ON HARMONIC TENSORS IN A COMPACT ALMOST-KÄHLERIAN SPACE

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1. Introduction. S.Kotō [1] proved that in a compact K-space, if a skew-symmetric pure tensor $T_{i_1 \dots i_p}$ [1)] is almost-analytic, then it is harmonic. This is an extension of Tachibana's result[2)] on an almost-analytic vector in a compact K-space to an almost-analytic tensor. The main purpose of this paper is to try an extension of the same Tachibana's result[3)] on an almost-analytic vector in a compact almost-Kählerian space to the case of a tensor.

In §2 we shall give some preliminary facts for later use. In §3 we shall prove some lemmas on almost-analytic tensors. Most part of the last section will be devoted to the proof of the main theorem.

2. Almost-Kahlerian spaces. [4)] Let X_{2n} be a $2n$-dimensional almost-complex space[5)] with local coordinates $\{x^i\}$ and $\varphi_j{}^i$ its almost-complex structure, then by definition we have

$$\varphi_j{}^r \varphi_r{}^i = -\delta_j^i. \tag{2. 1}$$

We define two linear operators

$$O_{ih}^{ml} = \frac{1}{2}(\delta_i^m \delta_h^l - \varphi_i{}^m \varphi_h{}^l), \quad {}^*O_{ih}^{ml} = \frac{1}{2}(\delta_i^m \delta_h^l + \varphi_i{}^m \varphi_h{}^l)^{6)}$$

and say a tensor is pure (hybrid) in two indices if it is annihilated by transvection of $^*O(O)$ on these indices and if a tensor is pure in every pair of indices, then it is called a pure tensor. For instance $\varphi_j{}^i$ is pure in j,i. In this place, the following properties can be easily verified.

If T_{ji} is pure (hybrid) in j,i, then we have

$$\varphi_i{}^r T_{jr} = \varphi_j{}^r T_{ri} \quad (\varphi_i{}^r T_{jr} = -\varphi_j{}^r T_{ri}).$$

If T_{ji} is pure in j,i and S^{ji} is hybrid in j,i, then we have

$$T_{ji} S^{ji} = 0.$$

If T_{ji} is pure in j,i and at the same time hybrid in j,i, then it vanishes.

If T_{ji} is pure in j,i and $S_j{}^i$ is pure (hybrid) in j,i, then $T_{jr} S_i{}^r$ is pure (hybrid) in j,i.

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1) As to the notations we follow S. Sawaki [2]. Indices run over $1, 2, \cdots, 2n$.
2) S. Tachibana [4].
3) S. Tachibana [3].
4), 5), 6) For example, see K. Yano [6].

For an arbitrary tensor T_{ji}, $T_{ji} + \varphi_j{}^a\varphi_i{}^b T_{ab}$ *is hybrid in* j, i, *and* $T_{ji} - \varphi_j{}^a\varphi_i{}^b T_{ab}$ *is pure in* j, i.

Throughout this paper, in every calculation concerning with purity and hybridity, these properties will be used.

Now, an almost-complex space with the structure $(\varphi_j{}^i, g_{ji})$ satisfying the following relations is called an almost-Kählerian space:

$$g_{ab}\varphi_j{}^a\varphi_i{}^b = g_{ji}, \tag{2. 2}$$

$$\nabla_j\varphi_{ih} + \nabla_i\varphi_{hj} + \nabla_h\varphi_{ji} = 0 \tag{2. 3}$$

where $\varphi_{ji} \equiv g_{ri}\varphi_j{}^r$ and ∇_j denotes the operator of Riemannian derivation.

