of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ such that $\boldsymbol{A}_1e_1$ and $\boldsymbol{A}_2e_2$ are smooth in $e_1\boldsymbol{M}e_1$ and $e_2e\boldsymbol{M}_2$ respecsively. $\boldsymbol{A}_1e_1$ and $\boldsymbol{A}_2e_2$ are similar if the simplifications $((e_1\boldsymbol{M}e_1)\hat{}, (\boldsymbol{A}_1e_1)\hat{}, \theta_1)$ and $((e_2\boldsymbol{M}e_2)\hat{}, (\boldsymbol{A}_2e_2)\hat{}, \theta_2)$ are unitarily equivalent in the sense that there exists a unitary u of the underlying Hilbert space $\mathfrak{K}_1$ of $(e_1\boldsymbol{M}e_1)\hat{}$ onto the one $\mathfrak{K}_2$ of $(e_2\boldsymbol{M}e_2)\hat{}$ such that $u(\boldsymbol{A}_1e_1)u^{-1} = (\boldsymbol{A}_2e_2)$, $u(e_1\boldsymbol{M}e_1)\hat{}\,u^{-1} = (e_2\boldsymbol{M}e_2)\hat{}$ and $\theta_2(uxu^{-1}) = \theta_1(x)$ for $x \in (e_1\boldsymbol{M}e_1)\hat{}\,'$.*

PROOF. Suppose that $\boldsymbol{A}_1e_1$ and $\boldsymbol{A}_2e_2$ are similar. Let E_1 and E_2 be the Borel sets in Γ_1 and Γ_2 associated with e_1 and e_2 respectively. Putting $\mathfrak{A}_i = \mathfrak{A}e_i$ and $\Phi^i e_i = \{\varphi^i_{\gamma_i} \in \Phi^i; \gamma_i \ni E_i\}$ $(i=1,2)$, $\mathfrak{R}^{e_1Me_1, \mathfrak{A}_1, \Phi^1 e_1}_{A_1e_1} = \mathfrak{R}_1$ and $\mathfrak{R}^{e_2Me_2, \mathfrak{A}_2, \Psi^2 e_2}_{A_2e_2} = \mathfrak{R}_2$ are the restrictions of $\mathfrak{R}^{M, \mathfrak{A}, \Phi^1}_{A_1}$ and $\mathfrak{R}^{M, \mathfrak{A}, \Phi^2}_{A_2}$ to E_1 and E_2 respectively. Let $(\widehat{E_1}, \hat{\mu}_1)$ and $(\widehat{E_2}, \hat{\mu}_2)$ be the quotient measure spaces of (E_1, μ_1) and (E_2, μ_2) by $\mathfrak{R}_1$ and $\mathfrak{R}_2$ respectively.

Let r_1 and r_2 be the associated canonical mappings of E_1 and E_2 onto $\widehat{E}_1$ and $\widehat{E}_2$ respectively. Since $\mathfrak{R}^{M,\mathfrak{A},\Phi_1}_{A_1}(\gamma_1,\gamma_1')$, $\mathfrak{R}^{M,\mathfrak{A},\Phi^1,\Phi^2}_{A_1A_2}(\gamma_1,\gamma_2)$ and $\mathfrak{R}^{M,\mathfrak{A},\Phi^2}_{A_2}(\gamma_2,\gamma_2')$ imply $\mathfrak{R}^{M,\mathfrak{A},\Phi^1,\Phi^2}_{A_1,A_2}(\gamma_1',\gamma_2')$ for $\gamma_1,\gamma_1' \in \Gamma_1$ and for γ_2, $\gamma_2' \in \Gamma_2$, the mapping e of $\widehat{E}_1$ to $\widehat{E}_2$ defined by

$$\hat{r}(\hat{\gamma}_1) = r_2\circ pr_2((r_1^{-1}(\hat{\gamma}_1)\times E_2)\cap R)$$

for $\hat{\gamma}_1 \in E_1$ is a one-to-one mapping, where R is the graph of $\mathfrak{R}^{M,\mathfrak{A},\Phi^1,\Phi^2}_{A_1,A_2}$ in $E_1 \times E_2$. By the similarity of A_1e_1 and A_2e_2 $\hat{r}$ is defined almost everywhere in $\widehat{E}_1$ and it has the range with null complement in $\widehat{E}_2$. Eliminating null sets from $\widehat{E}_1$ and $\widehat{E}_2$ respectively, we may assume that $\hat{r}$ is a one-to-one mapping of $\widehat{E}_1$ onto $\widehat{E}_2$. Since $\hat{r}^{-1}(\widehat{S}_2) = r_1\circ pr_1((E_1\times r_2^{-1}(\widehat{S}_2))\cap R)$, $\hat{r}^{-1}(\widehat{S}_2)$ is analytic in $\widehat{E}_1$ for each Borel subset $\widehat{S}_2 \subset \widehat{E}_2$, so that $\hat{r}$ is measurable. Similary $\hat{r}^{-1}$ is also measurable. Besides the similarity of A_1e_1 and A_2e_2 implies that $\hat{r}$ is an isomorphism of the measure space $(\widehat{E}_1,\hat{\mu}_1)$ onto the one $(\widehat{E}_2,\hat{\mu}_2)$. Let ϕ_1 and ϕ_2 be measurable mappings of E_1 and $\widehat{E}_2$ to E_1 and E_2 such that $\phi_1(\hat{\gamma}_1) \in r_1^{-1}(\hat{\gamma}_1)$ and $\phi_2(\hat{\gamma}_2) \in r_2^{-1}(\hat{\gamma}_2)$ for almost every $\hat{\gamma}_1$ and $\hat{\gamma}_2$ respectively. Then we have $\mathfrak{R}^{M,\mathfrak{A},\Phi^1\Phi,^2}_{A_1,A_2}(\phi_1(\hat{\gamma}_1),\phi_2(r(\hat{\gamma}_1))$for almost every $\hat{\gamma}_1 \in \widehat{E}_1$. Using the naturally corresponding notations in the proof of necessity of Theorem 3. 1, there exists a unitary u of $\mathfrak{K}_1(\hat{\gamma}_1) = \mathfrak{H}^1,(\phi_1(\hat{\gamma}_1))$ onto $\mathfrak{K}_2(\hat{r}(\hat{\gamma}_1)) = \mathfrak{H}^2(\phi_2(\hat{r}(\hat{\gamma}_2)))$ for almpst every $\hat{\gamma}_1 \in \widehat{E}_1$ such that $u^{-1}x^1(\phi_1(\hat{\gamma}_1))u = x^2(\phi_2(\hat{r}(\hat{\gamma}_1))$ for each $x \in \mathfrak{A}$. By Lemma 3. 1 there exists a family $\{u(\hat{\gamma}_1);\hat{\gamma}_1 \in \widehat{E}_1\}$ of unitaries from $\mathfrak{H}^2(\phi_2\circ\hat{r}(\hat{\gamma}_1))$ onto $\mathfrak{H}^1(\phi_1(\hat{\gamma}_1))$ such that $u(\hat{\gamma}_1)^{-1}x^1(\phi_1(\hat{\gamma}_1) = x^2(\phi_2\circ r(\hat{\gamma}_1))$ for each $x\in\mathfrak{A}$, which defines a unitary u of $\mathfrak{K}_2$ onto $\mathfrak{K}_1$ by

