

AN EXTENSION OF THE INTERPOLATION THEOREM OF MARCINKIEWICZ II

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1. Introduction. The purpose of this paper is to give the detailed proof of the theorem which is announced in my previous paper [2], that is, to show that Marcinkiewicz's theorem on the interpolation of operators (e.g. see A. Zygmund [6; Chap. XII]) holds good for Hardy class $H_p(p \geq 1)$.

H_p -class ($p > 0$) is the space of all functions analytic in the unit circle such that

$$\|\varphi\|_p = \lim_{r \rightarrow 1-0} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |\varphi(re^{i\theta})|^p d\theta \right\}^{1/p}$$

is finite. For $p \geq 1$ this class is equivalent to the space of functions in $L_p(-\pi, \pi)$ with the ordinary L_p -norm such that their Fourier expansion is power series type, that is,

$$\sum_{n=0}^{\infty} a_n e^{inx}. \tag{1.1}$$

Our method of proof depends on the real one and can be applied to some n -dimensional analogues of H_p -class.

Sections 2 and 3 contain the case of one variable H_p -space.

Sections 4 and 5 treat n -dimensional analogues of H_p -space.

Section 6 contains some applications to the theorems on Fourier series.

2. Two Lemmas. We begin by defining some notations. Let $f \in L_p(-\pi, \pi)$ ($p > 1$) be periodic with period 2π and its Fourier expansion be

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{inx},$$

then its conjugate function $\tilde{f}(x)$ is defined by

$$\tilde{f}(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \int_{\pi \geq |y| \geq \varepsilon} \frac{f(y)}{2 \tan(x-y)/2} dy \tag{2.1}$$

or equivalently

$$\tilde{f}(x) \sim - \sum_{n=-\infty}^{\infty} i(\text{sign } n) a_n e^{inx}. \tag{2.2}$$

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Therefore if we put

$$Kf = (f + i\tilde{f})/2 \tag{2.3}$$

for $f \in L_p(-\pi, \pi)$ ($p > 1$), then by expressions (1.1) and (2.2), we have $Kf \in H_p$ and in particular if $f \in H_p$ ($p \geq 1$), then we have $Kf = f$.

Our first lemma is a modification of that of L.Hörmander's [1; p. 115].

LEMMA 1. Let $f \in L_p(-\pi, \pi)$ ($\infty > p \geq 1$), $p \geq r \geq 1$ and let us define a_0 by

$$\frac{\pi}{2} = \frac{1}{a_0^r} \int_{-\pi}^{\pi} |f(x)|^r dx, \tag{2.4}$$

then for every $a > a_0$, the following decomposition of f is possible ;

- (i) $f = u + u', u' = v + w, w = \sum_{k=1}^{\infty} w_k,$
- (ii) $u = f, \text{ if } |f| < a \text{ and } u = 0 \text{ elsewhere,}$
- (iii) $|v(x)| \leq 2^{1/r} a \quad \text{for a.e. } x \text{ in } (-\pi, \pi),$
- (iv) $\int_{-\pi}^{\pi} |v(x)|^s dx \leq \int_{-\pi}^{\pi} |u'(x)|^s dx \quad \text{for each } s, 1 \leq s \leq p,$
- (v) $\sum_{k=1}^{\infty} \int_{-\pi}^{\pi} |w_k(x)|^s dx \leq 2^{s+1} \int_{-\pi}^{\pi} |u'(x)|^s dx \quad \text{for each } s, 1 \leq s \leq p,$
- (vi) *there exists a sequence $\{I_k\}$ of disjoint intervals such that the support of w_k is contained in I_k and*

$$\sum_{k=1}^{\infty} |I_k| \leq \frac{1}{a^r} \int_{-\pi}^{\pi} |u'(x)|^r dx,$$

$$\int_{-\pi}^{\pi} w_k(x) dx = 0, \quad k = 1, 2, 3, \dots \tag{vii}$$

PROOF. We define u by (ii) and put $u' = f - u$. Now we decompose u' . If we note that $a > a_0$, by the definition of a_0 we have

$$\frac{\pi}{2} = \frac{1}{a_0^r} \int_{-\pi}^{\pi} |f(x)|^r dx \geq \frac{1}{a^r} \int_{-\pi}^{\pi} |u'(x)|^r dx. \tag{2.5}$$

Let us divide the interval $(-\pi, \pi)$ into the four intervals of same length. The mean value of $|u'(x)|^r$ over every intervals is less than a^r by (2.5). Divide each interval into two equal intervals and let $I_{11}, I_{12}, I_{13}, \dots$ be those intervals over which the mean of $|u'|^r$ is not smaller than a^r . We have

$$a^r |I_{1k}| \leq \int_{I_{1k}} |u'(x)|^r dx \leq 2a^r |I_{1k}|. \tag{2.6}$$

For if I_{1k} is obtained by subdivision of I , by our construction

$$a^r |I_{1k}| \leq \int_{I_{1k}} |u'(x)|^r dx \leq \int_I |u'(x)|^r dx \leq a^r |I| = 2 a^r |I_{1k}|.$$

We set

$$v(x) = \frac{1}{|I_{1k}|} \int_I u'(y) dy \quad \text{for } x \text{ in } I_{1k}, \tag{2.7}$$

$$w_{1k}(x) = \begin{cases} u'(x) - v(x) & \text{for } x \text{ in } I_{1k} \\ 0 & \text{elsewhere,} \end{cases} \tag{2.8}$$