It is easily verified that $\nabla_j\varphi_{ih}$ is pure in i, h and therefore $\nabla_j\varphi_i{}^h$ is hybrid in i, h, that is,

$$*O_{ih}^{ab}\nabla_j\varphi_{ab} = 0, \quad O_{ib}^{ah}\nabla_j\varphi_a{}^b = 0. \tag{2. 4}$$

From (2.2), we have

$$\varphi_{ji} = -\varphi_{ij}. \tag{2. 5}$$

And in an almost-Kählerian space, we know that $\nabla_j\varphi_{ih}$ is a pure tensor and therefore $\nabla_j\varphi_i{}^h$ is hybrid in j, h, that is,

$$*O_{jh}^{ab}\nabla_a\varphi_{ib} = 0, \quad O_{jb}^{ah}\nabla_a\varphi_i{}^b = 0^{7)} \tag{2. 6}$$

from which it follows

$$\nabla_r\varphi_i{}^r = 0. \tag{2. 7}$$

Let $R_{kji}{}^h$ and $R_{ji} \equiv R_{rji}{}^r$ be Riemannian curvature tensor and Ricci tensor respectively, then by the Ricci's identity and (2.7), we have

$$\nabla^r\nabla_j\varphi_r{}^i = \frac{1}{2}\varphi^{rs}R_{rsj}{}^i + R_j{}^r\varphi_r{}^i \tag{2. 8}$$

where $\nabla^r \equiv g^{tr}\nabla_t$ and $\varphi^{rs} \equiv g^{tr}\varphi_t{}^s$.

3. Almost-analytic tensors. We say that a covariant pure tensor $T_{i_1\ldots i_p}$ in an almost-complex space is almost-analytic if it satisfies

$$\varphi_k{}^l\partial_l T_{i_1\ldots i_p} - \partial_k\widetilde{T}_{i_1\ldots i_p} + \sum_{r=1}^{p}(\partial_{i_r}\varphi_k{}^t)\, T_{i_1\ldots t\ldots i_p} = 0$$

where $\widetilde{T}_{i_1\ldots i_p} \equiv \varphi_{i_1}^t\, T_{ti_2\ldots i_p}$ and $\partial_i \equiv \partial/\partial x^i$.

This equation can be written in the tensor form

$$\varphi_k{}^l\nabla_l T_{i_1\ldots i_p} - \nabla_k\widetilde{T}_{i_1\ldots i_p} + \sum_{r=1}^{p}(\nabla_{i_r}\varphi_k{}^t)T_{i_1\ldots t\ldots i_p} = 0 \tag{3. 1}$$

and we notice $\varphi_{i_1}^l T_{li_2\ldots i_p} = \varphi_{i_r}^l T_{i_1\ldots l\ldots i_p}$ for every $r\,(1 \leqq r \leqq p)$.

7) For example, see S. Sawaki [2].

For almost-analytic tensors, the following lemma can be easily verified too by a straightforward calculation.

LEMMA 3.1. (*S.Tachibana* [5]) *In an almost-complex space, if* $T_{i_1\ldots i_p}$ *is a skew-symmetric pure tensor then* $\widetilde{T}_{i_1\ldots i_p}$ *is also a skew-symmetric pure tensor and if* $T_{i_1\ldots i_p}$ *is almost-analytic, then so is* $\widetilde{T}_{i_1\ldots i_p}$.

Moreover, since

$$\nabla_{i_r}(\varphi_k{}^t T_{i_1\ldots t\ldots i_p}) = (\nabla_{i_r}\varphi_k{}^t)T_{i_1\ldots t\ldots i_p} + \varphi_k{}^t\nabla_{i_r}T_{i_1\ldots t\ldots i_p}$$

we have

$$\sum_{r=1}^{p}(\nabla_{i_r}\varphi_k{}^t)T_{i_1\ldots t\ldots i_p} = \sum_{r=1}^{p}\nabla_{i_r}\widetilde{T}_{i_1\ldots k\ldots i_p} - \sum_{r=1}^{p}\varphi_k{}^t\nabla_{i_r}T_{i_1\ldots t\ldots i_p}.$$

But, if $T_{i_1\ldots i_p}$ is skew-symmetric, then by the above lemma, so is $\widetilde{T}_{i_1\ldots i_p}$ and hence (3.1) is equivalent to

$$\varphi_k{}^l\nabla_{[l}T_{i_1\ldots i_p]} - \widetilde{\nabla}_{[k}T_{i_1\ldots i_p]} = 0. \tag{3.2}$$

Thus we have the following

LEMMA 3.2. (S.Kotō [1]) *In an almost-complex space, if skew-symmetric pure tensors* $T_{i_1\ldots i_p}$ *and* $\widetilde{T}_{i_1\ldots i_p}$ *are both closed, then they are almost-analytic.*

Now we assume we are in an almost-Kählerian space and let $T_{i_1\ldots i_p}$ be an almost-analytic tensor.