$$u\,\xi = \int_{E_1}^{\oplus} u(\hat{\gamma}_1)\,\xi(\hat{r}(\hat{\gamma}_1))\sqrt{\frac{d\hat{\mu}^{-1}\hat{\mu}_2}{d\hat{\mu}_1}(\hat{\gamma}_1)}d\hat{\mu}^{-1}(\hat{\mu}_2)(\hat{\gamma}_1) \text{ for } \quad \xi \in \mathfrak{K}_2.$$

It is clear that u carries$(A_2e_2)^\wedge$ onto $(A_1 e_1)^\wedge$. Since $u\theta_2^{-1}(x)u^{-1} = \theta_1^{-1}(x)$ for each $x \in \mathfrak{A}$, u^{-1} induces the desired spatial isomorphism of $(e_1Me_1)^\wedge$ onto $(e_2Me_2)^\wedge$.

Conversely suppose that there exists a unitary u of $\mathfrak{K}_1$ onto $\mathfrak{K}_2$ satisfying the condition of our theorem. By [8 : Theorem 2. 7] and Lemma 3.1 there exists a one-to-one mapping r from a Borel suset of E_1 onto a Borel subset of E_2 with null complements such that $\hat{\gamma}(\tilde{\mu}_1)\approx\tilde{\mu}_2$ and there exists a family $\{u(\hat{\gamma}_1);\hat{\gamma}_1 \in \widehat{E}_1\}$ of unitaries from $\mathfrak{K}_2(\hat{r}(\hat{\gamma}_1))$ onto $\mathfrak{K}_1(\hat{\gamma}_1)$ such that

$$u\,\xi = \int_{E_1}^{\oplus} u(\hat{\gamma}_1)\,\xi(\hat{r}(\hat{\gamma}_1))\sqrt{\frac{d\hat{r}^{-1}(\hat{\mu}_2)}{d\hat{\mu}_1}(\hat{\gamma}_1)}d\hat{\mu}_1(\hat{\gamma}_1)$$

for $\xi \in \mathfrak{K}_2$ and $u(\hat{\gamma}_1)^{-1}\theta_1^{-1}(xe_1)(\hat{\gamma}_1)u(\hat{\gamma}_1) = \theta_2^{-1}(xe_2)(\hat{r}(\hat{\gamma}_1))$ for almost every $\hat{\gamma}_1 \in \widehat{E}_1$ and for each $x \in \mathfrak{A}$. By the proof of necessity in Theorem 3. 1 there exist unitaries u_1 and u_2 of $\mathfrak{H}^1(\gamma_1)$ and $\mathfrak{H}^2(\gamma_2)$ onto $\mathfrak{K}_1(r_1(\gamma_1))$ and $\mathfrak{K}_2(r_2(\gamma_2))$ for almost every $\gamma_1 \in E_1$ and $\gamma_2 \in E_2$ respectively such that

$$u_1x(\gamma_2)u_1^{-1} = \theta_1^{-1}(xe_1)(r_1(r_1)) \text{ and } u_2\,x(\gamma_2)u_2^{-1} = \theta_2^{-1}(xe_2)(r_2(\gamma_2))$$

for every $x \in \mathfrak{A}$. Theorefore we have $\mathfrak{R}^{M,\mathfrak{A},\Phi}_{A_1,A_2}(\gamma_1,\gamma_2)$ if $\hat{r} \circ r_1(\gamma_1) = r_2(\gamma_2)$ for almost every $\gamma_1 \in E_2$, and $\gamma_2 \ni E_2$ that is,

$pr_2((S_1 \times E_2) \cap R) \supset r_2^{-1}\hat{r}r_1(S_1)$ and $pr_1((E_1 \times S_2) \cap R) \supset r_1^{-1}\hat{r}r_2(S_2)$

for each subset $S_1 \subset E_1$ and $S_2 \subset E_2$ respectively. Since $r_1(\mu_1) = \hat{\mu}_1$ and $r_2(\mu_2) = \hat{\mu}_2$, we have

$$\mu_2(pr_2((S_1 \times E_2) \cap R)) > 0 \quad \text{and} \quad \mu_1(pr_1((E_1 \times S_2) \cap R)) > 0$$

for each non-negligible subsets $S_1 \subset E_1$ and $S_2 \subset E_2$, which implies the similarity of A_1e_1 and A_2e_2. This completes the proof.

Then we get the following

COROLLARY. *Similar simple maximal abelian subalgebras of a von Neumann algebra are unitarily equivalent.*

THEOREM 3.4. *Let A_1 and A_2 be two maximal abelian subalgebras of a von Neumann algebra M. Let e_1 and e_2 be non-zero projections of A_1 and A_2 respecively. If A_1e_1 and A_2e_2 are both smooth in e_1Me_1 and e_2Me_2 respectively, then there exist unique projections p_1 and p_2 of A_1 and A_2 majorized by e_1 and e_2 respectively such that A_1p_1 and A_2p_2 are similar and $A_1(e_1 - p_1)$ and $A_2(e_2 - p_2)$ are unrelated.*

PROOF. If A_1e_1 and A_2e_2 are unrelated, then our mention is trivial. So we assume A_1e_1 and A_2e_2 are related. We use the notation in the proof of Theorem 3.3.