$k = 1, 2, 3, \dots$

Next we make a new subdivision of the intervals which are not in $\{I_{1k}\}$. Denoting by I_{2k} the intervals over which the mean value of $|u'|$ is not smaller than a^r , we extend the definition (2.7) and (2.8) to those intervals. Continuing in this way, we get the sequences of functions w 's and intervals I 's; for simplicity we write them by $\{w_k\}$ and $\{I_k\}$. If we write $O = \bigcup_{k=1}^{\infty} I_k$ and define

$$v(x) = u'(x) \quad \text{for } x \text{ in } (-\pi, \pi) - O,$$

then it is clear that (i), (vi) and (vii) hold. By Hölder's inequality with exponents s and s' , $1/s + 1/s' = 1$, we have

$$|v(x)|^s \leq \frac{1}{|I_k|^s} \left(\int_{I_k} |u'(y)| dy \right)^s \leq \frac{|I_k|^{s/s'}}{|I_k|^s} \int_{I_k} |u'(y)|^s dy = \frac{1}{|I_k|} \int_{I_k} |u'(y)|^s dy \tag{2.8}$$

for x in I_k . Thus we have

$$\begin{aligned} \int_{-\pi}^{\pi} |v(x)|^s dx &= \left(\int_{CO} + \sum_{k=1}^{\infty} \int_{I_k} \right) |v(x)|^s dx \\ &\leq \int_{CO} |u'(x)|^s dx + \sum_{k=1}^{\infty} \int_{I_k} |u'(y)|^s dy = \int_{-\pi}^{\pi} |u'(y)|^s dy, \end{aligned}$$

where $CO = (-\pi, \pi) - O$ and

$$\begin{aligned} \int_{-\pi}^{\pi} |w_k(x)|^s dx &\leq 2^s \left\{ \int_{I_k} |u'(x)|^s dx + \int_{I_k} |v(x)|^s dx \right\} \\ &\leq 2^s \left\{ \int_{I_k} |u'(x)|^s dx + \int_{I_k} |u'(y)|^s dy \right\} = 2^{s+1} \int_{I_k} |u'(y)|^s dy, \end{aligned}$$

using (2.8) again. Therefore (iv) and (v) are proved. To prove (iii) we use (2.8) with $s = r$. If x belongs to some interval I_k , we have

$$|v(x)|^r \leq \frac{1}{|I_k|} \int_{I_k} |u'(y)|^r dy \leq 2a^r$$

by (2.6). On the other hand if x does not belong to any intervals I_k , there are arbitrarily small intervals containing x such that

$$\frac{1}{|I|} \int_I |u'(x)|^r dx \leq a^r,$$

by our choice of I_k . Hence $|v(x)| = |u'(x)| \leq a$ for a.e. x in CO .

Therefore our proof is completed.

LEMMA 2. *With the notations of (2.3) and Lemma 1, we have*

$$\sum_{k=1}^{\infty} \int_{CE} |Kw_k| dx \leq A \int_{-\pi}^{\pi} |u'| dx, \quad (2.9)$$

where E is the set obtained by expanding each I_k concentrically three times and $CE = (-\pi, \pi) - E$ and A is some constant¹⁾ not depending on f and $a > a_0$.

PROOF. This lemma is known, but for the sake of completeness we show it.

Let us fix k and put $I_k = I = (a - h, a + h)$ and $I_k^* = I^* = (a - 3h, a + 3h) \cap (-\pi, \pi)$, then we have

$$\int_{CE} |Kw_k| dx \leq \frac{1}{2} \int_{CI^*} |w_k| dx + \frac{1}{2} \int_{CI^*} |\tilde{w}_k| dx.$$

Using (vii) in Lemma 1, we have

$$\begin{aligned} \int_{CI^*} |\tilde{w}_k| &\leq \frac{1}{2\pi} \int_{I^*} dx \left| \int_{-\pi}^{\pi} \left(\frac{1}{\tan(x-y)/2} - \frac{1}{\tan(x-a)/2} \right) w_k(y) dy \right| \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |w_k(y)| dy \int_{CI^*} \left| \frac{1}{\tan(x-y)/2} - \frac{1}{\tan(x-a)/2} \right| dx \\ &\leq \frac{1}{2\pi} \int_{-h}^h |w_k(y+a)| dy \int_{(-\pi, \pi) - (-3h, 3h)} \left| \frac{1}{\tan(x-y)/2} - \frac{1}{\tan x/2} \right| dx \\ &\leq \frac{1}{2\pi} \int_{-h}^h |w_k(y+a)| dy \int_{(-\pi, \pi) - (-3h, 3h)} \left| \frac{\sin y/2}{(\sin(x-y)/2) \sin x/2} \right| dx. \end{aligned}$$

Noting that $0 < h \leq \pi/2$ by our construction, it is easily verified that the inner integral does not exceed

$$A' \int_{3h}^{\infty} \left| \frac{h}{(x-y)x} \right| dx < A'', \quad |y| \leq h.$$

Hence we get $\int_{CI^*} |\tilde{w}_k| dx \leq (A''/2) \int_I |w_k| dx$ and

1) A, A', A'' , are some constants and may be different in each occasion.

$$\int_{CE} |Kw_k| dx \leq (1/2 + A''/2) \int_{-\pi}^{\pi} |w_k| dx.$$

Summing up over k and using (v) of Lemma 1 with $s = 1$, we get

$$\begin{aligned} \sum_{k=1}^{\infty} \int_{CE} |Kw_k| dx &\leq (1/2 + A''/2) \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} |w_k| dx \\ &\leq (1 + A'') \int_{-\pi}^{\pi} |u'| dx, \end{aligned}$$

which prove the lemma.

3. Main Theorem. An operator T which maps into the scalar valued functions, is called quasi-linear if $T(f+g)$ is uniquely defined whenever Tf and Tg are defined, and if

$$|T(f + g)| \leq \kappa(|Tf| + |Tg|), \tag{3.1}$$

where κ is a constant independent on f and g .

THEOREM 1. *Suppose that a quasi-linear operator T of H_p , to μ -measurable functions satisfies*

$$\mu(\{s : |(T\varphi)(s)| > t\})^{1/q_i} \leq \frac{M_i}{t} \|\varphi\|_{p_i} \tag{3.2}$$

for all φ in H_{p_i} ($i = 0, 1$) and for all $t > 0$, where $1 \leq p_i \leq q_i < \infty$ ($i = 0, 1$), $p_0 \neq p_1$ and $q_0 \neq q_1$. Let us put

$$1/p = (1 - \theta)/p_0 + \theta/p_1 \text{ and } 1/q = (1 - \theta)/q_0 + \theta/q_1 \quad (0 < \theta < 1). \tag{3.3}$$