Transvecting (3.1) with $\varphi_h{}^k$, we have

$$\nabla_h T_{i_1\ldots i_p} + \varphi_h{}^k\nabla_k(\varphi_{i_1}{}^t T_{ti_2\ldots i_p}) - \sum_{r=1}^{p}\varphi_h{}^k(\nabla_{i_r}\varphi_k{}^t)T_{i_1\ldots t\ldots i_p} = 0$$

from which it follows

$$\begin{aligned}\nabla_h T_{i_1\ldots i_p} + \varphi_h{}^k\varphi_{i_1}{}^t\nabla_k T_{ti\ldots i_p} - \sum_{r=2}^{p}\varphi_h{}^k(\nabla_{i_r}\varphi_k{}^t)T_{i_1\ldots t\ldots i_p} \\ = \varphi_h{}^k(\nabla_{i_1}\varphi_k{}^t)T_{ti_2\ldots i_p} - \varphi_h{}^k(\nabla_k\varphi_{i_1}{}^t)T_{ti_2\ldots i_p}.\end{aligned} \tag{3.3}$$

In this equation, $\nabla_h T_{i_1\ldots i_p} + \varphi_h{}^k\varphi_{i_1}{}^t\nabla_k T_{ti_2\ldots i_p}$ is hybrid in h, i_1. And $\varphi_h{}^k(\nabla_{i_r}\varphi_k{}^t)$ $T_{i_1\ldots t\ldots i_p}(r \geqq 2)$ is also hybrid in h, i_1, because, by (2.4), $\varphi_h{}^k\nabla_{i_r}\varphi_k{}^t$ is hybrid in h, t and $T_{i_1\ldots t\ldots i_p}$ is pure in i_1, t. Hence the left-hand side of (3.3) is hybrid in h, i_1. Similarly, by (2.6) the right-hand side of (3.3) is pure in h, i_1.

Consequently, from (3.3) we have

$$\nabla_h T_{i_1\ldots i_p} + \varphi_h{}^k\varphi_{i_1}^t\nabla_k T_{ti_2\ldots i_p} - \sum_{r=2}^{p}\varphi_h{}^l(\nabla_{i_r}\varphi_l{}^t)T_{i_1\ldots t\ldots i_p} = 0 \tag{3.4}$$

and

$$\varphi_h{}^k(\nabla_{i_1}\varphi_k{}^t - \nabla_k\varphi_{i_r}^t)T_{ti_2\ldots i_p} = 0$$

or by (2.1) and (2.3), the latter is equivalent to

(3. 5) $$(\nabla^t \varphi_{hi_1}) T_{ti_2\ldots i_p} = 0.$$

Thus we have the following

LEMMA 3.3.[8)] *In an almost-Kählerian space, a pure tensor* $T_{i_1\ldots i_p}$ *is almost-analytic if and only if*

(1) $$\nabla_h T_{i_1\ldots i_p} + \varphi_h{}^l \varphi_{i_1}{}^t \nabla_l T_{ti_2\ldots i_p} - \varphi_h{}^l \sum_{r=2}^{p} (\nabla_{i_r} \varphi_l{}^t) T_{i_1\ldots t\ldots i_p} = 0,$$

(2) $$(\nabla^t \varphi_{hi_1}) T_{ti_2\ldots i_p} = 0.$$

Again we consider an almost-analytic skew-symmetric tensor $T_{i_1\ldots i_p}$. Since $\widetilde{T}^{i_1\ldots i_p} \equiv g^{i_1 r_1} \cdots g^{i_p r_p} \widetilde{T}_{r_1\ldots r_p}$ is pure in i_r, i_s $(r \neq s)$ and by (2. 6), $(\nabla_{i_r} \varphi_k{}^t) T_{i_1\ldots t\ldots i_p}$ is hybrid in i_r, i_s, we have