As in the proof of Theorem 3.3, there exists a one-to-one measurable mapping $\hat{r}$ from subset of $\hat{E}_1$ into $\hat{E}_2$, whose definition domain $\hat{F}_1$ and range $\hat{F}_2$ are given by

$\hat{F}_1 = r_1 \circ pr_1((E_1 \times E_2) \cap R)$ and $\hat{F}_2 = r_2 \circ pr_2((E_1 \times E_2) \cap R)$ respectively.

By the relatedness of A_1e_1 and A_2e_2 the measures $\hat{r}(\hat{\mu}_1|\hat{F}_1)$ and $\hat{\mu}_2|\hat{F}_2$ are not disjoint. Hence there exists a unique subset $\hat{P}_2 \subset \hat{F}_2$ up to μ_2-null set such that the measures $\hat{r}(\hat{\mu}_1|\hat{F}_1)|\hat{P}_2$ and $\hat{\mu}_2|\hat{P}_2$ are equivalent and the measures $\hat{r}(\hat{\mu}_1|\hat{F}_1)|(\hat{F}_2 - \hat{P}_2)$ and $\hat{\mu}_2|(\hat{F}_2 - \hat{P}_2)$ are disjoint. Putting $r_1^{-1}\circ\hat{r}^{-1}(\hat{P}_2) = P_1 \subset E_1$ and $r_2^{-1}(\hat{P}_2) = P_2 \subset E_2$, the projections p_1 and p_2, associated with P_1 and P_2 respectively, are required ones.

In closing this section we state the following interpretation in the representation theory.

COROLLARY. *Let φ_1 and φ_2 be two representations of an involutive separble Banach algebra $\mathfrak{B}$ over separable Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively. Let $A_1 = L^\infty(\Gamma_1, \mu_1)$ and $A_2 = L^\infty(\Gamma_2, \mu_2)$ be smooth maximal abelian subalgebras of $\varphi_1(\mathfrak{B})' = M_1$ and $\varphi_2(\mathfrak{B})' = M_2$ respectively. Decompos eφ_1 and φ_2 into direct integrals of irreducible representations over Γ_1 and Γ_2 with respect to A_1 and A_2 as follows respectively;*

$$\mathfrak{H}_1 = \int_{\Gamma_1}^{\oplus} \mathfrak{H}_1(\gamma_1)\, d\mu_1(\gamma_1), \qquad \mathfrak{H}_2 = \int_{\Gamma_2}^{\oplus} \mathfrak{H}_2(\gamma_2)\, d\mu_2(\gamma_2)$$

$$\varphi_1 = \int_{\Gamma_1}^{\oplus} \varphi_1(\gamma_1) d\mu_1(\gamma_1), \quad \text{and } \varphi_2 = \int_{\Gamma_2}^{\oplus} \varphi_2(\gamma_2)\, d\mu_2(\gamma_2).$$

Then there exist Borel subsets $P_1 \subset \Gamma_1$ *and* $P_2 \subset \Gamma_2$ *such that* $\varphi_1(\gamma_1) \not\approx \varphi_2(\gamma_2)$ *for every* $(\gamma_1, \gamma_2) \in \complement P_1 \times \complement P_2$ *and if* P_1 *and* P_2 *are non-negligible then for each non-negligible subset* $S_i \subset P_i$ *we have*

$$\mu_j(\{\gamma_j \in P_j;\, \varphi_1(\gamma_1) \cong \varphi_2(\gamma_2) \text{ for some } \gamma_i \ni S_i\}) > 0$$

$i,j = 1, 2, i \neq j$. *Bedides if* p_1 *and* p_2 *are the projections of* A_1 *and* A_2 *associated with* P_1 *and* P_2, *then* $\varphi_1^{p_1}$ *and* $\varphi_2^{p_2}$ *are quasi-equivalent*[5]. *Hence if* φ_1 *and* φ_2 *are disjoint, then* $\mu_1(P_1) = \mu_2(P_2) = 0$.

PROOF. Put

$$\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2,\ \varphi = \varphi_1 \oplus \varphi_2 \text{ and } \boldsymbol{A} = \boldsymbol{A}_1 \oplus \boldsymbol{A}_2 = L^\infty(\Gamma_1 \oplus \Gamma_2, \mu_1 \oplus \mu_2).$$

Then $\boldsymbol{A} \subset \boldsymbol{M}_1 \oplus \boldsymbol{M}_2 \subset \varphi(\mathfrak{B})' = \boldsymbol{M}$ and $\boldsymbol{A}$ becomes a maximal abelian subalgebra of $\boldsymbol{M}$. Let e_1 and e_2 be the projections of $\mathfrak{H}$ onto $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively. Then e_1 and e_2 belong to $\boldsymbol{A}$. Application of Theorem 3.3 and 3.4 to $\boldsymbol{M}$, $\boldsymbol{A}e_1$ and $\boldsymbol{A}e_2$ yields our mention.

REMARK. Unfortunately $\varphi_1^{(I-p_1)}$ and $\varphi_2^{(I-p_2)}$ need not be disjoint. Indeed, if $e_1 - p_1 \sim e_2 - p_2$ in $\boldsymbol{M}$ then $\varphi_1^{(I-p_1)}$ and $\varphi_2^{(I-p_2)}$ are unitarily equivalent. Such case often occurs if $\boldsymbol{M}$ is of continuous type. For instance, let $\boldsymbol{A}$ be a simple maximal abelian subalgebra of $\boldsymbol{M}$ and let e be a projection of $\boldsymbol{A}$ such $e \sim I - e$. Then φ^e and $\varphi^{(I-e)}$ are unitarily equivalent, though the decompositions of φ^e and $\varphi^{(I-e)}$ with respect to $\boldsymbol{A}e$ and $\boldsymbol{A}(I - e)$ has no common component.