Then T can be extended to an operator on H_p and satisfies

$$\|TKf\|_q \leq A(\kappa + 1)^2 M_0^{-\theta} M_1^{\theta} \|f\|_p$$

for all f in $L_p(-\pi, \pi)$, where A depends only on p_0, p_1, q_0, q_1 and θ , and

$$A^q = O[(p - 1)^{-q_0}(q - q_0)^{-1} + (q_1 - q)^{-1}]. \tag{3.4}$$

PROOF. First we give the following remarks (cf. [6; vol.2, p.112]). If f is a non-negative μ -measurable function and $p \geq 1$, we have

$$\int f^p d\mu = - \int_0^{\infty} y^p d\mu(\{s : f(s) > y\}) = p \int_0^{\infty} y^{p-1} \mu(\{s : f(s) > y\}) dy$$

by the definition of Lebesgue-Stieltjes integral and integration by parts. Therefore if we denote $\mu(\{s : |f(s)| > y\})$ by $\mu(|f| > y)$, then

$$\|TKf\|_q^q = q \int_0^{\infty} y^{q-1} \mu(|TKf| > y) dy.$$

Next as is easily verified, we have

$$\begin{aligned} & \mu(|T(f + g + h)| > 3\kappa(\kappa + 1)y) \\ & \leq \mu(|Tf| > y) + \mu(|Tg| > y) + \mu(|Th| > y), \end{aligned} \tag{3.6}$$

since $|T(f + g + h)| \leq \kappa(\kappa + 1)(|Tf| + |Tg| + |Th|)$.

From now we use the notations of Lemma 1: we put $r = (p + 1)/2$.

We consider separately the four cases.

Case (i). $1 = p_0 < p_1, q_0 < q_1$. Let $f \in L_p(-\pi, \pi)$, then by (3.5) we have

$$\begin{aligned} \|TKf\|_q^q &= q \left(\int_0^{3\kappa(\kappa+1)y_0} + \int_{3\kappa(\kappa+1)y_0}^\infty \right) y^{q-1} \mu(|TKf| > y) dy \\ &= q \int_0^{3\kappa(\kappa+1)y_0} t^{q-1} \mu(|TKf| > t) dt \\ &+ q(3\kappa(\kappa + 1))^q \int_{y_0}^\infty y^{q-1} \mu(|TKf| > 3\kappa(\kappa + 1)y) dy. \end{aligned}$$

We put

$$a = (y/B)^\lambda \text{ and } a_0 = (y_0/B)^\lambda, \tag{3.7}$$

where λ and B are constants determined later. We assume $\lambda > 0$. Since $a > a_0$ if $y > y_0$, we can decompose f into u, v and w by Lemma 1.

Hence using (3.6) we have

$$\begin{aligned} \|TKf\|_q^q &\leq q \int_0^{3\kappa(\kappa+1)y_0} t^{q-1} \mu(|TKf| > t) dt \\ &+ q(3\kappa(\kappa + 1))^q \left\{ \int_{y_0}^\infty y^{q-1} \mu(|TKu| > y) dy + \int_{y_0}^\infty y^{q-1} \mu(|TKv| > y) dy \right. \\ &\quad \left. + \int_{y_0}^\infty y^{q-1} \mu(|TKw| > y) dy \right\} \\ &= qI_1 + q(3\kappa(\kappa + 1))^q \{I_2 + I_3 + I_4\}, \text{ say.} \end{aligned}$$

Now we estimate I_1, I_2, I_3 and I_4 .

By the definition of y_0 and Hölder's inequality with exponents p/r and $p/(p - r)$, we have

$$y_0 = Ba_0^{1/\lambda} = B \left(\frac{2}{\pi} \int_{-\pi}^\pi |f|^r dx \right)^{1/r\lambda} \leq B \left(\frac{4^{p/r}}{2\pi} \int_{-\pi}^\pi |f|^p dx \right)^{1/p}. \tag{3.8}$$

By (3.2), we have

$$\begin{aligned} I_1 &\leq M_0^{q_0} \int_0^{3\kappa(\kappa+1)y_0} t^{q-q_0-1} \|Kf\|_{p_0}^{q_0} dt \\ &\leq M_0^{q_0} \|Kf\|_{p_0}^{q_0} \int_0^{3\kappa(\kappa+1)y_0} t^{q-q_0-1} dt. \end{aligned}$$

By Hölder's inequality and M.Riesz' inequality $\|Kf\|_p \leq A_p \|f\|_p$, $A_p = O\{(p-1)^{-1}\}$, $p > 1$, the last expression does not exceed

$$\begin{aligned} & M_0^{q_0} \|Kf\|_p^{q_0} \{3\kappa(\kappa+1)y_0\}^{q-q_0}/(q-q_0) \\ & \leq M_0^{q_0} A_p^{q_0} \|Kf\|_p^{q_0} \{3(\kappa+1)^2\}^q y_0^{q_0-q_0}/(q-q_0) \end{aligned}$$

Hence we get

$$I_1 \leq \left(\frac{4^{p/r}}{2\pi}\right)^{(q-q_0)/\lambda p} \{3(\kappa+1)^2\}^q \frac{M_0^{q_0} A_p^{q_0} B^{q-q_0}}{q-q_0} \|f\|_p^{q_0+(q-q_0)/\lambda} \tag{3.9}$$

By (3.2) and M.Riesz' inequality, we get

$$\begin{aligned} I_2 & \leq M_1^{q_1} \int_{y_0}^{\infty} y^{q-q_1-1} \|Ku\|_{p_1}^{q_1} dy \\ & \leq M_1^{q_1} A_{p_1}^{q_1} \int_{y_0}^{\infty} y^{q-q_1-1} \left\{ \int |u(x)|^{p_1} dx \right\}^{q_1/p_1} dy. \end{aligned}$$

Applying Minkowski's inequality $\left\{ \int_{\mathcal{F}} \left| \int_E h(x,y) dx \right|^r dy \right\}^{1/r} \leq \int_{\mathcal{F}} \left\{ \int_E |h(x,y)|^r dy \right\}^{1/r} dx$

($r \geq 1$), the last integral does not exceed

$$\begin{aligned} & \left[\int dx \left\{ \int_{y_0}^{\infty} y^{q-q_1-1} |u(x)|^{q_1} dy \right\}^{p_1/q_1 - q_1/p_1} \right] \\ & \leq \left[\int dx \left\{ \int_{|B|^{1/\lambda}}^{\infty} y^{q-q_1-1} |f|^{q_1} dy \right\}^{p_1/q_1 - q_1/p_1} \right] \\ & = \frac{B^{q-q_1}}{q_1-q} \left\{ \int |f|^{p_1+[(q-q_1)p_1/q_1\lambda]} dx \right\}^{q_1/p_1} \tag{3.10} \end{aligned}$$