(3. 6) $$\widetilde{T}^{i_1\ldots i_p} \sum_{r=1}^{p} (\nabla_{i_r} \varphi_k{}^t) T_{i_1\ldots t\ldots i_p} = 0.$$

Multiplying (3. 1) by $\widetilde{T}^{i_1\ldots i_p}$ and making use of (3. 6), we find

$$\varphi_k{}^l \widetilde{T}^{i_1\ldots i_p} \nabla_l T_{i_1\ldots i_p} - \widetilde{T}^{i_1\ldots i_p} \nabla_k \widetilde{T}_{i_1\ldots i_p} = 0.$$

Operating ∇^k to this equation and using (2.7), we have

(3. 7) $$\varphi_k{}^l (\nabla^k \widetilde{T}^{i_1\ldots i_p}) \nabla_l T_{i_1\ldots i_p} + \varphi_k{}^l \widetilde{T}^{i_1\ldots i_p} \nabla^k \nabla_l T_{i_1\ldots i_p}$$
$$- (\nabla^k \widetilde{T}^{i_1\ldots i_p}) \nabla_k \widetilde{T}_{i_1\ldots i_p} - \widetilde{T}^{i_1\ldots i_p} \nabla^k \nabla_k \widetilde{T}_{i_1\ldots i_p} = 0.$$

On the other hand, multiplying (3.1) by $\nabla^k \widetilde{T}^{i_1\ldots i_p}$, we have

(3. 8) $$\varphi_k{}^l \nabla^k (\widetilde{T}^{i_1\ldots i_p}) \nabla_l T_{i_1\ldots i_p} - (\nabla^k \widetilde{T}^{i_1\ldots i_p}) \nabla_k \widetilde{T}_{i_1\ldots i_p} + p (\nabla_{i_1} \varphi_k{}^t) T_{ti_2\ldots i_p} \nabla^k \widetilde{T}^{i_1\ldots i_p} = 0$$

Forming the difference (3. 7)$-$(3. 8), we have

(3. 9) $$\varphi_k{}^l \widetilde{T}^{i_1\ldots i_p} \nabla^k \nabla_l T_{i_1\ldots i_p} - \widetilde{T}^{i_1\ldots i_p} \nabla^k \nabla_k \widetilde{T}_{i_1\ldots i_p}$$
$$- p (\nabla_{i_1} \varphi_k{}^t) T_{ti_2\ldots i_p} \nabla^k \widetilde{T}^{i_1\ldots i_p} = 0.$$

For the first term of the left-hand side of (3.9), by virtue of the Ricci's identity, we have

$$\begin{aligned}
&\varphi_k{}^l \widetilde{T}^{i_1\ldots i_p} \nabla^k \nabla_l T_{i_1\ldots i_p} \\
&= \frac{1}{2} \varphi^{kl} \widetilde{T}^{i_1\ldots i_p} (\nabla_k \nabla_l T_{i_1\ldots i_p} - \nabla_l \nabla_k T_{i_1\ldots i_p}) \\
&= -\frac{1}{2} \varphi^{kl} \widetilde{T}^{i_1\ldots i_p} \sum_{r=1}^{p} R_{kli_r}{}^s T_{i_1\ldots s\ldots i_p} \\
&= -\frac{1}{2} p \varphi^{kl} \widetilde{T}^{i_1\ldots i_p} R_{kli_1}{}^s T_{si_2\ldots i_p}.
\end{aligned}$$

Hence (3. 9) can be written in the form

8) S. Sawaki [2].

$$\frac{1}{2}pp^{kl}\widetilde{T}^{i_1\ldots i_p}R_{kli_1}{}^{s}T_{si_2\ldots i_p}+\widetilde{T}^{i_1\ldots i_p}\nabla^k\nabla_k\widetilde{T}_{i_1\ldots i_p}$$
$$+p(\nabla_{i_1}\varphi_k{}^t)T_{ti_2\ldots i_p}\nabla^k\widetilde{T}^{i_1\ldots i_p}=0. \tag{3.10}$$