4. Simple maximal abelian subalgebras and completely rough maximal abelian subalgebras. In [3] Dixmier introduced the notions of regularity, semi-regularity and singularity for the maximal abelian subalgebra of factor. This section is devoted to the study of the relation between these algebraic properties and rather analytic properties : simplicity and complete roughness, of maximal subalgebras. First we shall slightly generalize the notion defined by Dixmier.

DEFINITION 4.1. Let $\boldsymbol{M}$ be a von Neumann algebra, $\boldsymbol{Z}$ its center and $\boldsymbol{A}$ a maximal abelian subalgebra. Let $\boldsymbol{P}$ be the von Neumann subalgebra of $\boldsymbol{M}$ generated by all unitaries of $\boldsymbol{M}$ safisfying the condition $uAu^{-1} = \boldsymbol{A}$. We call $\boldsymbol{A}$ regular, semi-regular or singular according to $\boldsymbol{P} = \boldsymbol{M}, \boldsymbol{P}' \cap \boldsymbol{M} = \boldsymbol{Z}$ or $\boldsymbol{P} = \boldsymbol{A}$.

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5) For a representation φ of $\mathscr{B}$ and a projection e of $\varphi(\mathscr{B})'$, φ^e is the representation of $\mathscr{B}$ to $e\,\mathfrak{H}$ defined by $\varphi^e(x)\xi = \varphi(x)e\xi$ for every $\xi \in e\mathfrak{H}$ and $x \in \mathscr{B}$.

THEOREM 4.1. *A simple maximal abelian subalgebra is singular.*

PROOF. Suppose that a maximal abelian subalgebra $\boldsymbol{A} = L^\infty(\Gamma, \mu)$ of a von Neumann algebra $\boldsymbol{M}$ is not singular. Then there exists a unitary u of $\boldsymbol{M}$ such that $u\boldsymbol{A}u^{-1} = \boldsymbol{A}$ and $u \notin \boldsymbol{A}$. By the maximality of $\boldsymbol{A}$, u does not commute with some element of $\boldsymbol{A}$, so that u induces a nontrivial automorphism θ of $\boldsymbol{A}$. Hence the associated mapping Θ of Γ onto Γ is not trivial, that is, there exists a Borel set E such that $\mu(E) > 0$ and $\Theta(\gamma) \neq \gamma$ for every $\gamma \in E$. By Theorem 1.2 we have $\mathfrak{R}^{\boldsymbol{M},\mathfrak{A},\Phi}_{\boldsymbol{A}}(\gamma, \Theta(\gamma))$ for almost every $\gamma \in E$, so that $\mathfrak{R}^{\boldsymbol{M},\mathfrak{A},\Phi}_{\boldsymbol{A}}(\gamma, \gamma')$ does not imply $\gamma = \gamma'$. Hence $\boldsymbol{A}$ is not simple.

THEOREM 4.2. *If there exists a semi-regular smooth maximal abelian subalgebra in a von Neumann algebra* $\boldsymbol{M}$, *then* $\boldsymbol{M}$ *is of type I.*

PROOF. Let $\boldsymbol{A} = L^\infty(\Gamma, \mu)$. Let G be a countable group of unitaries satisfying the condition $u\boldsymbol{A}u^{-1} = \boldsymbol{A}$ which generates $\boldsymbol{P}$. The existence of such group is guaranteed by the separability condition for $\boldsymbol{M}$. By the countability of G we may assume that the associated automorphism Θ_u in (Γ, μ) is defined over the whole space Γ for all $u \in G$ by elimination of some null Borel set from Γ.

Replacing the measure μ with an equivalent finite one, we assume that μ is a finite measure. We shall use the notations in the proof of 2° of Theorem 3.2. As in the proof of Theorem 3.2, we decompose μ over the measure space (Γ, μ) as follows; $\mu = \int \mu^{\hat\gamma} d\hat\mu(\hat\gamma)$ and $\mu^{\hat\gamma}$ is concentrated in $r^{-1}(\hat\gamma)$ for almost every γ. Putting

$$\mathfrak{H}(\hat\gamma) = \int^{\oplus}_{r^{-1}(\hat\gamma)} \mathfrak{H}(\gamma)\, d\mu^{\hat\gamma}(\gamma) \quad \text{and} \quad x(\gamma) = \int^{\oplus}_{r^{-1}(\hat\gamma)} x(\gamma) d\mu^{\hat\gamma}(\gamma)$$

for each $x \in \mathfrak{A}$, we have

$$\mathfrak{H}(\hat\gamma) = \mathfrak{H}(\gamma) \otimes L^2(\hat{r}^{-1}(\gamma), \mu^{\hat\gamma}),\ x(\hat\gamma) = x(\gamma) \otimes I$$

for some $\gamma \in r^{-1}(\hat\gamma)$ and

$$\mathfrak{H} = \int^{\oplus}_{\hat\Gamma} \mathfrak{H}(\hat\gamma)\, d\hat\mu(\hat\gamma), \qquad x = \int^{\oplus}_{\hat\Gamma} x(\hat\gamma)\, d\hat\mu(\hat\gamma)$$

for each $x \in \mathfrak{A}$ as in the proof of Theorem 3.2. Since $\boldsymbol{A}$ is maximal abelian in $\boldsymbol{M}$, $\{x(\gamma) : x \in \mathfrak{A}\}$ acts irreducibly on $\mathfrak{H}(\gamma)$ for almost every $\gamma \in \Gamma$, so that $\{x(\gamma) : x \in \mathfrak{A}\}$ generates a factor of type I on $\mathfrak{H}(\gamma)$. The associated diagonal algebra in the new decomposition of $\mathfrak{H}$ becomes $\widehat{\boldsymbol{A}} = L^\infty(\hat\Gamma, \hat\mu)$. On the other hand, Θ_u, $u \in G$, transforms each equivalence class onto itself, so that every element of $\boldsymbol{A}$ commutes with $u \in G$. Hence we have $\boldsymbol{A} = \boldsymbol{Z}$, since $\boldsymbol{P}' \cap \boldsymbol{M} = \boldsymbol{Z}$. Therefore the new decomposition of $\mathfrak{H}$ induces the central decomposition $\boldsymbol{M}' = \int^{\oplus}_{\hat\Gamma} \boldsymbol{M}'(\hat\gamma)\, d\hat\mu(\hat\gamma)$ of $\boldsymbol{M}'$. Since $\boldsymbol{M}'(\hat\gamma)$ is generated by $\{x(\hat\gamma) : x \in \mathfrak{A}\}$, $\boldsymbol{M}'(\hat\gamma)$ is of type I. Hence $\boldsymbol{M}'$ is of type I by [2: Chap. II, § 3 Prop. 3], so that $\boldsymbol{M}$ is of type I. This completes the proof.