Using (3.2) and M.Riesz' inequality again we have

$$I_3 \leq M_1^{q_1} A_1^{q_1} \int_{y_0}^{\infty} y^{q-q_1-1} \left\{ \int |v(x)|^{p_1} dx \right\}^{q_1/p_1} dy.$$

Since $|v(x)|^{p_1} \leq 2^{(p_1-1)/r} B^{-\lambda(p_1-1)} y^{\lambda(p_1-1)} |v(x)|$ by (iii) of Lemma 1, the last integral is bounded by

$$2^{q_1(p_1-1)/r} B^{-\lambda(p_1-1)q_1/p_1} \int_{y_0}^{\infty} y^{q-q_1-1+[(p_1-1)q_1\lambda/p_1]} \left\{ \int |v(x)| dx \right\}^{q_1/p_1} dy.$$

Applying (iv) of Lemma 1 with $s = 1$ and Minkowski's inequality, the above integral does not exceed

$$\begin{aligned} & \int_{y_0}^{\infty} y^{q-q_1-1+[(p_1-1)q_1\lambda/p_1]} \left\{ \int |u'(x)| dx \right\}^{q_1/p_1} dy \\ & \leq \left[\int dx \left\{ \int_{y_0}^{\infty} y^{q-q_1-1+[(p_1-1)q_1\lambda/p_1]} |u'(x)|^{q_1/p_1} dy \right\}^{p_1/q_1 - q_1/p_1} \right] \end{aligned}$$

$$\begin{aligned} &\leq \left[\int dx \left\{ \int_0^{B|f|^{1/\lambda}} y^{q-q_1-1+[(p_1-1)q_1\lambda/p_1]} |f|^{q_1/p_1} dy \right\}^{p_1/q_1} \right]^{q_1/p_1} \\ &= \frac{B^{q-q_1+[(q-q_1)p_1/\lambda q_1]}}{q-q_1+[(p_1-1)q_1\lambda/p_1]} \left\{ \int |f|^{[(q-q_1)p_1/\lambda q_1]+p_1} dx \right\}^{q_1/p_1}. \end{aligned}$$

Hence we get

$$I_3 \leq 2^{q_1(p_1-1)/rp_1} \frac{M_1^{q_1} A_{p_1}^{q_1} B^{q-q_1}}{q-q_1+[(p_1-1)q_1\lambda/p_1]} \left\{ \int |f|^{[(q-q_1)p_1/\lambda q_1]+p_1} dx \right\}^{q_1/p_1}. \tag{3.11}$$

By our hypotheses (3.2) and $p_0 = 1$, we have

$$\begin{aligned} I_4 &\leq M_0^{q_0} \int_{y_0}^{\infty} y^{q-q_0-1} \|K\tau w\|_{p_0}^{q_0} dy \\ &\leq 2^{q_0} M_0^{q_0} \left\{ \int_{y_0}^{\infty} y^{q-q_0-1} \left(\int_E |K\tau w| dx \right)^{q_0} dy + \int_{y_0}^{\infty} y^{q-q_0-1} \left(\int_{CE} |K\tau w| dx \right)^{q_0} dy \right\} \\ &= 2^{q_0} M_0^{q_0} (J_1 + J_2), \end{aligned} \tag{3.12}$$

say, where E is the set defined in Lemma 2.

For J_2 , we have

$$\begin{aligned} J_2 &\leq A^{q_0} \int_{y_0}^{\infty} y^{q-q_0-1} \left\{ \int |u'(x)| dx \right\}^{q_0} dy \\ &\leq A^{q_0} \left[\int dx \left\{ \int_y^{\infty} y^{q-q_0-1} |u'(x)|^{q_0} dy \right\}^{1/q_0-q_0} \right]^{q_0} \\ &\leq A^{q_0} \left[\int dx \left\{ \int_0^{B|f|^{1/\lambda}} y^{q-q_0-1} |f(x)|^{q_0} dy \right\}^{1/q_0} \right]^{q_0} \\ &= \frac{A^{q_0} B^{q-q_0}}{q-q_0} \left\{ \int |f|^{[(q-q_0)/\lambda q_0]+1} dx \right\}^{q_0}, \end{aligned} \tag{3.13}$$

using Lemma 2 for the first inequality and Minkowski's inequality for the second. On the other hand, by Hölder's inequality, we get

$$J_1 \leq \int_y^{\infty} y^{q-q_0-1} |E|^{q_0/r'} \left\{ \int |K\tau w|^r dx \right\}^{q_0/r} dy,$$

where $1/r + 1/r' = 1$. Applying (vi) of Lemma 1 for $|E|$ and M.Riesz' inequality for the inner integral, we have

$$J_1 \leq 3^{q_0/r'} A_r^{q_0} B^{\lambda q_0/r'} \int_{y_1}^{\infty} y^{q-q_0-1-[q_0\lambda r/r']} \left\{ \int |u'|^r dx \right\}^{q_0/r'} \left\{ \int |\tau w|^r dx \right\}^{q_0/r'} dy.$$

By (iv) of Lemma 1 with $s = r$, above integral is not greater than

$$2^{(r+1)q_0} \int_{y_0}^{\infty} y^{q-q_0-1-[q_0\lambda r/r']} \left\{ \int |u'|^r dx \right\}^{q_0} dy.$$

If we assume that

$$q - q_0 - [\lambda q_0 r / r'] > 0, \tag{3.14}$$

then by the same way as in estimating J_2 , the integral does not exceed

$$\begin{aligned} & \left[\int dx \left\{ \int_{y_0}^{\infty} y^{q-q_0-1-[\lambda q_0 r / r']} |u'(x)|^{q_0 r} dy \right\}^{q_0} \right]^{1/q_0} \\ & \leq \left[\int dx \left\{ \int_0^{B|f|^{1/\lambda}} y^{q-q_0-1-[\lambda q_0 r / r']} |u'(x)|^{q_0 r} dy \right\}^{q_0} \right]^{1/q_0} \\ & = \frac{B^{q-q_0-[\lambda q_0 r / r']}}{q-q_0-[\lambda q_0 r / r']} \left\{ \int |f|^{[(q-q_0)/q_0 \lambda]+1} dx \right\}^{q_0}. \end{aligned}$$