But, since $\widetilde{T}_{i_1\ldots i_p}$ is also almost-analytic, from (2) of Lemma 3. 3, we have

$$(\nabla^{i_1}\varphi_{kt})\widetilde{T}_{i_1\ldots i_p}=0 \text{ or } (\nabla_{i_1}\varphi_k{}^t)\widetilde{T}^{i_1\ldots i_p}=0$$

and then operating ∇^k to the last equation, we get

$$(\nabla^k\nabla_{i_1}\varphi_k{}^t)\widetilde{T}^{i_1\ldots i_p}+(\nabla_{i_1}\varphi_k{}^t)\nabla^k\widetilde{T}^{i_1\ldots i_p}=0$$

or making use of (2.8)

$$\frac{1}{2}\varphi^{ab}R_{abi_1}{}^t+R_{i_1}{}^s\varphi_s{}^t)\widetilde{T}^{j_1\ldots i_p}+(\nabla_{i_1}\varphi_k{}^t)\nabla^k\widetilde{T}^{i_1\ldots i_p}=0. \tag{3.11}$$

Accordingly, forming the difference (3. 10) $-\ pT_{ti_2\ldots i_p}\times$ (3. 11), we get

$$\widetilde{T}^{i_1\ldots i_p}\nabla^k\nabla_k\widetilde{T}_{i_1\ldots i_p}-p\widetilde{T}^{j_1\ldots i_p}R_{i_1}{}^s\varphi_s{}^tT_{ti_2\ldots i_p}=0$$

that is,

$$(\nabla^k\nabla_k\widetilde{T}_{i_1\ldots i_p}-pR_{i_1}{}^s\widetilde{T}_{si_2\ldots i_p})\widetilde{T}^{i_1\ldots i_p}=0. \tag{3.12}$$

Thus if we use the relation $T_{i_1\ldots i_p}=-\widetilde{\widetilde{T}}_{i_1\ldots i_p}$, then by Lemma 3.1 we have the following

LEMMA 3.4. *In an almost-Kählerian space, if $T_{i_1\ldots i_p}$ is a skew-symmetric almost-analytic tensor, then we have*

$$(\nabla^l\nabla_lT_{i_1\ldots i_p}-pR_{i_1}{}^sT_{si_2\ldots i_p})T^{i_1\ldots i_p}=0.$$

4. Main theorem.

THEOREM 4.1. *In a compact almost-Kählerian space, if a skew-symmetric pure tensor $T_{i_1\ldots i_p}$ is almost-analytic, then it is harmonic.*

PROOF. From Lemma 3.3, we have

$$\nabla_hT_{i_1\ldots i_p}+\varphi_h{}^l\varphi_{i_1}{}^t\nabla_lT_{ti_2\ldots i_p}-\varphi_h{}^l\sum_{r=2}^{p}(\nabla_{i_r}\varphi_l{}^t)T_{i_1\ldots t\ldots i_p}=0, \tag{4. 1}$$

$$(\nabla^t\varphi_{hi_1})T_{ti_2\ldots i_p}=0. \tag{4. 2}$$

Operating ∇^{i_2} to (4.2), we find

$$(\nabla^{i_2}\nabla^t\varphi_{hi_1})T_{ti_2\ldots i_p}+(\nabla^t\varphi_{hi_1})\nabla^{i_2}T_{ti_2\ldots i_p}=0. \tag{4. 3}$$

On the other hand, by the Ricci's identity

$$(\nabla^a\nabla^b\varphi_{hi_1})T_{abi_3\ldots i_p}=\frac{1}{2}(\nabla^a\nabla^b\varphi_{hi_1}-\nabla^b\nabla^a\varphi_{hi_1})T_{abi_3\ldots i_p}$$

$$= -\frac{1}{2}(R^{ab}{}_{h}{}^{t}\varphi_{ti_1} + R^{ab}{}_{i_1}{}^{t}\varphi_{ht})T_{abi_3\ldots i_r}$$

and therefore (4.3) turns to

$$(R^{ab}{}_{h}{}^{t}\varphi_{ti_1} + R^{ab}{}_{i_1}{}^{t}\varphi_{ht})T_{abi_3\ldots i_p} + 2(\nabla^t\varphi_{hi_1})\nabla^{i_2}T_{ti_2\ldots i_p} = 0. \tag{4.4}$$