COROLLARY. *Every semi-regular maximal abelian subalgebra $\boldsymbol{A}$ of a von Neumann algebra $\boldsymbol{M}$ of continuous type is completely rough.*

PROOF. Let $\boldsymbol{A} = L^\infty(\Gamma, \mu)$. Suppose that there exists a non-zero projection e of $\boldsymbol{A}$ such that $\boldsymbol{A}e$ is a smooth maximal abelian subalgebra of $e\boldsymbol{M}e$. Let E be the Borel subset of Γ associated with e. Then we have $\boldsymbol{A}e = L^\infty(E, \mu)$. Denote $\mathfrak{R} = \mathfrak{R}^{\boldsymbol{M}, \mathfrak{A}, \Phi}_{\boldsymbol{A}}$ and $\mathfrak{R}^e = \mathfrak{R}^{e\boldsymbol{M}e, \mathfrak{A}e, \Phi e}_{\boldsymbol{A}e}$ in Γ and E respectively. By Lemma 2.1 there exists a Borel subset $S \subset E$ which has one and only one element in common with each $\mathfrak{R}^e$-equivalence class, eliminating a null set.

Now let G and $\{\Theta_u : u \in G\}$ be the groups of unitaries of $\boldsymbol{M}$ and of transformations in (Γ, μ) defined in the proof of Theorem 4.2 respectively. Putting $\cup\{\Theta_u E; u \in G\} = Z$, the projection z of $\boldsymbol{A}$ associated with Z commutes with every $u \in G$. Hence z is a non-zero central projection. Since $\mathfrak{R}^e$ is the restriction of $\mathfrak{R}$ to E, S has one and only one element in common with each $\mathfrak{R}^z$-equivalence class in Z where $\mathfrak{R}^z$ means $\mathfrak{R}^{\boldsymbol{M}z, \mathfrak{A}z, \Phi z}_{\boldsymbol{A}z}$. Hence $\boldsymbol{A}z$ is a smooth maximal abelian subalgebra of $\boldsymbol{M}z$ by Lemma 2.1. Moreover the semi-regularity of $\boldsymbol{A}$ yields the semi-regularity of $\boldsymbol{A}z$ in $\boldsymbol{M}z$. An application of Theorem 4.2. to $\boldsymbol{A}z$ and $\boldsymbol{M}z$ yields our mention.

THEOREM 4.3. *A completely rough maximal abelian subalgebra and a smooth one are unrelated.*

PROOF. Let $\boldsymbol{A}_1 = L^\infty(\Gamma_1, \mu_1)$ be a completely rough maximal abelian subalgebra of a von Neumann algebra $\boldsymbol{M}$ and $\boldsymbol{A}_2 = L^\infty(\Gamma_2, \mu_2)$ a smooth one of $\boldsymbol{M}$. Put $\mathfrak{R} = \mathfrak{R}^{\boldsymbol{M}, \mathfrak{A}, \Phi^1, \Phi^2}_{\boldsymbol{A}_1, \boldsymbol{A}_2}$, $\mathfrak{R}_1 = \mathfrak{R}^{\boldsymbol{M}, \mathfrak{A}, \Phi^1}_{\boldsymbol{A}_1}$, and $\mathfrak{R}_2 = \mathfrak{R}^{\boldsymbol{M}, \mathfrak{A}, \Phi^2}_{\boldsymbol{A}_2}$. Suppose that $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are related. Let R be the graph of $\mathfrak{R}$ in $\Gamma_1 \times \Gamma_2$. Put $E_1 = pr_1(R)$ and $E_2 = pr_2(R)$. By Theorem 1.3, E_1 and E_2 are analytic subsets of Γ_1 and Γ_2 respectively. Eliminating a null set from Γ_2, we may assume that $\hat{\Gamma}_2$ is a standard Borel space, so that $\widehat{E_2} = r_2(E_2)$ is an analytic subset of $\hat{\Gamma}_2$. Since $\mathfrak{R}(\gamma_1, \gamma_2)$, $\mathfrak{R}_1(\gamma_1, \gamma_1')$ and $\mathfrak{R}_2(\gamma_2, \gamma_2')$ imply $\mathfrak{R}(\gamma_1', \gamma_2')$, $f(\gamma_1) = r_2 \circ pr_2(\{\gamma_1\} \times \Gamma_2) \cap R)$ defines a mapping of E_1 into $\hat{\Gamma}_2$ whose range is $\widehat{E_2}$. For each Borel set $\widehat{S_2} \subset \hat{\Gamma}_2$, $f^{-1}(\widehat{S_2}) = pr_1(\{\Gamma_1 \times r_2^{-1}(\widehat{S_2})\} \cap R)$ is an analytic subset of E_1. Since $f^{-1}(\widehat{S_2})$ and $f^{-1}(\complement \widehat{S_2})$ are analytic and complementary subsets of E_1, $f^{-1}(\widehat{S_2})$ is a relatively Borel subset of E_1. Hence f is a Borel mapping of E_1 into $\hat{\Gamma}_2$. The measures $f(\mu_1)$ and $\hat{\mu}_2$ are not disjoint by the relatedness of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, so that there exists a Borel subset $\widehat{F_2} \subset \hat{\Gamma}_2$ contained in $\widehat{E_2}$ such that $f(\mu_1)|\widehat{F_2}$ and $\hat{\mu}_2|\widehat{F_2}$ are equivalent. Putting $F_1 = f^{-1}(\widehat{F_2})$, F_1 is a relatively Borel subset of E_1, so that f defines a Borel mapping from the analytic Borel space F_1 onto the standard one $\widehat{F_2}$. Besides, for each $\hat{r}_2 \in \widehat{F_2}$, $f^{-1}(\hat{r}_2)$ becomes a $\mathfrak{R}_1$-equivalence class in F_1, so that the mapping ϕ defined by $\phi(\hat{\gamma}_1) = f(r_1^{-1}(\hat{\gamma}))$ becomes a well defined, one-to-one