Hence we get

$$J_1 \leq 3^{q_0/r'} 2^{(r+1)q_0} \frac{A_r^{q_0} B^{(q-q_0)}}{q-q_0-[\lambda q_0 r / r']} \left\{ \int |f|^{[(q-q_0)/q_0 \lambda]+1} dx \right\}^{q_0}.$$

Therefore we have

$$I_4 \leq 2^{q_0} M_0^{q_0} \left(\frac{3^{q_0/r'} 2^{(r+1)q_0} A_r^{q_0} B^{(q-q_0)}}{q-q_0-[\lambda q_0 r / r']} + \frac{A_r^{q_0} B^{q-q_0}}{q-q_0} \right) \left(\int |f|^{(q-q_0)/\lambda q_0+1} dx \right)^{q_0}. \tag{3.15}$$

Now if we put

$$\lambda = \frac{p_0(q-q_0)}{q_0(p-p_0)} = \frac{p_1(q-q_1)}{q_1(p-p_1)},$$

$q - q_0 - [\lambda q r / r'] = (q - q_0)/2 > 0$ which is the assumption (3.14) and $|f|$ in I_1, I_2, I_3 and I_4 contain the same power p . Next we set

$$B = M_0^\rho M_1^\sigma \|f\|_p^\tau$$

and select ρ, σ and τ so that the powers of M_0, M_1 and $\|f\|_p$ in I 's are same, that is, we put

$$\rho = -q_0/(q_1 - q_0), \sigma = -q_1/(q_0 - q_1) \text{ and } \tau = (p_1 q - p q_1)/p_1(q - q_1).$$

Then we obtain by (3.9), (3.10), (3.11) and (3.15)

$$\|TKf\|_a^q \leq (\kappa+1)^{2q} A' M_0^{(1-\theta)q} M_1^{\theta q} \left(\frac{1}{q-q_0} + \frac{1}{q_1-q} + \frac{1}{p-1} + \frac{1}{(q-q_0)(p-1)^{q_0}} \right) \|f\|_p^q,$$

where we used that $A_r, A_p = O(p-1)$ and A' is a constant depending only on p_0, p_1, q_0 and q_1 .

Therefore we get Theorem in this case.

Case (ii). $1 = p_0 < p_1, q_1 < q_0$. We can prove this case by the same way. Since $\lambda < 0$ in this case, $a > a_0$ if $y < y_0$, therefore we put for f in $L_p(-\pi, \pi)$

$$\begin{aligned} \|TKf\|_q^q &= q \left(\int_{3\kappa(\kappa+1)y_0}^\infty + \int_0^{3\kappa(\kappa+1)y_0} \right) y^{q-1} \mu(|TKf| > y) dy \\ &\leq q \int_{3\kappa(\kappa+1)y_0}^\infty t^{q-1} \mu(|TKf| > t) dt + q(3\kappa(\kappa+1))^q \\ &\left\{ \int_0^{y_0} y^{q-1} \mu(|TKu| > y) dy + \int_0^{y_0} y^{q-1} \mu(|TKv| > y) dy + \int_0^{y_0} y^{q-1} \mu(|TKw| > y) dy \right\} \\ &= qI_1 + q(3\kappa(\kappa+1))^q(I_2 + I_3 + I_4), \end{aligned}$$

say. I_1, I_2, I_3 and I_4 can be estimated by the same way as previous case but we must replace the domains of integrals $(0, B|f|^{1/\lambda})$, and $(B|f|^{1/\lambda}, \infty)$ by $(B|f|^{1/\lambda}, \infty)$ and $(0, B|f|^{1/\lambda})$ respectively, and in the concluding expressions (3.9), (3.10), (3.11) and (3.15) the sign of the denominators will be changed. Further the inequality (3.8) will be replaced by

$$y_0 \geq B \left(\frac{4^{p/r}}{2\pi} \int |f|^p dx \right)^{1/\lambda p}$$

Case (iii). $1 < p_0 < p_1, q_0 < q_1$. In this case our theorem degenerates to the special case of that of Marcinkiewicz's.

For every $a > 0$, we put

$$f = u + u' \quad \text{and} \quad a = (y/B)^\lambda,$$

where $u = f$ if $|f| < a$ and $u = 0$ otherwise, and λ, B are the same constants as in case (i). (3.5) and similar inequality to (3.6) show that

$$\begin{aligned} \|TKf\|_q^q &= q \int_0^\infty y^{q-1} \mu(|TKf| > y) dy \\ &= (2\kappa)^q q \int_0^\infty y^{q-1} \mu(|TKf| > 2\kappa y) dy \\ &\leq (2\kappa)^q q \left\{ \int_0^\infty y^{q-1} \mu(|TKu| > y) dy + \int_0^\infty y^{q-1} \mu(|TKu'| > y) dy \right\} \\ &= (2\kappa)^q q(I_1 + I_2), \end{aligned} \tag{3.16}$$

say. By the assumption (3.2) and M.Riesz' inequality, we have

$$\begin{aligned} I_1 &\leq M_1^{q_1} \int_0^\infty y^{q-q_1-1} \|Ku\|_{p_1}^{q_1} dy \\ &\leq A_{p_1}^{q_1} M_1^{q_1} \int_0^\infty y^{q-q_1-1} \|u\|_{p_1}^{q_1} dy. \end{aligned}$$

Remarking that $\lambda > 0$ and arguing in the same way as (3.10), we get

$$\begin{aligned}
 I_1 &\leq \frac{A_{p_1}^{q_1} M_1^{q_1} B^{q_1 - q}}{q_1 - q} \left\{ \int |f|^{p_1 + [(q-p_1)p_1/q_1\lambda]} dx \right\}^{q_1/p_1} \\
 &= \frac{A_{p_1}^{q_1} M_0^{q_1(1-\theta)} M_1^{q_1\theta}}{q_1 - q} \|f\|_p^q.
 \end{aligned}
 \tag{3.17}$$