Transvecting (4. 4) with $\varphi_c{}^{i_1}$, we get

$$(R^{ab}{}_{hc} - \varphi_h{}^t\varphi_c{}^{i_1}R^{ab}{}_{ti_1})T_{abi_3\ldots i_p} + 2\varphi_c{}^{i_1}(\nabla^t\varphi_{hi_1})\nabla^{i_2}T_{ti_2\ldots i_p} = 0$$

i.e.

$$-(R^{ab}{}_{hc} + \varphi_h{}^t\varphi_c{}^{i_1}R^{ab}{}_{ti_1})T_{abi_3\ldots i_p} + 2R^{ab}{}_{hc}T_{abi_3\ldots i_p} + 2\varphi_c{}^{i_1}(\nabla^t\varphi_{hi_1})\nabla^{i_2}T_{ti_2\ldots i_p} = 0. \tag{4.5}$$

But, we have

$$(R^{ab}{}_{hc} + \varphi_h{}^t\varphi_c{}^{i_1}R^{ab}{}_{ti_1})T_{abi_3\ldots i_p}T^{hci_3\ldots i_p} = 0$$

because $R^{ab}{}_{hc} + \varphi_h{}^t\varphi_c{}^{i_1}R^{ab}{}_{ti_1}$ is hybrid in h, c and $T^{hci_3\ldots i_p}$ is pure in h, c. Hence, multiplying (4.5) by $T^{hci_3\ldots i_p}$, we find

$$[R^{ab}{}_{hc}T_{abi_3\ldots i_p} + \varphi_c{}^{i_1}(\nabla^t\varphi_{hi_1})\nabla^{i_2}T_{ti_2\ldots i_p}]\, T^{hci_3\ldots i_p} = 0. \tag{4.6}$$

In this place, since by (2.6) $\nabla_t\varphi_{hi_1}$ is pure in t, i_1, we have

$$\varphi_c{}^{i_1}\nabla_t\varphi_{hi_1} = \varphi_t{}^{i_1}\nabla_{i_1}\varphi_{hc}$$

and consequently (4.6) can be written in the form

$$[R^{ab}{}_{hc}T_{abi_3\ldots i_p} + \varphi^{ti_1}(\nabla_{i_1}\varphi_{hc})\nabla^{i_2}T_{ti_2\ldots i_p}]T^{hci_3\ldots i_p} = 0. \tag{4.7}$$

In the next place, transvecting (4.1) with $g^{i_1i_2}$ and taking account of skew-symmetricity of $T_{i_1\ldots i_p}$, we get

$$\varphi_h{}^l\varphi^{i_2t}\nabla_l T_{ti_2\ldots i_p} = \varphi_h{}^l(\nabla^{i_1}\varphi_l{}^t)T_{i_1ti_3\ldots i_p}$$

and again transvecting the last equation with $\varphi_k{}^h$, we have

$$\varphi^{i_2t}\nabla_k T_{ti_2\ldots i_p} = (\nabla^{i_1}\varphi_k{}^t)T_{i_1ti_3\ldots i_p}. \tag{4.8}$$

Since, by (4.2) the right-hand side of (4.8) vanishes, we have

$$\varphi^{ti_2}\nabla_k T_{ti_2\ldots i_p} = 0. \tag{4.9}$$

On the other hand, $\varphi^{ti_2}T_{ti_2\ldots i_p} = 0$ because φ^{ti_2} is hybrid in t, i_2 and $T_{ti_2\ldots i_p}$ is pure in t, i_2. Operating ∇_k to the last equation, we get

$$\varphi^{ti_2}\nabla_k T_{ti_2\ldots i_p} + (\nabla_k\varphi^{ti_2})T_{ti_2\ldots i_p} = 0$$

from which and (4.9), it follows

$$(\nabla_k\varphi^{ti_2})T_{ti_2\ldots i_p} = 0. \tag{4.10}$$

Accordingly from (4.7), we have

$$R^{ab}{}_{hc}T_{abi_3\ldots i_p}T^{hci_3\ldots i_p} = 0. \tag{4.11}$$

Hence, if in an almost-Kählerian space a skew-symmetric tensor $T_{i_1\ldots i_p}$ is almost-analytic, then by Lemma 3.4 and (4.11) we find