Borel mapping of $F_1/\mathfrak{R}_1 = \widehat{F_1}$ onto $\widehat{F_2}$ because of $r_1^{-1}\phi^{-1}(\widehat{S_2}) = f^{-1}(\widehat{S_2})$ for each Borel set $\widehat{S_2} \subset \widehat{F_2}$, where r_1 means the canonical mapping of F_1 onto $\widehat{F_1}$. For each Borel set $\widehat{S_1} \subset \widehat{F_1}$, $\phi(\widehat{S_1}) = f(r_1^{-1}(\widehat{S_1}))$ is analytic in $\widehat{F_2}$ and $\phi(\complement \widehat{S_1})$ and $\phi(\widehat{S_1})$ are complementary subsets of $\widehat{F_2}$, so that ϕ becomes a Borel isomorphism of $\widehat{F_1}$ onto $\widehat{F_2}$. Hence $\widehat{F_1}$ is a standard Borel space. Since F_1 is analytic, there exists a relatively Borel null set $N_1 \subset F_1$ such that $F_1 - N_1$ is standard, that is, $F_1 - N_1$ is a Borel subset of Γ_1, $(F_1 - N_1)/\mathfrak{R}_1 = (F_1 - N_1)^{\wedge}$ is analytic since $(F_1 - N_1)^{\wedge} \cong r_1(F_1 - N_1) \subset \widehat{F_1}$. Hence if f_1 is the projection of $\boldsymbol{A}_1$ associated with $F_1 - N_1$then $\boldsymbol{A}_1 f_1$ becomes smooth in $f_1 \boldsymbol{M} f_1$, but f_1 does not vanish by the definitions of F_1 and F_2. This contradicts to the complete roughness of $\boldsymbol{A}_1$ in $\boldsymbol{M}$.

Combining Corollary of Theorem 4. 2 and Theorem 4. 3 we assert the following

COROLLARY. *In the von Neumann algebra of continuous type, a smooth maximal abelian subalgebra and a semi-regular one are unrelated.*

THEOREM 4. 4. *A smooth singular maximal abelian subalgebra is simple.*

PROOF. Let $\boldsymbol{A}$ be a smooth singular maximal abelian subalgebra of a von Neumann algebra $\boldsymbol{M}$. Let $\{p_n\}_{n=0,1,\dots,\infty}$ be the family of projections appeared in Theorem 3.2. If $p_n \neq 0$ for some $n \neq 1$, it is clear that there exists a unitary u_n of $p_n \boldsymbol{M} p_n$ such that $u_n \boldsymbol{A} p_n u_n^{-1} = \boldsymbol{A} p_n$ and $u_n \notin \boldsymbol{A} p_n$. Putting $u = u_n + (I - p_n)$, then $u \boldsymbol{A} u^{-1} = \boldsymbol{A}, u \in \boldsymbol{M}$ and $u \notin \boldsymbol{A}$. Hence $\boldsymbol{A}$ is not singular. Therefore we have $p_n = 0$ for each $n \neq 1$, which implies the simplicity of $\boldsymbol{A}$.

Throughout the discussion of § 3 and § 4 the following natural questions arise for us: Are there algebraic characterizations of simple, smooth or completely rough maximal abelian subalgebra? In particular, is any singular maximal abelian subalgebra simple? Indeed, as shown in the next §, every already known example of singular maximal abelian subalgebra is simple.

5. Examples. In [8, Chap. III § 5] Mackey gave an example of unrelated pair of maximal abelian subalgebras in a factor of type II_1 which consists of simple one and regular. Besides his arguments show that the example of singular maximal abelian subalgebra of hyperfinite factor constructed by Dixmier [3] is simple. So in this section we shall give an example of simple maximal abelian subalgebra in a factor of type III by showing the example of singular one in a factor of type III constructed by Pukanszky [12] to be simple.

Let G be an arbitrary countably infinite discrete abelian group. For each element $g \in G$ we associate the cyclic group $\Omega_g = \{0, 1\}$ of order 2. By Ω we denote the product compact group of $\{\Omega_g; g \in G\}$. Δ is the subgroup of Ω composed of the element α such that $\alpha(g) = 0$ except for a finite member of g's. For $0 < p \leqq 1/2$ we define the measure μ_g in Ω_g by $\mu_g(\{0\}) = p$ and

$\mu_g(\{1\}) = 1 - p$ and the measure μ in Ω by $\mu = \prod_{g \in G} \mu_g$. For $g_0 \in G$ we define an automorphism of Ω by $\omega^{g_0}(g) = \omega(g_0 g)$. Putting $\mathfrak{G} = G \times \Delta$, we define the product in $\mathfrak{G}$ by $(g_1, \alpha_1)(g_2, \alpha_2) = (g_1 g_2, \ \alpha_1^{g_2} + \alpha_2)$. We canonically identify G and Δ with $G \times \{0\}$ and $\{e\} \times \Delta$ respectively. Next we define the action of $\mathfrak{G}$ on Ω by $\omega s = \omega^g + \alpha$ for $s = g\alpha \in \mathfrak{G}$. Then the measure μ becomes quasi-invariant under the action of $\mathfrak{G}$ by [12 : p. 144]. Putting $\frac{d\mu_s}{d\mu}(\omega) = \rho(\omega, s)$, where μ_s means the measure defined by $\mu_s(E) = \mu(Es)$, we have $\rho(\omega, g\alpha) = \rho(\omega g, \alpha)$ for $g \in G$ and $\alpha \in \Delta$.