For I_2 we have

$$\begin{aligned}
 I_2 &\leq M_0^{q_0} \int_0^\infty y^{q_0 - q_0 - 1} \|Ku'\|_{p_0}^{q_0} dy \\
 &\leq M_0^{q_0} A_{p_0}^{q_0} \int_0^\infty y^{q_0 - q_0 - 1} \|u'\|_{p_0}^{q_0} dy \\
 &\leq M_0^{q_0} A_{p_0}^{q_0} \left[\int dx \left\{ \int_0^\infty y^{q_0 - q_0 - 1} |u'(x)|^{q_0} dy \right\}^{p_0/q_0} \right]^{q_0/p_0} \\
 &= M_0^{q_0} A_{p_0}^{q_0} \left[\int dx \left\{ \int_0^{B|f|^{1/\lambda}} y^{q_0 - q_0 - 1} |u'(x)|^{q_0} dy \right\}^{p_0/q_0} \right]^{q_0/p_0} \\
 &= \frac{M_0^{q_0} A_{p_0}^{q_0} B^{q_0 - q_0}}{q - q_0} \left\{ \int |f|^{[(q-q_0)p_0/\lambda q_0] + 1} dx \right\}^{q_0/p_0} \\
 &= \frac{M_0^{(1-\theta)q} M_1^{q_0\theta}}{q - q_0} \|f\|_p^q.
 \end{aligned}$$

By (3.16), (3.17) and (3.18), we get the theorem.

Case (iv). $1 < p_0 < p_1, q_1 < q_0$. We can prove this case by the same way as case (iii). But since $\lambda < 0$, the integration domains $(0, B|f|^{1/\lambda})$ and $(B|f|^{1/\lambda}, \infty)$ in (3.17) and (3.18) must be interchanged.

4. Several Variables Case. We consider the n -dimensional analogue of §§2 and 3. We use the following notations; Euclidean space of n -dimension is denoted by E_n , its points $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)$, etc. by X, Y etc. and the element of volume $dx_1 dx_2 \dots dx_n$ by dX .

Now we define an analogue of H_p -class.

We put

$$\tilde{f}(X) = \lim_{\epsilon \rightarrow 0} \int_{1/\epsilon \geq |Y| \geq \epsilon} L(Y) f(X - Y) dY,$$

where $L(Y)$ is Calderón-Zygmund type kernel, that is, $L(Y)$ is locally integrable except the origin and satisfies the inequality

$$\int_{|x| > 4|Y|} |L(X - Y) - L(X)| dX \leq A' \tag{4.1}$$

for all Y in E_n and there exists $q > 1$ such that

$$\|\tilde{f}\|_q \leq A'' \|f\|_q$$

for all f in $L_q(E_n)$.

We define vector $\mathfrak{R}f$ by

$$\mathfrak{R}f = (f, \tilde{f})$$

and $\mathfrak{H}_p(p \geq 1)$ as the space of all vectors such that $\mathfrak{R}f$ belongs to $L_p(E_n)$. Then we have

$$\|\mathfrak{R}f\|_p \leq A_p \|f\|_p \quad \text{for all } f \text{ in } L_p(E_n), \quad (4.2)$$

where $p > 1$ and $A_p = O\{(p-1)^{-1}\}$ (cf. J. Schwartz [3]).

Specially the case $L(X) = X/c_n |X|^{n+1}$ was studied in detail by E.M. Stein-G. Weiss [5], where $c_n = \pi^{(n+1)/2} / \Gamma((n+1)/2)$.

Next two lemmas correspond to Lemmas 1 and 2, respectively.

LEMMA 3. *Let $f \in L_p(E_n)$ ($\infty > p \geq 1$) and $p \geq r \geq 1$, then for each $a > 0$ the following decomposition of f is possible;*

- (i) $f = u + u', u' = v + w, w = \sum_{k=1}^{\infty} w_k,$
- (ii) $u = f, \text{ if } |f| < a \text{ and } u = 0 \text{ elsewhere,}$
- (iii) $|v(X)| \leq 2^{n/r} a \quad \text{for a. e. } X \text{ in } E_n,$
- (iv) $\int_{E_n} |v(X)|^s dX \leq \int_{E_n} |u'(X)|^s dX \quad \text{for each } s, 1 \leq s \leq p,$
- (v) $\sum_{k=1}^{\infty} \int_{E_n} |w_k(X)|^s dX \leq 2^{s+1} \int_{E_n} |u'(X)|^s dX \quad \text{for each } s, 1 \leq s \leq p,$
- (vi) *there exists a sequence $\{I_k\}$ of disjoint cubes such that the support of w_k is contained in I_k and*

$$\sum_{k=1}^{\infty} |I_k| \leq \frac{1}{a^r} \int_{E_n} |u'(X)|^r dX,$$

- (vii) $\int_{E_n} w_k(X) dX = 0, \quad k = 1, 2, 3, \dots$

PROOF. We define u by (ii) and put $u' = f - u$. Divide the space E_n into a mesh of cubes of volume $> a^{-r} \int_{E_n} |u'|^r dX$. Next divide each cube into 2^n equal cubes and let $I_{11}, I_{12}, I_{13}, \dots$ be those cubes over which the mean value of $|u'|^r$ is not smaller than a^r . Then we have the analogous inequality to (2.6) and the rest of arguments proceeds as in Lemma 1.

The following lemma will be proved by the same way as in Lemma 2, if we note the assumption (4.1).

LEMMA 4. *With the notations of Lemma 3, we have*

$$\sum_{k=1}^{\infty} \int_{CE} |\mathfrak{R}w_k| dX \leq A \int_{E_n} |u'| dX,$$

where E is the set obtained by expanding each I_k concentrically A times and $CE = E_n - E$.

5. n-dimensional Version of Theorem 1.

THEOREM 2. *Suppose that a quasi-linear operator T of \mathfrak{S}_{p_i} to μ -measurable functions satisfies (3.1) and suppose that we have*

$$\mu\{s: |(TF)(s)| > t\}^{1/q_i} \leq \frac{M_i}{t} \|F\|_{p_i} \tag{5.1}$$

for all F in \mathfrak{S}_{p_i} ($i = 0,1$) and $t > 0$, where $1 \leq p_i \leq q_i < \infty$ ($i = 0,1$), $p_0 \neq p_1$, $q_0 \neq q_1$. Let us define p and q by (3.1).

Then T can be extended to an operator on $L_p(E_n)$ and satisfies

$$\|T\mathfrak{R}f\|_q \leq A\kappa(\kappa + 1)M_0^{1-\theta}M_1^\theta \|f\|_p \tag{5.2}$$

for all f in $L_p(E_n)$, where A is a constant independent on f and of the order of (3.4).