$$(4.12) \qquad (\Delta T_{i_1\dots i_p})T^{i_1\dots i_p}$$
$$= (\nabla^h\nabla_h T_{i_1\dots i_p} - \sum_{r=1}^{p} R_{i_r}{}^t T_{i_1\dots t\dots i_p} - \sum_{t<s}^{p} R^{ab}{}_{i_t i_s} T_{i_1\dots a\dots b\dots i_p})T^{i_1\dots i_p}$$
$$= (\nabla^h\nabla_h T_{i_1\dots i_p} - pR_{i_1}{}^t T_{ti_2\dots i_p} - \frac{p(p-1)}{2} R^{ab}{}_{i_1 i_2} T_{abi_3\dots i_p})T^{i_1\dots i_p} = 0$$

where $\Delta T_{i_1\dots i_p}$ is the Laplacian of $T_{i_1\dots i_p}$.

Thus, from the well known integral formula

$$(4.13) \qquad \int_{X_{2n}} \Big[(\Delta T_{i_1\dots i_p})T^{i_1\dots i_p} + (p+1)\nabla^{[h}T^{i_1\dots i_p]}\nabla_{[h}T_{i_1\dots i_p]} + p(\nabla_l T^{li_2\dots i_p})\nabla^t T_{ti_2\dots i_p}\Big] d\sigma = 0$$ [9],

we can deduce $\nabla_{[h}T_{i_1\dots i_p]} = 0$ and $\nabla^t T_{ti_2\dots i_p} = 0$, that is, $T_{i_1\dots i_p}$ is a harmonic tensor. q.e.d.

Moreover, according to this theorem, Lemma 3.1 and Lemma 3.2, we have the following

THEOREM 4.2.[10] *In a compact almost-Kählerian space, a necessary and sufficient condition that a skew-symmetric pure tensor $T_{i_1\dots i_p}$ be almost-analytic is that $T_{i_1\dots i_p}$ and $\widetilde{T}_{i_1\dots i_p}$ are both harmonic.*

We conclude this section with the following two theorems.

THEOREM 4.3. *In an almost-Kählerian space, if a skew-symmetric almost-analytic tensor $T_{i_1\dots i_p}$ is closed, then $T_{i_1\dots i_p}$ and $\widetilde{T}_{i_1\dots i_p}$ are both harmonic.*

PROOF. Transvecting (4.1) with g^{hi_1}, we get

$$(4.14) \qquad 2\nabla^h T_{hi_2\dots i_p} = \sum_{r=2}^{p} \varphi^{hl}(\nabla_{i_r}\varphi_l{}^t)T_{hi_2\dots t\dots i_p}$$

but since $\nabla_{i_r}\varphi_l{}^t$ is pure in i_r, l, we have $\varphi^{hl}\nabla_{i_r}\varphi_l{}^t = \varphi_{i_r}{}^l\nabla_l\varphi^{ht}$ and therefore (4.14) turns to

$$(4.15) \qquad 2\nabla^h T_{hi_2\dots i_p} = \sum_{r=2}^{p} \varphi_{i_r}{}^l(\nabla_l\varphi^{ht})T_{hi_2\dots t\dots i_p}.$$

From (4.15), we have

$$\nabla^h T_{hi_2\dots i_p} = 0$$

because by virtue of (4.10), the right-hand side of (4.15) vanishes. Hence for $\widetilde{T}_{i_1\dots i_p}$ also, we have

9) K. Yano and S. Bochner [7].
10) For a vector, see S. Tachibana [3].

$$\nabla^{h'}\widetilde{T}_{hi_2\ldots i_p} = 0$$

and if $T_{i_1\ldots i_p}$ is closed, then by (3.2) $\widetilde{T}_{i_1\ldots i_p}$ is also closed. q.e.d.

By this theorem and Lemma 3.2, we have

THEOREM 4.4. *In an almost-Kählerian space, if skew-symmetric pure tensors* $T_{i_1\ldots i_p}$ *and* $\widetilde{T}_{i_1\ldots i_p}$ *are both closed, then they are both harmonic.*

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