Let $\mathfrak{H}_\Delta = L^2(\Omega \times \Delta, \mu \times \delta)$, where δ is the discrete measure in Δ. Let Γ be the dual group of G with Hear measure ν. For each $\gamma \in \Gamma$ and $\xi \in \mathfrak{H}_\Delta$, defining

$$(\hat{u}_\Delta^\gamma(g\alpha)\xi)(\omega, \beta) = \gamma(g)\rho(\omega, \alpha)^{1/2}\xi(\omega^g + \alpha, \beta^g + \alpha) \text{ for } g \in G \text{ and } \alpha \in \Delta,$$

$$(l_\Delta^\gamma(a)\xi)(\omega, \beta) = a(\omega)\xi(\omega, \beta) \qquad \text{for } a \in L^\infty(\Omega, \mu),$$

$$(\hat{v}_\Delta(\alpha)\xi)(\omega, \beta) = \xi(\omega, \beta - \alpha) \qquad \text{for } \alpha \in \Delta$$

and

$$(m_\Delta(a)\xi)(\omega, \beta) = a(\omega - \beta)\xi(\omega, \beta) \qquad \text{for } a \in L^\infty(\Omega, \mu),$$

we get bounded operators $\hat{u}_\Delta^\gamma(s)$, $l_\Delta^\gamma(a)$, $\hat{v}_\Delta(\alpha)$ and $m_\Delta(a)$ on $\mathfrak{H}_\Delta$ for $s \in \mathfrak{G}$, $\alpha \in \Delta$ and $a \in L^\infty(\Omega, \mu)$. Besides $\hat{u}_\Delta^\gamma(s)$ becomes a strongly continuous operator valued function over Γ and $l_\Delta^\gamma(a)$ becomes a constant function, so that we can define operators $\hat{v}(s)$ and $l(a)$ on $\mathfrak{H} = \mathfrak{H}_\Delta \otimes L^2(\Gamma, \nu)$ by

$$\hat{u}(s) = \int_\Gamma^\oplus \hat{u}_\Delta^\gamma(s) d\nu(\gamma) \qquad \text{and} \qquad l(a) = \int_\Gamma^\oplus l_\Delta^\gamma(a) d\nu(\gamma).$$

Of course, we have $l(a) = l_\Delta^\gamma(a) \otimes I$.

Let U^γ be the unitary representation of $\mathfrak{G}$ induced by the one-dimensional representation γ of the subgroup G and let u be the unitary representation of $\mathfrak{G}$ on $L^2(\Omega, \mu)$ defined by

$$(u(s)\xi)(\omega) = \rho(\omega, s)^{1/2}\xi(\omega s) \qquad \text{for } s \in \mathfrak{G} \text{ and } \xi \in L^2(\Omega, \mu).$$

Then we have, for $s \in \mathfrak{G}$ and $a \in L^\infty(\Omega, \mu)$,

$$u_\Delta^\gamma(s) = u(s) \otimes U^\gamma(s) \qquad \text{and} \qquad l_\Delta^\gamma(a) = a \otimes I \text{ on } \mathfrak{H}_\Delta = L^2(\Omega, \mu) \otimes l^2(\Delta).$$

Since

$$\mathfrak{H} = \mathfrak{H}_\Delta \otimes L^2(\Gamma, \nu) = \mathfrak{H}_\Delta \otimes l^2(G) = L^2(\Omega, \mu) \otimes l^2(\mathfrak{G})$$

under the natural identification and the right regular representation R of $\mathfrak{G}$ is decomposed into the direct integral $R = \int_\Gamma^\oplus U^\gamma(s) d\nu(\gamma)$ by [7 : Cor. of Theorem

10. 1], we have

$$u(s) = u(s) \otimes R(s) \text{ and } l(a) = a \otimes I \text{ on } \mathfrak{H} = L^2(\Omega.\mu) \otimes l^2(\mathfrak{G}),$$

for $s \in \mathfrak{G}$ and $a \in L^\infty(\Omega, \mu)$. Since the diagonal algebra in the decomposition $R = \int_\Gamma^\oplus U^\gamma d\nu(\gamma)$is generated by the image of G under the left regular representation of $\mathfrak{G}$, the diagonal algebra $\boldsymbol{A}$ in the decomposition $u(s) = \int_\Gamma^\oplus u_\Delta^\gamma(s)\, d\nu(\gamma)$ is generated by the image of G under the respresentation v of $\mathfrak{G}$ defined by

$$(v(s_0)\xi)(\omega, s) = \xi(\omega, s_0^{-1}s) \qquad \text{for } s_0,\ s \in \mathfrak{G}$$

and $\xi \in \mathfrak{H} = L^2(\Omega \times \mathfrak{G}, \mu \times \delta)$. Let $\boldsymbol{M}_\Delta$ be the von Neumann algebra generated by $\{v_\Delta(\alpha),\ m_\Delta(a) : \alpha \in \Delta,\ a \in L^\infty(\Omega, \mu).\}$ Then for every $\gamma \in \Gamma$ $\{u_\Delta^\gamma(\alpha),\ l_\Delta^\gamma(a) : \alpha \in \Delta \text{ and } a \in L^\infty(\Omega, \mu)\}$ generates $\boldsymbol{M}_\Delta'$. Let $\boldsymbol{M}$ be the von Neumann algebra acting on $\mathfrak{H}$ generated by $\{v(s),\ m(a) : s \in \mathfrak{G} \text{ and } a \in L^\infty(\Omega, \mu)\}$, where $m(a)$ is defined by $(m(a)\xi)(\omega, s) = a(\omega s^{-1})\xi(\omega, s)$ for $\xi \in \mathfrak{H} = L^2(\Omega \times \mathfrak{G},\ \mu \times \delta)$. Then $\boldsymbol{M}$ and $\boldsymbol{M}_\Delta$ becomes a factor of type II or type III according to the choice of p and $\boldsymbol{A}$ is a singular maximal abelian subalgebra of $\boldsymbol{M}$ by [12]. We shall show that $\boldsymbol{A}$ is simple.