PROOF. Our proof proceeds as Theorem 1. We consider separately four cases too:

- (i) $1 = p_0 < p_1, q_0 < q_1,$ (ii) $1 = p_0 < p_1, q_1 < q_0,$
- (iii) $1 < p_0 < p_1, q_0 < q_1,$ (iv) $1 < p_0 < p_1, q_1 < q_0.$

But since the proofs of Cases (iii) and (iv) are not different from the corresponding cases of Theorem 1, we treat only Cases (i) and (ii).

We use the notations of Lemmas 3 and 4.

Case (i). $1 = p_0 < p_1, q_0 < q_1$. Let $f \in L_p(E_n)$. Using (3.5) replaced by and (3.6), we have

$$\begin{aligned} \|T\mathfrak{R}f\|_q^q &= q \int_0^\infty y^{q-1} \mu(|T\mathfrak{R}f| > y) dy \\ &\leq q(3\kappa(\kappa + 1))^q \left\{ \int_0^\infty y^{q-1} \mu(|T\mathfrak{R}u| > y) dy + \right. \\ &\quad \left. \int_0^\infty y^{q-1} \mu(|T\mathfrak{R}v| > y) dy + \int_0^\infty y^{q-1} \mu(|T\mathfrak{R}w| > y) dy \right\} \\ &= q(3\kappa(\kappa + 1))^q (I_1 + I_2 + I_3), \end{aligned} \tag{5.3}$$

say. We put $a = (y/B)^\lambda$ where λ and B are same as before.

For I_1 using the arguments of I_2 of Case (i) in Theorem 1 but applying Calderón-Zygmund inequality (4.2), we have

$$\begin{aligned}
 I_1 &\leq M_1^{q_1} \int_0^\infty y^{q-q_1-1} \|\mathfrak{R}u\|_{p_1}^{q_1} dy \\
 &\leq M_1^{q_1} A_{p_1}^{q_1} \int_0^\infty y^{q-q_1-1} \left\{ \int |u(X)|^{p_1} dX \right\}^{q_1/p_1} dy \\
 &\leq M_1^{q_1} A_{p_1}^{q_1} \left[\int dX \left\{ \int_0^\infty y^{q-q_1-1} |u|^q dy \right\}^{p_1/q_1 - q_1/p} \right] \\
 &= M_1^{q_1} A_{p_1}^{q_1} \left[\int dX \left\{ \int_{|B|^{1/\lambda}} y^{q-q_1-1} |f|^{q_1} dy \right\}^{p_1/q_1 - q_1/p} \right] \\
 &= \frac{M_1^{q_1} A_{p_1}^{q_1} B^{q-q_1}}{q_1 - q} \left\{ \int |f|^{p_1 + [(q-q_1)p_1/q_1\lambda]} dX \right\}^{q_1/p_1} \\
 &= \frac{A_{p_1}^{q_1} M_0^{(1-\theta)q} M_1^{\theta q}}{q_1 - q} \|f\|_p^q. \tag{5.4}
 \end{aligned}$$

By (5.1), (4.2) and (iii) of Lemma 3, we get

$$\begin{aligned}
 I_2 &\leq M_1^{q_1} A_{p_1}^{q_1} \int_0^\infty y^{q-q_1-1} \left\{ \int |v(X)|^{p_1} dX \right\}^{q_1/p_1} dy \\
 &\leq M_1^{q_1} A_{p_1}^{q_1} 2^{n(p_1-1)q_1/rp_1} B^{-\lambda(p_1-1)q_1/p_1} \int_0^\infty y^{q-q_1-1 + [(p_1-1)q_1\lambda/p_1]} \left\{ \int |v(X)| dX \right\}^{q_1/p_1} dy \\
 &\leq M_1^{q_1} A_{p_1}^{q_1} 2^{n(p_1-1)q_1/rp_1} B^{-\lambda(p_1-1)q_1/p_1} \left[\int dX \left\{ \int_0^\infty y^{q-q_1-1 + [(p_1-1)q_1\lambda/p_1]} |u'(X)|^{q_1/p_1} dy \right\}^{p_1/q_1 - q_1/p_1} \right] \\
 &\leq M_1^{q_1} A_{p_1}^{q_1} 2^{n(p_1-1)q_1/rp_1} B^{-\lambda(p_1-1)q_1/p_1} \left[\int dX \left\{ \int_0^{|B|^{1/\lambda}} y^{q-q_1-1 + [(p_1-1)q_1\lambda/p_1]} |f|^{q_1/p_1} dy \right\}^{p_1/q_1 - q_1/p_1} \right] \\
 &= \frac{M_1^{q_1} A_{p_1}^{q_1} 2^{n(p_1-1)q_1/rp_1} B^{-\lambda(p_1-1)q_1/p_1}}{q - q_1 + [(p_1 - 1)q_1\lambda/p_1]} \left\{ \int |f|^{p_1 + [(q-q_1)p_1/\lambda q_1]} dX \right\}^{q_1/p_1} \\
 &= 2^{n(p_1-1)q_1/rp_1} A_{p_1}^{q_1} M_0^{(1-\theta)q} M_1^{\theta q} \frac{(p_1 - p)}{(q_1 - q)(p - 1)} \|f\|_p^q. \tag{5.5}
 \end{aligned}$$

For I_3 we put as (3.12)

$$\begin{aligned}
 I_3 &\leq M_0^{q_0} \int_0^\infty y^{q-q_0-1} \|\mathfrak{R}v\|_{p_0}^{q_0} dy \\
 &\leq 2^{q_0} M_0^{q_0} \left\{ \int_0^\infty y^{q-q_0-1} \left(\int_E |\mathfrak{R}v| dX \right)^{q_0} dy + \int_0^\infty y^{q-q_0-1} \left(\int_{cE} |\mathfrak{R}v| dX \right)^{q_0} dy \right\} \\
 &= 2^{q_0} M_0^{q_0} (J_1 + J_2), \tag{5.6}
 \end{aligned}$$

say, where E is the set defined in Lemma 4. By the last lemma, we get

$$J_2 \leq A^{q_0} \int_0^\infty y^{q-q_0-1} \left\{ \int_{E_n} |u'(X)| dX \right\}^{q_0} dy$$

$$\begin{aligned}
 &\cong A^{q_0} \left[\int dX \left\{ \int_0^\infty y^{q-q_0-1} |u'(X)|^{q_0} dy \right\}^{1/q_0 - q_0} \right] \\
 &\cong A^{q_0} \left[\int dX \left\{ \int_0^{B|f|^{1/\lambda}} y^{q-q_0-1} |f(X)|^{q_0} dy \right\}^{1/q_0 - q_0} \right] \\
 &= \frac{A^{q_0} B^{q-q_0}}{q - q_0} \left\{ \int |f|^{[(q-q_0)/\lambda q_0] + 1} dX \right\} \\
 &= \frac{A^{q_0} M_0^{(1-\theta)q} M_1^{\theta q}}{q - q_0} \|f\|_p^q. \tag{5.7}
 \end{aligned}$$

By Hölder's inequality and (vi) of Lemma 3, we have

$$\begin{aligned}
 J_1 &\cong \int_0^\infty y^{q-q_0-1} |E|^{q_0/r'} \left\{ \int |\mathfrak{R}w|^r dX \right\}^{q_0/r'} dy \\
 &\cong A^{q/r'} A_r^{q_0} B^{\lambda q_0 r'/r'} \int_0^\infty y^{q-q_0-1 - [q_0 \lambda r'/r']} \left\{ \int |u'|^r dX \right\}^{q_0/r'} \left\{ \int |w|^r dX \right\}^{q_0/r'} dy,
 \end{aligned}$$

where $1/r + 1/r' = 1$. By virtue of (v) of Lemma 3,

$$\begin{aligned}
 J_1 &\cong A^{q_0/r'} 2^{(r+1)q_0/r} A_r^{q_0} B^{\lambda q_0 r'/r'} \int_0^\infty y^{q-q_0-1 - [q_0 \lambda r'/r']} \left\{ \int |u'|^r dX \right\}^{q_0} dy \\
 &\cong A^{q_0/r'} 2^{(r+1)q_0/r} A_r^{q_0} B^{\lambda q_0 r'/r'} \left[\int dX \left\{ \int_0^\infty y^{q-q_0-1 - [q_0 \lambda r'/r']} |u'(X)|^{q_0 r} dy \right\}^{q_0} \right]^{1/q_0} \\
 &\cong A^{q_0/r'} 2^{(r+1)q_0/r} A_r^{q_0} B^{\lambda q_0 r'/r'} \left[\int dX \left\{ \int_0^{B|f|^{1/\lambda}} y^{q-q_0-1 - [q_0 \lambda r'/r']} |u'(X)|^{q_0 r} dy \right\}^{q_0} \right]^{1/q_0} \\
 &\cong \frac{A' B^{q-q_0 - [\lambda q_0 r'/r']}}{(r-1)q_0 \{q - q_0 - [\lambda q_0 r'/r']\}} \left\{ \int |f|^{[(q-q_0)/q_0 \lambda] + 1} dX \right\}^{q_0} \\
 &\cong \frac{A''}{(p-1)^{q_0} (q - q_0)} M_0^{(1-\theta)q} M_1^{\theta q} \|f\|_p^q, \tag{5.8}
 \end{aligned}$$

where A' and A'' depend only on p_0, p_1, q_0 and q_1 .

Thus we obtain the estimation

$$\|T_{\mathfrak{R}} f\|_q^q \cong \kappa^q (\kappa + 1)^q A''' M_0^{(1-\theta)q} M_1^{\theta q} \left(\frac{1}{q_1 - q} + \frac{1}{(p-1)^{q_0} (q - q_0)} \right) \|f\|_p^q,$$

A''' depending only on p_i and q_i ($i = 0, 1$).

Therefore a proof of Case (i) is completed.

Case (ii). $1 = p_0 < p_1, q_1 < q_0$. A proof of this case may proceed as above case. But the domains of integrals $(0, B|f|^{1/\lambda})$ and $(B|f|^{1/\lambda}, \infty)$ in I_k ($k = 1, 2, 3$) must be interchanged.

6. Some Applications. For the sake of convenience we restate the examples of application of our theorem.

Littlewood-Paley function g^* is defined by

$$g^*(\theta, \varphi) = \left\{ \sum_{n=1}^{\infty} \frac{|S_n(\theta) - \sigma_n(\theta)|^2}{n} \right\}^{1/2},$$

where $S_n(\theta)$ and $\sigma_n(\theta)$ are n -th partial sum and $(C, 1)$ -mean of Fourier series of φ in H_1 respectively. E.M. Stein [4] proved that this operator is weak type (1,1) for H_1 -class (on the other hand g^* is not weak type (1,1) for $L_1(-\pi, \pi)$, see [6, vol. II, p. 315]), and it is rather easily verified that g^* is strong type (2, 2). Therefore we get the well-known inequality applying Theorem 1;

$$\|g^*\|_p \leq A_p \|f\|_p \quad \text{for all } f \text{ in } L_p(-\pi, \pi),$$

where $A_p = O\{(p-1)^{-2}\}$ and $2 \geq p > 1$.

Another example is the operator $(T\varphi)(\theta) = S_{n(\theta)}(\theta)$, where $n(\theta)$ is any integral valued measurable function. This operator T is strong type (1,1) for H_1 and strong type (2, 2) for $L_2(-\pi, \pi)$ (see [6, Chap. XIII]), when $d\mu(\theta) = d\theta/\log(|n(\theta)| + 2)$ with the notation of Theorem 1. Since the factorization of functions in H_1 is possible by real methods (see e.g., W. Rudin; Fourier Analysis on Groups, Interscience, New York, 1962, p.205), our theorem gives a real proof of Littlewood-Paley inequality

$$\int_{-\pi}^{\pi} \sup_{n \geq 0} \left\{ \frac{|S_n(x)|^p}{\log(n+2)} \right\} dx \leq A_p \int_{-\pi}^{\pi} |f(x)|^p dx,$$

for all f in $L_p(-\pi, \pi)$ ($2 \geq p > 1$), where A_p is a constant.

CORRECTION: The estimation of the norm of T of the Theorem in the previous paper, S. Igari [2] is incorrect, that is, $(p-1)^{-1}$ in A^q must be replaced by $(p-1)^{-q_0}$.

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