Let $\mathfrak{A}$ be the C^*-subalgebra of $\boldsymbol{M}'$ generated by $\{u(s),\ l(a) : s \in \mathfrak{G}$ and $a \in C(\Omega)\}$. Then $\mathfrak{A}$ is a uniformly separable weakly dense subalgebra of $\boldsymbol{M}$ by [12]. For γ_1 and γ_2 of Γ suppose that there exists a bounded operator x on $\mathfrak{H}_\Delta$such that

$$u_\Delta^{\gamma_1}(s)x = xu_\Delta^{\gamma_2}(s) \qquad \text{and} \qquad l_\Delta^{\gamma_2}(a)x = xl_\Delta^{\gamma_1}(a)$$

for every $s \in \mathfrak{G}$ and for every $a \in C(\Omega)$. Then x belongs to $\boldsymbol{M}_\Delta'$, so that x can be expressed by $x = \sum_{\alpha \in \Delta} m_\Delta(x_\alpha)x(\alpha)$ in the strong operator topology. For each $g \in G$ we have

$$\begin{aligned}(u_\Delta^{\gamma_1}(g)x\xi)(\omega, \alpha) &= \gamma_1(g)(x\xi)(\omega^g, \alpha^g)\\ &= \gamma_1(g)\sum_{\beta \in \Delta} x_\beta((\omega - \alpha)g)\,\xi(\omega^g, \alpha^g - \beta)\end{aligned}$$

and

$$\begin{aligned}(xu_\Delta^{\gamma_2}(g)\xi)\,(\omega, \alpha) &= \sum_{\beta \in \Delta} x_\beta(\omega - \alpha)\,(u_\Delta^{\gamma_2}\,(g)\xi)\,(\omega, \alpha - \beta)\\ &= \gamma_2(g)\sum_{\beta \in \Delta} x_\beta(\omega - \alpha)\,\xi\,(\omega^g,\ \alpha^g - \beta^g)\end{aligned}$$

for every $\xi \in \mathfrak{H}_\Delta$. Hence we have

$$\gamma_1(g)\sum_{\beta \in \Delta} x_\beta(\omega^g - \alpha^g)\,\xi\,(\omega^g, \alpha^g - \beta)$$

$$= \gamma_2(g)\sum_{\beta\in\Delta} x_\beta(\omega-\beta)\,\xi(\omega^g, \alpha^g-\beta^g)$$

tor every $\xi \in \mathfrak{H}_\Delta$. Putting $\xi_0(\omega,\alpha)=1$ if $\alpha=0,=0$, if $\alpha\neq 0$ we have

$$\gamma_1(g)x_{\alpha^g}(\omega^g-\alpha^g) = \gamma_2(g)\,x_\alpha(\omega-\alpha)$$

for every $\alpha\in\Delta$, $g\in G$ and for almost every $\omega\in\Omega$. Hence

$$\gamma_1(g)\,x_\alpha^g(\omega^g) = \gamma_2(g)\,x_\alpha(\omega).$$

for every $\alpha\in\Delta$, $g\in G$ and for almost every $\omega\in\Omega$. It follows that

$$\int_\Omega |x_{\alpha^g}(\omega^g)|^2 d\mu(\omega) = \int_\Omega |x_\alpha^g(\omega)|^2 d\mu(\omega) = \int_\Omega |x_\alpha(\omega)|^2 d\mu(\omega).$$

Since $\sum_{\alpha\in\Delta}\int_\Omega |x_\alpha(\omega)|^2 d\mu(\omega) = (xx^*\xi_0,\xi_0) < +\infty$ and the set $\{\alpha^g : g\in G\}$ has infinitely many elements if $\alpha\neq 0$, we have $x_\alpha(\omega)=0$ almost everywhere for $\alpha\neq 0$. Putting

$$\xi_\alpha(\omega) = (-1)^{\sum_{g\in G}\alpha(g)\omega(g)} \prod_{g\in G}(p/(1-p))^{(\omega(g)-1/2)\alpha(g)}$$

for $\alpha\in\Delta$, $\{\xi_\alpha; \alpha\in\Delta\}$ becomes a complete orthonormalized system of $L^2(\Omega,\mu)$ by [12: Lemma 4]. Putting

$$c_\alpha = (x_0,\ \xi_\alpha) = \int_\Omega x_0(\omega)\xi_\alpha(\omega)d\mu(\omega),$$

we have

$$c_\alpha^g = \int_\Omega x_0(\omega)\,\xi_\alpha^g(\omega)\ d\mu(\omega) = \int_\Omega x_0(\omega)\,\xi_\alpha(\omega^{g^{-1}})\,d\mu(\omega)$$

$$= \int_\Omega x_0(\omega^g)\xi_\alpha(\omega)\ d\mu(\omega) = \int_\Omega \frac{\gamma_2(g)}{\gamma_1(g)}\,x_0(\omega)\,\xi_\alpha(\omega)\,d\mu(\omega)$$

$$= \frac{\gamma_2(g)}{\gamma_1(g)}\,c_\alpha,$$

which implies $|c_\alpha^g| = |c_\alpha|$. Hence $c_\alpha=0$ if $\alpha\neq 0$. This means that $x_0(\omega)$ is a constant. That is, x becomes a scalar. Therefore, if γ_1 and γ_2 are different characters of G then $x=0$. If $\gamma_1=\gamma_2$, say γ, then $\{u_\Delta^\gamma(s)$ and $l_\Delta^\gamma(a)(a) : s\in\mathfrak{G}$ and $a\in C(\Omega)\}$, which generates the γ-component of $\mathfrak{A}$, acts irreducibly on $\mathfrak{H}_\Delta$ Hence $\mathfrak{R}_A^{M,\mathfrak{A},\Phi}$ (γ_1,γ_2) implies $\gamma_1=\gamma_2$ for $\gamma_1,\gamma_2\in\Gamma$, that is, $\boldsymbol{A}$ is a simple maximal abelian subalgebra of $\boldsymbol{M}$, where Φ means the set of representations of $\mathfrak{A}'$ defined by its γ's-compoments.

After all, we get the following

THEOREM 5.1. *Hyperfinite factor has a simple maximal abelian subalgebra and a completely rough one simultaneously. There exists a factor of type* III *which has a simple maximal abelian subalgebra and a completely rough one simultaneously.*

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